A NOTE ON EUCLIDEAN ORDER TYPES

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Abstract. Euclidean functions with values in an arbitrary well-ordered set were first considered by Motzkin in 1949 and studied in more detail by Samuel and Nagata in the 1970's and 1980's. Here these results are revisited, simplified, and extended. The main themes are (i) consideration of **Ord**-valued functions on an Artinian poset and (ii) use of ordinal arithmetic, including the Hessenberg-Brookfield ordinal sum. To any Euclidean ring we associate an ordinal invariant, its **Euclidean order type**, and we initiate a study of this invariant, especially for Euclidean rings which are not domains.

Throughout, "a ring" means a commutative ring with identity. We denote by $R^\bullet$ the set $R \setminus \{0\}$ and by $R^\times$ the group of units of R. We denote by $\mathbb{N}$ the natural numbers, including 0. When we are thinking of $\mathbb{N}$ as the least infinite ordinal, we denote it by ω. For a class S equipped with a "zero element" – here, the least element of an ordered class or the identity element of a monoid – we put $S^\bullet = S \setminus \{0\}$.

By an **ordered class** X we mean a pair $(X, \leq)$ with X a class and $\leq$ a reflexive, anti-symmetric, transitive relation on X (i.e., what is often called a *partial ordering*). For $x \in (X, \leq)$ we define the **principal downsets**[1]

$$D^\circ(x) = \{y \in X \mid y < x\},\ D(x) = \{y \in X \mid y \leq x\}.$$

We will encounter some ordinal arithmetic, and it is important to remember that the "ordinary" sum and product of transfinite ordinal numbers need not be commutative. The literature seems to agree that for $\alpha, \beta \in \mathbf{Ord}$, $\alpha + \beta$ should be the order type of a copy of β placed *above* a copy of α, so that $\omega + 1 > \omega$, $1 + \omega = \omega$. However, both conventions on $\alpha\beta$ seem to be in use. We take the one in which $2\omega = \omega + \omega$, *not* the one in which $2\omega = 2 + 2 + \ldots = \omega$.

Introduction

What is a Euclidean ring? In the classical literature, a Euclidean ring is a commutative domain R admitting a *Euclidean function* $\varphi : R^\bullet \to \mathbb{N}$, i.e., such that for all $a \in R$ and $b \in R^\bullet$, there are $q, r \in R$ with $a = qb + r$ and either $r = 0$ or $\varphi(r) < \varphi(b)$. It was observed by T. Motzkin [Mo49] that one can instead take the codomain of the Euclidean function to be any well-ordered set. In [Sa71] P. Samuel observed that much of the theory goes through for commutative rings with zero divisors.[2] In all of these contexts a Euclidean ring remains a **principal ring**: every ideal is singly generated.

Let us give a simple example of the usefulness of admitting Euclidean rings which

Thanks to John Doyle, David Krumm and Robert Varley for inspiring this work.

[1]Perhaps we should say "downclass", but though we will consider some proper classes, in cases of interest to us, each $D(x)$ will be a set.

[2]Non-commutative Euclidean rings have also been pursued, but will not be here.

are not domains.[3] One would like to know (e.g. when studying the action of $\mathrm{SL}_2(\mathbb{Z})$ on the Riemann sphere by linear fractional transformations) that for all $m, n \in \mathbb{Z}^+$, the *reduction map* $r : \mathrm{SL}_m(\mathbb{Z}) \to \mathrm{SL}_m(\mathbb{Z}/n\mathbb{Z})$ is surjective. A natural proof uses the following idea: if I is an ideal in a ring R, then $r : \mathrm{SL}_m(R) \to \mathrm{SL}_m(R/I)$ is surjective when $\mathrm{SL}_m(R/I)$ is generated by **transvections** – a transvection is a matrix $E_{ij}(\alpha)$ for $1 \leq i \neq j \leq m$, $\alpha \in R$ obtained from the identity matrix by changing the (i, j) entry from 0 to α – since r is visibly surjective on transvections.

Theorem. *If $\mathfrak{r}$ is Euclidean, $\mathrm{SL}_m(\mathfrak{r})$ is generated by transvections for all $m \in \mathbb{Z}^+$.*

This result is part of the classical literature – see e.g. [vdW] – when R is a domain with $\mathbb{N}$-valued Euclidean function, and the argument goes over verbatim to the more general definition given here: one uses the Euclidean algorithm and elementary row and column operations to reduce any matrix $M \in \mathrm{GL}_n(\mathfrak{r})$ to a diagonal matrix D with $D_{ii} = 1$ for all $i > 1$.

To complete the argument we must verify that $\mathbb{Z}/m\mathbb{Z}$ is Euclidean. It is easy to build a Euclidean function on $\mathbb{Z}/m\mathbb{Z}$ from a Euclidean function on $\mathbb{Z}$: this is an embryonic form of Theorem 2.35. However, this answer requires us to use that $\mathbb{Z}$ is Euclidean, whereas in fact such surjectivity results hold much more generally.

Theorem. *Let R be a Dedekind domain, I a nonzero ideal of R and $m \in \mathbb{Z}^+$. Then the reduction map $r : \mathrm{SL}_m(R) \to \mathrm{SL}_m(R/I)$ is surjective.*

Proof. Since R is Dedekind we may write $I = \mathfrak{p}_1^{a_1} \cdots \mathfrak{p}_r^{a_r}$ as a product of powers of distinct prime ideals. Let $S = R \setminus \bigcup_{i=1}^r \mathfrak{p}_i$. The localization $S^{-1}R$ is Dedekind with finitely many nonzero primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, hence a PID. Thus $R/I \cong S^{-1}R/IS^{-1}R$ is an Artinian principal ring, hence Euclidean (Theorem 3.1), so $\mathrm{SL}_m(R/I)$ is generated by transvections. As above, this implies the surjectivity of r. $\square$

In any Euclidean ring there is a *bottom Euclidean function*, which corresponds to the most efficient Euclidean algorithm in a sense made precise in Lemmas 2.22 and 2.23. The set of values taken by the bottom Euclidean function on nonzero elements of R is an ordinal number $e(R)$, and thus we have associated to any Euclidean ring an ordinal invariant, the **Euclidean order type**.

All the prerequisites for such a definition appear in the work of Motzkin and Samuel, but because such a definition was not given, until now almost all work has focsed on the dichotomy $e(R) = \omega$ versus $e(R) > \omega$. By a result of C.R. Fletcher, a Euclidean ring which is neither a domain nor Artinian must have $e(R) > \omega$, thus for such rings the dichotomy is decided...but it is still interesting to ask about the Euclidean order type! The Euclidean order type of a Euclidean ring which is neither a domain nor an Artinian ring is studied here for the first time.

By a result of Zariski and Samuel, every principal ring is the product of finitely many PIDs and a principal Artinian ring. Thus much of our work turns on understanding the bottom Euclidean function and the Euclidean order type of a product in terms of the bottom Euclidean function and the order type of its factors. Samuel and Nagata each showed that a finite product of Euclidean rings is Euclidean. When one

[3]This example came up in the problem session of a course taught by R. Varley and the author in 2012. John Doyle and David Krumm, (then) graduate students, presented solutions in which these ideas appeared.

analyzes these constructions quantitatively, it turns out that for Euclidean rings R_1 and R_2 the natural upper bound on $e(R_1 \times R_2)$ is not the usual ordinal sum $e(R_1)+e(R_2)$ but the **Hessenberg sum** $e(R_1) \oplus e(R_2)$ (defined in § 1.2). Although our initial goal was to compute $e(R_1 \times R_2)$ in terms of $e(R_1)$ and $e(R_2)$, one of the main results of this paper (Theorem 2.40) is the pair of inequalities

$$\max e(R_1) + e(R_2), e(R_2) + e(R_1) \leq e(R_1 \times R_2) \leq e(R_1) \oplus e(R_2).$$

We can show that equality holds in many cases, and we do not know of any cases of inequality: see Question 4.4 and the following discussion.

Identifying the Samuel-Nagata product construction with the Hessenberg sum uses work of G. Brookfield [Br02]. More generally [Br02] develops the notion of the transfinite length function λ_X on a downward-small Artinian class X. The more we examined the Motzkin-Samuel-Nagata theory of transfinite Euclidean rings, the more we found connections to transfinite length functions. Especially, any Euclidean function on a Noetherian ring R must be at least as large as the transfinite length function ℓ_R on its dual ideal lattice and thus $e(R)$ must be at least as large as the tranfinite length $\operatorname{len} R$. It is interesting to consider the case of equality $\varphi_R = \ell_R$ – we call these ℓ-Euclidean rings and study them in § 3.1 – as well as the case of equality $e(R) = \operatorname{len} R$ – we call these small Euclidean rings and study them in § 3.2.

This paper is partially expository: we have reproduced much of the foundational theory of Motzkin, Samuel and Nagata, in a version which emphasizes connections to transfinite length functions. (An earlier version of this paper was more streamlined – concentrating only on novel results – but was found difficult to read.)

We believe that we have included attributions of all previously known results. In several cases the result that we present is technically novel but is a modest generalization of a previously known result: we indicate this by prefacing the attribution with "c.f.". Nevertheless this paper contains a number of new results, including Theorems 2.17, 2.31, 2.35, 2.40, 2.41, 3.2, 3.4, 3.6 and 3.7.

1. Ordered Classes, Isotone Maps, and Length Functions

1.1. Isotone Maps and Artinian Ordered Classes.

For a set X, let $\mathbf{Ord}^X$ denote the class of all maps $f : X \to \mathbf{Ord}$, ordered by $f \leq g \iff \forall x \in X,\ f(x) \leq g(x)$.[4] Note that every nonempty subclass $\mathcal{C} = \{f_c\}$ has an infimum in $\mathbf{Ord}^X$: that is, there is a largest element $f \in \mathbf{Ord}^X$ with the property that $f \leq f_c$ for all $f_c \in \mathcal{C}$. Indeed, we may take $f(x) = \min_c f_c(x)$.

An ordered class $\mathcal{C}$ is **downward small** if for all $x \in \mathcal{C}$, $\{y \in X \mid y \leq x\}$ is a set. For any set X, $\mathbf{Ord}^X$ is downward small.

For ordered classes X and Y, let $X \times Y$ be the Cartesian product, with $(x_1, y_1) \leq (x_2, y_2) \iff x_1 \leq x_2$ and $y_1 \leq y_2$. A map $f : X \to Y$ is **weakly isotone** (resp. **isotone**) if $x_1 \leq x_2 \in X \implies f(x_1) \leq f(x_2)$ (resp. $x_1 < x_2 \implies f(x_1) < f(x_2)$). A map $f : X \to Y$ is **exact** if for all $x \in X$, $D(f(x)) \subseteq f(D(x))$.

[4] We hasten to reassure the reader that this is the limit of our set-theoretic ambitiousness: we will never consider the collection of all maps between two proper classes!

The composite of weakly isotone (resp. isotone, resp. exact) maps is weakly isotone (resp. isotone, resp. exact).

An ordered class X is **Noetherian** (resp. **Artinian**) if there is no isotone map $f : \mathbb{Z}^+ \to X$ (resp. $f : \mathbb{Z}^- \to X$).

For an ordered *set* X, we define $\mathrm{Iso}(X) \subset \mathbf{Ord}^X$ to be the subclass of isotone maps, with the induced partial ordering.

Lemma 1.1. *Let X be an ordered set. Then every nonempty subclass of* $\mathrm{Iso}(X)$ *has an infimum in* $\mathrm{Iso}(X)$.

Proof. Let $I = \{f_i\}$ be a nonempty subclass of $\mathrm{Iso}(X)$, and let f be the infimum in $\mathbf{Ord}^X$; it suffices to show that f is isotone. If $x < y$ in X, then $f_i(x) < f_i(y)$ for all $i \in I$, so $f(x) = \min_{i \in I} f_i(x) < \min_{i \in I}(f_i(x) + 1) \leq \min_{i \in I} f_i(y) = f(y)$. □

Theorem 1.2. *For a downward small ordered class X, the following are equivalent:*
*(i) There is an isotone map $f : X \to$ **Ord**.*
(ii) There is an Artinian ordered set Y and an isotone map $f : X \to Y$.
(iii) X is Artinian.

Proof. (i) $\implies$ (ii): If $f : X \to \mathbf{Ord}$ is an isotone map, then $f : X \to f(X)$ is an isotone map with codomain an Artinian ordered set.
(ii) $\implies$ (iii): Suppose not: then there is an isotone map $\iota : \mathbb{Z}^- \to X$. But then $f \circ \iota : \mathbb{Z}^- \to Y$ is isotone, so Y is not Artinian.
(iii) $\implies$ (i) [Na85, Prop. 4], [Br02, Thm. 2.5]: We will construct $\lambda_X \in \mathrm{Iso}(X)$ by a transfinite process. For $\alpha \in \mathbf{Ord}$, at the αth stage we assign some subset $X_\alpha \subset X$ the value α. Namely, take X_0 to be the set of minimal elements of X, and having defined X_β for all $\beta < \alpha$, assign the value α to all minimal elements of $X \setminus \bigcup_{\beta < \alpha} X_\beta$. Thus $\lambda_X(x) = \alpha$ iff α is the least number greater than $\lambda_X(y)$ for all $y < x$. Let $X' = \bigcup_{\alpha \in \mathbf{Ord}} X_\alpha$. We claim that since X is Artinian and downward small, $X' = X$: if not, let x be a minimal element of $X \setminus X'$. Then λ_X is defined on $D^\circ(x) = \{y \in X \mid y < x\}$, so if $\alpha = \sup \lambda_X(D^\circ(x))$, then $\lambda_X(x) = \alpha$ (if $\alpha \notin \lambda_X(D^\circ(x))$ or $\lambda_X(x) = \alpha + 1$ (if $\alpha \in \lambda_X(D^\circ(x))$). □

1.2. **Length Functions.**

By Theorem 1.2 and Lemma 1.1, for any Artinian ordered set X, $\mathrm{Iso}(X)$ has a bottom element. In fact, the map $\lambda_X : X \to \mathbf{Ord}$ constructed in the proof of Theorem 1.2 is the bottom element of $\mathrm{Iso}(X)$: let $\lambda : X \to \mathbf{Ord}$ be any isotone map. If it is not the case that $\lambda_X \leq \lambda$, then $\{x \in X \mid \lambda(x) < \lambda_X(x)\}$ is nonempty so has a minimal element x_0. Then for all $x < x_0$, $\lambda_X(x) \leq \lambda(x)$. Since $\lambda_X(x_0)$ is the least ordinal strictly greater than $\lambda_X(x)$ for all $x < x_0$ and $\lambda(x_0) < \lambda(x_0)$, there is $x < x_0$ with $\lambda(x_0) \leq \lambda_X(x) \leq \lambda(x)$, so λ is not isotone: contradiction.

Following [Br02], we call λ_X the **length function**.

Remark 1.3. *The length function λ_X is also characterized among all elements of* $\mathrm{Iso}(X)$ *by being exact* [Br02, Thm. 2.3].

If X has a top element T, we define the **length of X** to be $\mathrm{len}(X) = \lambda_X(T)$.

Example 1.4. *For $\alpha \in$ **Ord** and $x \leq \alpha$, $\lambda_{\alpha+1}(x) = x$, so* $\mathrm{len}(\alpha + 1) = \alpha$.

Example 1.5. *For $m, n \in \mathbb{Z}^+$, let $X_1 = \{0, \ldots, m\}$ and $X_2 = \{0, \ldots, n\}$, and let $X = X_1 \times X_2$. Then for all $(i, j) \in X$, $\lambda_X(i, j) = i + j$ and $\mathrm{len}(X) = m + n$.*

The partially ordered class $\mathbf{Ord} \times \mathbf{Ord}$ is Artinian and downward small, so it has a length function $\lambda_{\mathbf{Ord}\times\mathbf{Ord}}$. For $\alpha, \beta \in \mathbf{Ord}$, we define the **Brookfield sum**

$$\alpha \oplus_B \beta = \lambda_{\mathbf{Ord}\times\mathbf{Ord}}(\alpha, \beta) = \mathrm{len}((\alpha + 1) \times (\beta + 1)).$$

We recall the following, a version of the **Cantor normal form**: for any $\alpha, \beta \in \mathbf{Ord}^\bullet$ there are $\gamma_1, \ldots, \gamma_r \in \mathbf{Ord}$, $r \in \mathbb{Z}^+$ and $m_1, \ldots, m_r, n_1, \ldots, n_r \in \mathbb{N}$ with

$$\alpha = m_1\omega^{\gamma_1} + \ldots + m_r\omega^{\gamma_r},\ \ \beta = n_1\omega^{\gamma_1} + \ldots + n_r\omega^{\gamma_r}.$$

This representation of the pair (α, β) is unique if we require $\max(m_i, n_i) > 0$ for all i. We may then define the **Hessenberg sum**

$$\alpha \oplus_H \beta = (m_1 + n_1)\omega^{\gamma_1} + \ldots + (m_r + n_r)\omega^{\gamma_r}.$$

(For all $\alpha \in \mathbf{Ord}$, we define $\alpha \oplus_H 0 = 0 \oplus_H \alpha = \alpha$.)

Theorem 1.6. *For all $\alpha, \beta \in \mathbf{Ord}$, $\alpha \oplus_B \beta = \alpha \oplus_H \beta$.*

Proof. See [Br02, Thm. 2.12]. □

In view of Theorem 1.6 we write $\alpha \oplus \beta$ for $\alpha \oplus_B \beta = \alpha \oplus_H \beta$ and speak of the **Hessenberg-Brookfield sum**. This operation is well-known to the initiates of ordinal arithmetic, who call it the "natural sum". The next result collects facts about $\alpha \oplus \beta$ for our later use.

Proposition 1.7. *Let $\alpha, \beta, \gamma \in \mathbf{Ord}$.*
a) We have $\alpha \oplus \beta = \beta \oplus \alpha$ and $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$.
b) If $\beta < \omega$, then $\alpha + \beta = \alpha \oplus \beta$.
c) If $\alpha, \beta > 0$, then we have

$$\max(\alpha + \beta, \beta + \alpha) \leq \alpha \oplus \beta \leq \alpha\beta + \beta\alpha. \tag{1}$$

Proof. Left to the reader. □

The following result of Brookfield is a generalization of Example 1.5.

Theorem 1.8. *(*[Br02, Thm. 2.9]*) Let X and Y be Artinian ordered sets. Then*

$$\lambda_{X\times Y} = \lambda_X \oplus \lambda_Y.$$

In particular, if X and Y have top elements, then

$$\mathrm{len}(X \times Y) = \mathrm{len}(X) \oplus \mathrm{len}(Y).$$

Proof. Define $\lambda : X \times Y \to \mathbf{Ord}$ by $\lambda(x, y) = \lambda_X(x) \oplus \lambda_Y(y)$. Then

$$\lambda = \lambda_{\mathbf{Ord}\times\mathbf{Ord}} \circ (\lambda_X \times \lambda_Y).$$

As a composite of exact, isotone maps, λ is exact and isotone. By Remark 1.3,

$$\lambda = \lambda_{X\times Y}.$$

□

1.3. **The length function on a Noetherian ring.**

In this section we closely follow work of Gulliksen [Gu73] and Brookfield [Br02].

Let R be a ring, and let $\mathcal{I}(R)$ be the lattice of ideals of R. Then $\mathcal{I}(R)$ is Noetherian (resp. Artinian) iff R is Noetherian (resp. Artinian). Thus the dual lattice $\mathcal{I}^{\vee}(R)$ is Artinian (resp. Noetherian) iff R is Noetherian (resp. Artinian).

Henceforth we suppose R is Noetherian, so $\mathcal{I}^{\vee}(R)$ is Artinian with top element (0). By the results of §1 there is a least isotone map $\ell_R : \mathcal{I}^{\vee}(R) \to \mathbf{Ord}$, the **length function** ℓ_R of R, and we define the **length of R** as $\mathrm{len}(R) = \ell_R((0))$.

For any ideal I of R, R/I is Noetherian, so $\ell_{R/I}$ and $\mathrm{len}(R/I)$ are well-defined. If we denote the quotient map $R \to R/I$ by q, then the usual pullback of ideals q^* identifies $\mathcal{I}(R/I)$ with an ordered subset of $\mathcal{I}(R)$ and hence also $\mathcal{I}^{\vee}(R/I)$ with an ordered subset of $\mathcal{I}^{\vee}(R)$, and it is easy to see that under this identification we have

$$\ell_R|_{\mathcal{I}^{\vee}(R/I)} = \ell_{R/I},$$

and thus also

$$\mathrm{len}(R/I) = \ell_{R/I}((0))) = \ell_R(q^*((0))) = \ell_R(I).$$

(To ease notation, for $x \in R$ we will write $\ell_R(x)$ for $\ell_R((x))$.)

Lemma 1.9. *a) If R is a Noetherian ring, then $\mathrm{len}(R) < \omega \iff R$ is Artinian.*
b) Suppose R is a PID. For any $x = u\pi_1 \cdots \pi_r \in R^{\times}$ – here $u \in R^{\times}$ and each π_i is a prime element – we have $\ell_R(x) = r$. Moreover $\mathrm{len}\, R = \ell_R(0) = \omega$.

Proof. Left to the reader. □

Theorem 1.10. *Let $R_1, \ldots, R_n$ be rings. Put $R = \prod_{i=1}^n R_i$, and let $e_i = (0, \ldots, 1$ in the ith place $, 0, \ldots, 0)$.*
a) The map $\mathcal{I}^{\vee}(R) \to \prod_{i=1}^n \mathcal{I}^{\vee}(R)$, $I \mapsto (e_1 I, \ldots, e_n I)$ is an order isomorphism.
b) R is Noetherian iff R_i is Noetherian for all i.
c) If R is Noetherian, then under the isomorphism of part a),

$$\ell_R = \bigoplus_{i=1}^n \ell_{R_i}. \tag{2}$$

d) If R is Noetherian, then $\mathrm{len}\, R = \bigoplus_{i=1}^n \mathrm{len}\, R_i$.

Proof. Part a) is left to the reader. Part b) follows: a finite product $\prod_{i=1}^n X_i$ of nonempty ordered sets is Noetherian iff each X_i is Noetherian. Part c) follows immediately from Theorem 1.8. For part d), evaluate (2) at the zero ideal. □

2. Euclidean Functions

2.1. **Preliminaries on Principal Rings.**

Theorem 2.1. *a) If $R = \prod_{i=1}^r R_i$, then R is principal iff R_i is principal for all i.*
b) For every principal ring R there is $n \in \mathbb{N}$, a finite set of principal ideal domains $R_1, \ldots, R_n$ and a principal Artinian ring A such that

$$R \cong \prod_{i=1}^n R_i \times A. \tag{3}$$

The isomorphism classes of the R_i's (up to reordering) and A are uniquely determined by R. We call A the ***Artinian part*** *of R.*
c) A ring is Artinian principal iff it is isomorphic to a finite product of local *Artinian principal rings.*

Proof. See [ZS, p. 245]. $\square$

For a property P of rings, we will say that a ring R is **residually P** if for every nonzero ideal I of R, R/I has property P.

Proposition 2.2. *For a principal ring R the following are equivalent:*
(i) R is residually Artinian.
(ii) R is either a PID or an Artinian ring.
(iii) $\operatorname{len} R \leq \omega$.

Proof. Left to the reader. $\square$

2.2. **Basic Definitions.**

Let R be a ring (commutative, and with multiplicative identity as always).

A **Euclidean function** is a function $\varphi : R^\bullet \to \mathbf{Ord}$ such that for all $a \in R$, $b \in R^\bullet$, there are $q, r \in R$ with $a = qb + r$ and ($r = 0$ or $\varphi(r) < \varphi(b)$). A ring is **Euclidean** if it admits a Euclidean function.

Let $\operatorname{Euc} R \subset \mathbf{Ord}^{R^\bullet}$ be the subclass of Euclidean functions $\varphi : R^\bullet \to \mathbf{Ord}$, with the induced partial ordering. Thus R is Euclidean iff $\operatorname{Euc} R \neq \varnothing$.

Example 2.3. *Let $R = \mathbb{Z}$. Then $n \in \mathbb{Z}^\bullet \mapsto |n|$ is a Euclidean function.*

Example 2.4. *Let k be a field and let $R = k[t]$. Then $f \in k[t]^\bullet \mapsto \deg f$ is a Euclidean function.*

Example 2.5. *Let R be a local Artinian principal ring with maximal ideal (π). Then every element $x \in R^\bullet$ may be written in the form $u\pi^n$ for $u \in R^\times$ and $0 \leq n < \omega$. We claim that $\varphi : x \mapsto n$ is a Euclidean function. Indeed, let $a \in R$ and $b = u_b\pi_b^n \in R^\bullet$. We may assume $b \nmid a$ and thus $a = u_a\pi^{n_a}$ with $n_a < n_b$. Then $a = 0 \cdot b + a$ with $\varphi(a) < \varphi(b)$.*

Example 2.6. [Sa71, Prop. 5] *Let R be a PID with at least one and finitely many nonassociate maximal elements $\pi_1, \ldots, \pi_r$. Then every $x \in R^\bullet$ may be uniquely expressed as $u\pi_1^{x_1} \cdots \pi_r^{x_r}$ with $u \in R^\times$ and $x_i \in \mathbb{N}$. We claim that $\varphi : x \mapsto x_1 + \ldots + x_r$ is a Euclidean function on R. Indeed:*

Let $a \in R$ and $b = u_b\pi_1^{b_1} \cdots \pi_r^{b_r} \in R^\bullet$. We may assume $b \nmid a$. Then $a = u_a\pi_1^{a_1} \cdots \pi_r^{a_r} \in R^\bullet$ and the set

$$I = \{1 \leq i \leq r \mid a_i < b_i\}$$

is nonempty. By the Chinese Remainder Theorem, there is $r \in R$ with

$$r \equiv \begin{cases} a \pmod{\pi^{b_i}} & i \in I \\ b \pmod{\pi^{b_i+1}} & i \in \{1, \ldots, r\} \setminus I. \end{cases}$$

Then

$$\varphi(r) = \sum_{i \in I} a_i + \sum_{i \notin I} b_i < \sum_{i=1}^{r} b_i = \varphi(b).$$

Moreover $r - a$ is divisible by π^{b_i} for all $1 \leq i \leq r$ and hence of the form qb: this is clear if $i \in I$, while if $i \notin I$ then since $a_i \geq b_i$ we have $a \equiv 0 \equiv b \equiv r \pmod{\pi_i^{b_i}}$.

Proposition 2.7. *(*[Sa71, Prop. 3]*)*
*a) Let $\varphi : R^\bullet \to$ **Ord** be a Euclidean function. For every nonzero ideal I of R, let $x \in I^\bullet$ be such that $\varphi(x) \leq \varphi(y)$ for all $y \in I^\bullet$. Then $I = \langle x \rangle$.*
b) In particular, a Euclidean ring is principal.

Proof. Left to the reader. □

2.3. **Extension at Zero.**

It will be convenient to define our Euclidean functions at the zero element of R. There are several reasonable ways to do this. Although the initially appealing one is to take $\varphi(0) = 0$ and require φ to take nonzero values on $R^\bullet$, for us it will turn out to be useful (even critical, at times) to do exactly the opposite: we allow Euclidean functions to take the value zero at nonzero arguments, and we define $\varphi(0)$ to be the bottom element of $\mathbf{Ord} \setminus \varphi(R^\bullet)$. This is actually not so strange: after all, 0 is the *top* element of R with respect to the divisibility quasi-ordering.

Remark 2.8. *With the above convention, if $\varphi : R \to$ **Ord** is a Euclidean function, then for* all *$a, b \in R$, there are $q, r \in R$ with $a = qb + r$ and ($r = 0$ or $\varphi(r) < \varphi(b)$, i.e., this is true even when $b = 0$: if $a \neq 0$, take $r = a$ so $\varphi(r) = \varphi(a) < \varphi(0)$; if $a = 0$, take $r = 0$. (Having verified this, henceforth we may assume that $b \in R^\bullet$ when verifying a function is Euclidean.)*

Remark 2.9. *In* [Sa71] *a Euclidean function is defined to be a function $\varphi : R \to$ **Ord** such that for all $a \in R$ and $b \in R^\bullet$, there are $q, r \in R$ such that $a = qb + r$ and $\varphi(r) < \varphi(b)$. This definition yields $\varphi(0) < \varphi(x)$ for all $x \in R^\bullet$* [Sa71, Prop. 1].

We make the corresponding adjustment in our definition of $\operatorname{Euc} R$: it is now the ordered subclass of $\mathbf{Ord}^R$ consisting of Euclidean functions.

2.4. **Generalized Euclidean Functions.**

A **generalized Euclidean function** is a function φ from $R^\bullet$ to an Artinian ordered class X such that for all $a, b \in R$, there are $q, r \in R$ with $a = qb + r$ such that either $r = 0$ or $\varphi(r) < \varphi(b)$.

Lemma 2.10. *(c.f.* [Na85]*) Let X, Y be Artinian ordered classes, $\varphi : R \to X$ a generalized Euclidean function and $f : X \to Y$ an isotone map. Then $f \circ \varphi : R \to Y$ is a generalized Euclidean function.*

Proof. Left to the reader. □

Corollary 2.11. *(*[Sa71, Prop. 11], [Na85, Prop. 4]*) A ring which admits a generalized Euclidean function is Euclidean.*

Proof. If $\varphi : R \to X$ is generalized Euclidean, $\lambda_X \circ \varphi : R \to \mathbf{Ord}$ is Euclidean. □

Corollary 2.11 may suggest that there is nothing to gain in considering Euclidean functions with values in a non-well-ordered sets. But this is not the case!

Theorem 2.12. *(*[Na85, Thm. 2]*) Let R_1, R_2 be commutative rings, X_1, X_2 be Artinian ordered classes, and $\varphi_1 : R_1 \to X_1$, $\varphi_2 : R_2 \to X_2$ be generalized Euclidean functions. Then $\varphi_1 \times \varphi_2 : R_1 \times R_2 \to X_1 \times X_2$ is a generalized Euclidean function.*

Proof. Put $R = R_1 \times R_2$, and let $x = (x_1, x_2) \in R$, $y = (y_1, y_2) \in R^\bullet$. We may assume $y \nmid x$, so $x \in R^\bullet$. Since φ_1 and φ_2 are Euclidean, for $i = 1, 2$ there are $q_i, r_i \in R_i$ such that $x_i = q_i y_i + r_i$ and ($r_i = 0$ or $\varphi_i(r_i) < \varphi_i(y_i)$). Since $y \nmid x$, $r_1 \neq 0$ or $r_2 \neq 0$. Now:
Case 1: Suppose that any one of the following occurs:
(i) $r_1 \neq 0$ and $r_2 \neq 0$.
(ii) $r_1 = y_1 = 0$, and thus $r_2, y_2 \neq 0$.
(iii) $r_2 = y_2 = 0$, and thus $r_1, y_1 \neq 0$.
Put $q = (q_1, q_2)$ and $r = (r_1, r_2)$, so $x = qy + r$ and

$$\varphi(r) = (\varphi(r_1), \varphi(r_2)) < (\varphi(y_1), \varphi(y_2)) = \varphi(y).$$

Case 2: Suppose $r_1 = 0$, $y_1 \neq 0$, and thus $r_2 \neq 0$. Then take $q = (q_1 - 1, q_2)$ and $r = (y_1, r_2)$, so $x = qy + r$ and

$$\varphi(r) = (\varphi(y_1), \varphi(r_2)) < (\varphi(y_1), \varphi(y_2)) = \varphi(y).$$

Similarly, if $r_2 = 0$, then $y_2 \neq 0$.
Case 3: Suppose $r_2 = 0$, $y_2 \neq 0$, and thus $r_1 \neq 0$. Put $q = (q_1, q_2 - 1)$ and $r = (r_1, y_2)$, so $x = qy + r$ and

$$\varphi(r) = (\varphi(r_1), \varphi(y_2)) < (\varphi(y_1), \varphi(y_2)) = \varphi(y). \qquad \square$$

Theorem 2.13. *Let $R_1, \ldots, R_n$ be rings, and put $R = \prod_{i=1}^n R_i$.*
a) (c.f. [Na85]*) If $\varphi_i \in \operatorname{Euc} R_i$ for all i, then $\bigoplus_{i=1}^n \varphi_i \in \operatorname{Euc} R$.*
b) (c.f. [Fl71]*) The ring $R = \prod_{i=1}^n R_i$ is Euclidean iff each R_i is Euclidean.*

Proof. a) An easy induction reduces us to the $n = 2$ case. We have $\varphi_1 \oplus \varphi_2 = \lambda_{\mathbf{Ord} \times \mathbf{Ord}} \circ (\varphi_1 \times \varphi_2)$. Apply Theorem 2.12 and Lemma 2.10.
b) If each R_i is Euclidean, then by part a) R is Euclidean. Conversely, let $\varphi \in \operatorname{Euc} R$. Then $\varphi_i : x_i \mapsto \varphi(0, \ldots, 0, x_i, 0, \ldots, 0)$ lies in $\operatorname{Euc} R_i$. As we will later prove a more general result (Theorem 2.35), we leave the verification of this to the reader. $\square$

2.5. **Isotone Euclidean Functions.**

For a ring R, the divisibility relation is a **quasi-ordering** – i.e., reflexive and transitive but not necessarily anti-symmetric. If X and Y are quasi-ordered sets, we can define an isotone map $f : X \to Y$ just as above: if $x_1 < x_2 \implies f(x_1) < f(x_2)$.

Lemma 2.14. *Let X be a quasi-ordered set and Y be an ordered set. Suppose that $f : X \to Y$ is an isotone map. Then X is ordered.*

Proof. By contraposition: suppose there are $x_1, x_2 \in X$ such that $x_1 < x_2$, $x_2 < x_1$. Since f is isotone, this implies $f(x_1) < f(x_2)$, $f(x_2) < f(x_1)$. $\square$

If X is a quasi-ordered set, it has an ordered completion: i.e., an ordered set $\overline{X}$ and a weakly isotone map $X \to \overline{X}$ which is universal for weakly isotone maps from X into an ordered set: take the quotient of X under the equivalence relation $x_1 \sim x_2$ if $x_1 \leq x_2$ and $x_2 \leq x_1$. If we do this for the divisibility relation on R, we get the ordered set $\operatorname{Prin} R$ of principal ideals of R. This motivates the following definition: a Euclidean function $\varphi : R \to \mathbf{Ord}$ is **weakly isotone** (resp. **isotone**) if whenever x divides y, $\varphi(x) \leq \varphi(y)$ (resp. whenever x strictly divides y, $\varphi(x) < \varphi(y)$).

Much of the literature includes weak isotonicity as part of the definition of a Euclidean function. Doing so would not change the class of Euclidean rings. Indeed:

Theorem 2.15. *(c.f.* [Sa71, Prop. 4]*) Let $\varphi : R \to \mathbf{Ord}$ be a Euclidean function. Then the set of isotone Euclidean functions $\psi \leq \varphi$ has a maximal element*

$$\underline{\varphi} : x \in R \mapsto \min_{y \in (x)} \varphi(y).$$

Proof. Step 1: We show $\underline{\varphi}$ is Euclidean. Let $a, b \in R$. Then there exists $c \in R$ such that $\underline{\varphi}(b) = \varphi(bc)$. Since φ is Euclidean, there are $q, r \in R$ with $a = qbc + r$ and either $r = 0$ – so $b \mid a$– or $\underline{\varphi}(r) \leq \varphi(r) < \varphi(bc) = \underline{\varphi}(b)$.
Step 2: We show $\underline{\varphi}$ is isotone. For all $c \in R$,

$$\underline{\varphi}(a) = \min_{y \in R} \varphi(ay) \leq \min_{cy,\ y \in R} \varphi(acy) = \min_{y \in R} \varphi(acy) = \underline{\varphi}(ac).$$

Step 3: By construction $\underline{\varphi} \leq \varphi$. Moreover, if $\psi \leq \varphi$ is an isotone Euclidean function, then for all $a, c \in R$, $\psi(a) \leq \psi(ac) \leq \varphi(ac)$, so $\psi(a) \leq \underline{\varphi}(a)$. ☐

Corollary 2.16. *A Euclidean function is weakly isotone iff it is isotone.*

Proof. Of course any isotone function is weakly isotone. Conversely, let $\varphi : R \to \mathbf{Ord}$ be a weakly isotone Euclidean function, let $a, c \in R$, and suppose $\varphi(ac) = \varphi(c)$. Write $a = qac + r$ with $r = 0$ or $\varphi(r) < \varphi(ac) = \varphi(a)$. If $r \neq 0$, then $\varphi(r) = \varphi(a(1 - qc)) \geq \varphi(a)$, contradiction. So $r = 0$ and $(a) = (ac)$. ☐

The following result is a transfinite generalization of [Sa71, Cor. 2].

Theorem 2.17. *Let $\varphi \in \operatorname{Euc} R$. Then $\ell_R \leq \varphi$.*

Proof. By Theorem 2.15 we may assume φ is isotone. Since R is Euclidean it is principal and thus $\mathcal{I}^{\vee}(R) = R/R^{\times}$. So both ℓ and φ induce well-defined isotone functions on $R/R^{\times}$. But ℓ_R is the *least* isotone function on $\mathcal{I}^{\vee}(R)$, so $\ell_R \leq \varphi$. ☐

2.6. The Bottom Euclidean Function and the Euclidean Order Type.

Lemma 2.18. *Let R be a commutative ring. Then every nonempty subclass of $\operatorname{Euc} R$ has an infimum in $\operatorname{Euc}(R)$.*

Proof. Let $\mathcal{C} = \{\varphi_c\}$ be a nonempty subclass of $\operatorname{Euc}(R)$, and let φ be the infimum in $\mathbf{Ord}^R$; it suffices to show that φ is Euclidean. Let $a, b \in R$ with $b \nmid a$. Choose $i \in I$ such that $\varphi(b) = \varphi_i(b)$. Since φ_i is Euclidean, there are $q, r \in R$ such that $\varphi_i(r) < \varphi_i(b)$, and then $\varphi(r) \leq \varphi_i(r) < \varphi_i(b) = \varphi(b)$. ☐

Theorem 2.19. *(Motzkin-Samuel) Let R be a Euclidean ring.*
a) The class $\operatorname{Euc}(R)$ of all Euclidean functions on R has a bottom element φ_R.
b) The bottom Euclidean function φ_R is isotone.
c) The set $\varphi_R(R)$ is an ordinal.

Proof. a) This is immediate from Lemma 2.18.
b) Since φ_R is the bottom Euclidean function, we must have $\varphi_R = \underline{\varphi_R}$.
c) It's enough to show $\varphi_R(R)$ is downward closed: we need to rule out the existence of $\alpha < \beta \in \mathbf{Ord}$ such that $\varphi_R(R)$ contains β but not α. Let α' be the least element of $\varphi_R(R)$ exceeding α. If we redefine φ_R to take the value α whenever φ_R takes the value α', we get a smaller Euclidean function than φ_R: contradiction. ☐

For a Euclidean ring R we define the **Euclidean order type** $e(R) = \varphi_R(0) \in \mathbf{Ord}$.

Theorem 2.20. *(Fletcher* [Fl71]*) Let R be a Euclidean ring.*
a) R is Artinian iff $e(R) < \omega$.
b) If $e(R) = \omega$, then R is a PID.

Proof. a) Since φ_R is isotone, its value on $x \in R$ depends only on the ideal (x). But an Artinian principal ring has only finitely many ideals! So $e(R) < \omega$. If R is not Artinian, then there is an infinite descending chain $\{(x_i)\}_{i=1}^{\infty}$ of ideals, and by isotonicity we have $\varphi(x_1) < \varphi(x_2) < \ldots \varphi(x_n) < \ldots$, so $e(R) \geq \omega$.
b) We will show the contrapositive: suppose R is not a PID. If R is Artinian then $e(R) \neq \omega$ by part a); otherwise, by Proposition 2.2, R is not residually Artinian: there is $b \in R^{\bullet}$ and an infinite descending chain of ideals $\{(x_i)\}_{i=1}^{\infty}$ with $(x_i) \supset (b)$ for all i. Isotonicity gives $\varphi_R(b) \geq \omega$ and thus $e(R) = \varphi_R(0) > \omega$. □

The following result shows that the converse of Theorem 2.20b) does not hold.

Theorem 2.21. *(Hiblot* [Hi75], [Hi77]*, Nagata* [Na78]*) There is a Euclidean domain R with $e(R) > \omega$.*

2.7. **The Motzkin-Samuel Process.**

Our proof that a Euclidean ring admits a minimal Euclidean function is maximally inexplicit: it yields no information on what the minimal Euclidean function may be, nor does it give any information on whether a given ring admits a Euclidean function. In the breakthrough paper [Mo49], T. Motzkin gave an explicit transfinite construction of φ_R. Motzkin's construction was refined by Samuel in [Sa71], so we speak of the **Motzkin-Samuel Process**. As we will need to make use of this process in our treatment of the quotient Euclidean function we give it here in full detail. Our treatment is close to Samuel's except for minor differences stemming from our convention on $\varphi(0)$: c.f. Remark 2.9.

For any set R and map $\varphi : R \to \mathbf{Ord}$, we define the **open segments**
$$\Phi_\alpha^\circ = \{x \in R \mid \varphi(x) < \alpha\}$$
and the **closed segments**
$$\Phi_\alpha = \{x \in R \mid \varphi(x) \leq \alpha\}.$$
These satisfy the following properties:
$$\forall \alpha \in \mathbf{Ord}, \Phi_\alpha^\circ = \bigcup_{\beta < \alpha} \Phi_\beta, \ \forall \alpha \in \mathbf{Ord}, \Phi_\alpha = \Phi_{\alpha+1}^\circ,$$
$$\forall \beta \leq \alpha \in \mathbf{Ord}, \ \Phi_\beta^\circ \subset \Phi_\alpha^\circ, \ \Phi_\beta \subset \Phi_\alpha.$$
The following lemma, whose proof is immediate, gives a sense in which the smaller Euclidean function is "more efficient": its segments fill up R more rapidly.

Lemma 2.22. *For $\varphi, \psi : R \to \mathbf{Ord}$, we have:*
$\varphi \leq \psi \iff \forall \alpha \in \mathbf{Ord}, \Psi_\alpha^\circ \subset \Phi_\alpha^\circ \iff \forall \alpha \in \mathbf{Ord}, \Psi_\alpha \subset \Phi_\alpha$.

Let R be any commutative ring. We define subsets $R_{[\alpha]}^\circ$ and $R_{[\alpha]}$ of R for all $\alpha \in \mathbf{Ord}$ by transfinite induction, as follows:
- For $\alpha \in \mathbf{Ord}$, $R_{[\alpha]}^\circ = \bigcup_{\beta < \alpha} R_{[\beta]}$.
- For all α,
$$\begin{aligned} R_{[\alpha]} &= \{b \in R \mid \text{the composite map } R_{[\alpha]}^\circ \cup \{0\} \hookrightarrow R \to R/(b) \text{ is surjective } \} \\ &= \{b \in R \mid \forall a \in R, \ \exists q \in R, r \in R_{[\alpha]}^\circ \cup \{0\} \text{ such that } a = qb + r\}. \end{aligned}$$
Some immediate consequences:
- $R_{[0]}^\circ = \varnothing$,

• $R_{[0]} = R^{\times}$, and
• For every $x \in R_{[1]} \setminus R_{[0]}$, (x) is a maximal ideal of R.

Lemma 2.23. *Let R be a commutative ring.*
*a) For all $\beta < \alpha \in$ **Ord**, $R_{[\beta]} \subset R_{[\alpha]}$.*
*b) For all $\alpha \in$ **Ord**, $R^{\circ}_{[\alpha]} \subset R_{[\alpha]}$.*
*c) There is a least $\alpha_0 \in$ **Ord** such that $R_{[\alpha_0]} = R_{\alpha_0+1}$. For all $\alpha \geq \alpha_0$, $R_{[\alpha]} = R_{[\alpha_0]}$.*
*d) For all $\varphi \in \operatorname{Euc} R$ and $\alpha \in$ **Ord**, $\Phi_\alpha \subset R_{[\alpha]}$.*

Proof. a) If $\beta < \alpha$, then $R^{\circ}_{[\beta]} \subset R^{\circ}_{[\alpha]}$, and it follows that for all $\beta < \alpha$, $R_{[\beta]} \subset R_{[\alpha]}$.
b) Using part a), $R^{\circ}_{[\alpha]} = \bigcup_{\beta<\alpha} R_{[\beta]} \subset R_{[\alpha]}$.
c) For a set S, a weakly isotone function $\mathbf{Ord} \to 2^S$ takes at most $2^{\#S}$ distinct values so is eventually stable: thus there is a least α_0 such that $R_{[\alpha_0]} = R_{[\alpha_0+1]}$. If it is not the case that $R_{[\alpha]} = R_{[\alpha_0]}$ for all $\alpha \geq \alpha_0$, then let α be minimal such that $R_{[\alpha_0]} \subsetneq R_{[\alpha]}$. Then $R^{\circ}_{[\alpha]} = \bigcup_{\beta<\alpha} R_{[\beta]} = R_{[\alpha_0]}$ and thus $R_{[\alpha]} = R_{[\alpha_0+1]} = R_{[\alpha_0]}$.
d) By transfinite induction: fix $\alpha \in \mathbf{Ord}$ and suppose that $\Phi_\beta \subset R_{[\beta]}$ for all $\beta < \alpha$. Then $\Phi^{\circ}_\alpha = \bigcup_{\beta<\alpha} \Phi_\beta \subset \bigcup_{\beta<\alpha} R_{[\beta]} = R^{\circ}_{[\alpha]}$. Let $b \in \Phi_\alpha$. Then for all $a \in R$, there is $q \in R$ and $r \in \Phi^{\circ}_\alpha \cup \{0\} \subset R^{\circ}_{[\alpha]} \cup \{0\}$ with $a = qb + r$. So $b \in R_{[\alpha]}$. □

Theorem 2.24. *(Motzkin-Samuel Process* [Mo49], [Sa71]*)*
a) For a commutative ring, the following are equivalent:
(i) R is Euclidean.
(ii) $R_{[\alpha_0]} = R$.
(iii) $0 \in R_{[\alpha_0]}$.
*b) When the equivalent conditions of part a) are satisfied, define $\psi : R \to$ **Ord** as follows: for $x \in R$, let α be the least ordinal number such that $x \in R_{[\alpha]}$, and put $\psi(x) = \alpha$. Then $\psi = \varphi_R$ is the bottom Euclidean function on R.*

Proof. a) (i) $\implies$ (ii): If $\varphi \in \operatorname{Euc} R$ then by Lemma 2.23d), $R = \Phi_{\varphi(0)} \subset R_{\varphi(0)}$. It follows that the eventual value of $R_{[\alpha]}$ is R.
(ii) $\implies$ (iii) is immediate.
(iii) $\implies$ (ii): If $0 \in R_{[\alpha_0]}$, then for all $a \in R^\bullet$ there is $q \in R$ and $r \in R^{\circ}_{[\alpha_0]} \cup \{0\}$ such that $a = qb + r = r$: i.e., $R^\bullet \subset R^{\circ}_{[\alpha_0]}$. Thus $R_{[\alpha_0]} = R$.
(ii) $\implies$ (i): Suppose $R_{[\alpha_0]} = R$. For each $x \in R^\bullet$, there is a least α such that $x \in R_{[\alpha]}$, and thus $x \in R_{[\alpha]} \setminus R^{\circ}_{[\alpha]}$. We put $\psi(x) = \alpha$. Then ψ is a Euclidean function on R: let $a \in R$, $b \in R^\bullet$, and put $\alpha = \psi(b)$. There is $r \in R^{\circ}_{[\alpha]} = \bigcup_{\beta<\alpha} R_{[\beta]}$ such that $r \equiv a \pmod{b}$, i.e., there is $q \in R$ such that $a = qb + r$ with $r = 0$ or $\psi(r) < \psi(b)$.
b) Suppose the conditions of part a) hold, and let φ be the Euclidean function constructed in the proof of (ii) $\implies$ (i). The closed segments of φ are the subsets $R_{[\alpha]}$. Let $\varphi \in \operatorname{Euc} R$ be any Euclidean function. By Lemma 2.23d), $\Phi_\alpha \subset R_{[\alpha]} = \Psi_\alpha$, so by Lemma 2.22, $\psi \leq \varphi$. Thus $\psi = \varphi_R$ is the bottom Euclidean function. □

The following results illustrate the Motzkin-Samuel process: especially, how it can be used both to explicitly determine the bottom Euclidean function of a known Euclidean ring and to show that certain principal rings are not Euclidean.

Example 2.25. [Sa71, p. 289] *Let $R = \mathbb{Z}$. It is easy to see by induction that for all $n < \omega$, $R_{[n]} = \{k \in \mathbb{Z}^\bullet \mid |k| < 2^{n+1}\}$, so the bottom Euclidean function is*

$$\varphi_{\mathbb{Z}}(n) = \begin{cases} \lfloor \log_2(n) \rfloor & n \neq 0 \\ \omega & n = 0 \end{cases}$$

Example 2.26. [Sa71, p. 290] *Let k be a field and $R = k[t]$. It is easy to see by induction that for all $n < \omega$, $R_{[n]} = \{f \in k[t]^\bullet \mid \deg f \leq n\}$, so the bottom Euclidean function is*

$$\varphi_{k[t]}(n) = \begin{cases} \deg f & f \neq 0 \\ \omega & f = 0 \end{cases}$$

Proposition 2.27. [Sa71] *Let R be a ring with finite unit group, of cardinality u.*
a) If $x \in R_{[1]} \setminus R_{[0]}$, then $\#R/(x) \leq u + 1$.
b) If R is Noetherian (e.g. if it is principal), then for all $n < \omega$, $R_{[n]}$ is finite.
c) If $R_{[\omega]} = R$, then R is residually finite.

Proof. a) This is immediate. b) The key is a result of Samuel [Sa71, Prop. 13]: in any Noetherian ring R, for all $n \in \mathbb{Z}^+$, there are only finitely many ideals I with $\#R/I \leq n$. The result follows easily from this by induction. c): If $x \in R^\bullet$, then by hypothesis $\varphi_R(x) = n < \omega$. Then $\#R/(x) \leq \#R^\circ_{[n]} + 1 \leq \#R_{[n]} + 1 < \aleph_0$. □

Example 2.28. [Sa71, pp. 286-287] *Let $R = \mathbb{Z} \times \mathbb{Z}$. By Theorem 2.12, R is Euclidean. However it has a finite unit group and is not residually finite, so by Proposition 2.27c) its Euclidean order type exceeds ω.*

Proposition 2.29. *(c.f.* [Sa71, Prop. 15]*) Let R be Euclidean and residually finite.*
a) $e(R) \leq \omega$.
b) R is either a PID or an Artinian principal ring.

Proof. a) If $e(R) > \omega$, there is $b \in R^\bullet$ with $\varphi_R(b) = \omega$. Then $R/(b)$ is finite and $\varphi_R(b)$ is the least ordinal greater than the minimum value taken on each nonzero coset of $R/(b)$. In other words, ω is the least ordinal which is larger than a finite set of finite ordinals: contradiction.
b) Combine part a) with Theorem 2.20. □

Example 2.30. *For $S \subset \mathbb{Z}^+$, let us say a ring R is **S-free** if there are no $b \in R$ with $\#R/(b) \in S$. Suppose R has finite unit group of cardinality u and R is $\{2, \ldots, u+1\}$-free. Then by Proposition 2.27, $R_{[1]} = R_{[0]} = R^\times$, so R is Euclidean iff it is a field.*

*Let d be a squarefree negative integer, let $K = \mathbb{Q}(\sqrt{d})$ and let R be the ring of integers of K. If $d \in \{-3, -1\}$, then the **norm** $x \mapsto N_{K/\mathbb{Q}}(x)$, $0 \mapsto \omega$ is a Euclidean function on R. Suppose now that $d < -3$: then $R^\times = \{\pm 1\}$, so if R is $\{2, 3\}$-free then it is* not *Euclidean.*

Basic algebraic number theory shows that R_d is $\{2, 3\}$-free if $d \equiv 5 \pmod{24}$. By the Heegner-Baker-Stark Theorem, there are seven values of $d < -3$ such that R is a PID: $-2, -7, -11, -19, -43, -67, -163$. Thus for $d \in \{-19, -43, -67, -163\}$, R is a non-Euclidean PID. This argument of Motzkin [Mo49] *provided the first examples of non-Euclidean PIDs. For $d \in \{-2, -7, -11\}$, the norm is Euclidean.*

Theorem 2.31. *Let R_1 and R_2 be Euclidean rings. Put $R = R_1 \times R_2$, and write $\varphi, \varphi_1, \varphi_2$ for the bottom Euclidean functions on R, R_1 and R_2.*
a) For all $x \in R_1$ and $u \in R_2^\times$, $\varphi((x, u)) = \varphi_1(x)$.
*b) For all $\alpha, \beta, \gamma \in$ **Ord** with $\alpha \oplus \beta \leq \gamma$, we have $(R_1)_{[\alpha]} \times (R_2)_{[\beta]} \subset R_{[\gamma]}$.*

c) Let $x \in R_1$ and $y \in R_2$ with $\varphi_2(y) < \omega$. Then $\varphi(x,y) = \varphi_1(x) + \varphi_2(y)$.
d) If $e(R_1) = e(R_2) = \omega$, then $\varphi = \varphi_1 \oplus \varphi_2$ and thus $e(R) = 2\omega$.

Proof. a) If $u \in R_2^\times$, then the cosets of $(x,u)R$ in R correspond naturally to the cosets of xR_1 in R_1. The result follows easily from this.
b) For all $\gamma \in \mathbf{Ord}$, the closed segment Ω_γ corresponding to the Euclidean function $\varphi_1 \oplus \varphi_2$ is precisely $\bigcup_{\alpha,\beta \mid \alpha\oplus\beta\leq\gamma}(R_1)_{[\alpha]} \times (R_2)_{[\beta]}$. Now apply Lemma 2.23d).
c) Seeking a contradiction, we suppose there are $(x,y) \in R$ such that $\varphi_2(y) = \alpha_2 < \omega$ and $\varphi((x,y)) < \varphi_1(x) + \varphi_2(y)$; among all such, choose (x,y) with $\varphi((x,y))$ minimal. If $\varphi_2(y) = 0$ then $y \in R_2^\times$ and the result is part a), so we may assume that $0 < \alpha_2 < \omega$. Then there is $y' \in R_2$ such that $\varphi_2(y') = \alpha_2 - 1$ and for all $q_y \in R_2$, $\varphi_2(y' - q_y y) \geq \alpha_2 - 1$. Apply the Euclidean property with $a = (x,y)$, $b = (x,y')$: there are $q_x, r_x \in R_1$ and $q_y, r_y \in R_2$ such that

$$(x,y) = (q_x x + r_x, q_y y' + r_y)$$

and $\varphi((x - q_x x, y - q_y y')) < \varphi(x,y)$. By minimality of $\varphi(x,y)$, we have

$$\varphi(x - q_x x, y - q_y y')) = \varphi_1(x - q_x x) + \varphi_2(y - q_y y')) \geq \varphi_1(x) + \alpha_2 - 1.$$

Thus

$$\varphi_1(x) + \varphi_2(x) - 1 < \varphi((x,y)),$$

so

$$\varphi_1(x) + \varphi_2(x) \leq \varphi((x,y)),$$

contradiction.
d) Applying part c) we get $\varphi((x,y)) = \varphi_1(x) + \varphi_2(y) = \varphi_1(x) \oplus \varphi_2(y)$ for all $x \in R_1$ and $y \in R_2^\bullet$. Applying part c) with R_1 and R_2 reversed we get $\varphi((x,y)) = \varphi_2(y) + \varphi_1(x) = \varphi_1(x) \oplus \varphi_2(y)$ for all $x \in R_1^\bullet$ and $y \in R_2$. It follows that $\varphi((0,0)) = 2\omega$. $\square$

Example 2.32. *By Theorem 2.31d), the bottom Euclidean function on $\mathbb{Z} \times \mathbb{Z}$ is:*

$$\varphi_{\mathbb{Z}\times\mathbb{Z}}((x,y)) = \begin{cases} \lfloor \log_2 x \rfloor + \lfloor \log_2 y \rfloor & x, y \neq 0 \\ \omega + \lfloor \log_2 y \rfloor & x = 0, y \neq 0 \\ \omega + \lfloor \log_2 x \rfloor & x \neq 0, y = 0 \\ 2\omega & x = y = 0 \end{cases}$$

2.8. **The Localized Euclidean Function.**

Theorem 2.33. *(Motzkin-Samuel) Let R be a Euclidean domain, and let $S \subset R$ be multiplicatively closed. Then $S^{-1}R$ is Euclidean and $e(S^{-1}R) \leq e(R)$.*

Proof. Without changing $S^{-1}R$ we may assume that the multiplicatively closed set S is *saturated*: if $x \in S$ and $y \mid x$ then $y \in S$. Then every element $x \in S^{-1}R$ may be written as $x = \frac{s}{t}x'$ with $s, t \in S$ and x' prime to all elements of S, and in such a representation x' is well-determined up to a unit. Let $\varphi : R^\bullet \to \mathbf{Ord}$ be an isotone Euclidean function. Then $\varphi_S : S^{-1}R \to \mathbf{Ord}$ by $\varphi_S(x) = \varphi(x')$ is well-defined; to complete the proof it suffices to show that φ_S is Euclidean: let $a, b \in (S^{-1}R)^\bullet$ and write $b = \frac{s}{t}b'$ as above. Since the canonical map $R/(b') \to S^{-1}R/(b)$ is an isomorphism, there is $a' \in R$ such that $\frac{t}{s}a \equiv a' \pmod{bS^{-1}R}$ and thus $a \equiv \frac{s}{t}a'$ $\pmod{bS^{-1}R}$. Since φ is Euclidean, we may write $a' = qb' + r$ with $q, r \in R$ and $\varphi(r) < \varphi(b')$. Multiplying through by $\frac{s}{t}$ we get $a = qb + \frac{s}{t}r$ and thus

$$\varphi_S\left(\frac{s}{t}\right) = \varphi_S(r) = \varphi_S(s'r') = \varphi_S(r') = \varphi(r') \leq \varphi(r) < \varphi(b') = \varphi(b).$$

$\square$

Remark 2.34. *If in the proof of Theorem 2.33 we take φ to be the bottom Euclidean function on R, then φ_S need not be the bottom Euclidean function on $S^{-1}R$. For instance let $R = \mathbb{Z}$, let $p \geq 5$ be a prime number, and let $S = \mathbb{Z} \setminus p\mathbb{Z}$. Then $S^{-1}R$ is a local PID, so by Example 2.6 the bottom Euclidean function takes the value* 1 *at p, while $\varphi_S(p) = \varphi_{\mathbb{Z}}(p) = \lfloor \log_2 p \rfloor \geq 2$.*

2.9. **The Quotient Euclidean Function.**

Theorem 2.35. *Let $\varphi : R \to \mathbf{Ord}$ be a Euclidean function. Let $b \in R^\bullet$, and let $f : R \to R/(b)$ be the quotient map. For $x \in R/(b)$, let $\tilde{x} \in f^{-1}(x)$ be any element such that $\varphi(\tilde{x}) \leq \varphi(y)$ for all $y \in f^{-1}(x)$.*
a) Then $\varphi' : x \in (R/(b))^\bullet \mapsto \varphi(\tilde{x})$ is a Euclidean function.
b) For the bottom Euclidean function φ_R, we have

$$\varphi'_R(0) = \sup_{x \in (R/(b))^\bullet} (\varphi'_R(x) + 1) = \varphi_R(b). \tag{4}$$

Proof. a) For $x \in R/(b)$, $y \in (R/(b))^\bullet$, $\tilde{x} \in R$, $\tilde{y} \in R^\bullet$, so there are $q, r \in R$ with $\tilde{x} = q\tilde{y} + r$ and $\varphi(r) < \varphi(\tilde{y})$. Then $x = f(q)y + f(r)$ and hence

$$\varphi'(f(r)) \leq \varphi(r) < \varphi(\tilde{y}) = \varphi'(y).$$

b) The first equality in (4) is the definition of the extension to 0 of any Euclidean function. The second equality follows from the description of the bottom Euclidean function given in the Motzkin-Samuel process: its value at b is the least ordinal which is larger than the minimum value taken on each nonzero coset of (b). $\square$

Remark 2.36. *The function φ'_R need not be the bottom Euclidean function on $R/(b)$. If it were, then for every domain R and $b \in R^\bullet$, $\varphi_R(b) = \varphi'_R(0) = e(R/(b)) < \omega$, so $e(R) \leq \omega$, contradicting Theorem 2.21.*

Corollary 2.37. *If R is Euclidean, so is every quotient ring R', and $e(R') \leq e(R)$.*

2.10. **The Product Theorem.**

The following is a standard piece of ordinal arithmetic. Because of its importance in the proof of Theorem 2.40, we will give a complete proof.

Lemma 2.38. *(Ordinal Subtraction)*
a) For $\alpha \leq \beta \in \mathbf{Ord}$, there is a unique $\gamma \in \mathbf{Ord}$ such that $\alpha + \gamma = \beta$. We may therefore define

$$-\alpha + \beta = \gamma.$$

b) Suppose we have ordinals α, β, γ such that $\gamma \leq \alpha < \beta$. Then $-\gamma + \alpha < -\gamma + \beta$.

Proof. a) Existence of γ: If $\alpha = \beta$, then we take $\gamma = 0$. Otherwise, $\alpha \subsetneq \beta$; let x_0 be the least element of $\beta \setminus \alpha$ and let γ be the order type of $\{x \in \beta \mid x \geq x_0\}$.
Uniqueness of γ: suppose we have two well-ordered sets W_1 and W_2 such that $\alpha + W_1$ is order-isomorphic to $\alpha + W_2$. Then the unique order-isomorphism between them induces an order-isomorphism from W_1 to W_2.
b) For if not, $-\gamma + \beta \leq -\gamma + \alpha$, and then $\beta = \gamma + (-\gamma + \beta) \leq \gamma + (-\gamma + \alpha) = \alpha$. $\square$

Remark 2.39. *Ordinal subtraction does not work for the Hessenberg-Brookfield sum: e.g. there is no ordinal γ with $1 \oplus \gamma = \omega$ or $\omega \oplus \gamma = \omega^2$.*

Theorem 2.40. *(Product Theorem) Let $R_1, \ldots, R_n$ be Euclidean rings. Then*

$$e(R_1) + \ldots + e(R_n) \leq e(\prod_{i=1}^{n} R_i) \leq e(R_1) \oplus \ldots \oplus e(R_n). \tag{5}$$

Proof. We may assume $n = 2$. By Theorem 2.13b), $R = R_1 \times R_2$ is Euclidean.
Step 1: Let $b = (0,1)$, so $R/(b) = R_1$. By Theorem 2.35b), $\varphi_R(b) \geq e(R_1)$. For $y \in R_2$, $b = (0,1) \mid (0,y)$, so by Theorem 2.15b), $\varphi_R((0,1)) \leq \varphi_R((0,y))$. By Lemma 2.38 we may put

$$\psi(y) = -\varphi_R((0,1)) + \varphi_R((0,y)).$$

We CLAIM $\psi : R_2 \to \mathbf{Ord}$ is a Euclidean function. Granting this for the moment, it then follows that $\psi \geq \varphi_{R_2}$, so

$$e(R) = \varphi_R((0,0)) = \varphi_R((0,1)) + \psi(0) \geq e(R_1) + e(R_2).$$

PROOF OF CLAIM: Let $x \in R_2,\ y \in R_2^{\bullet}$; as usual, we may assume $y \nmid x$. Since φ_R is Euclidean, there are $q = (q_1, q_2),\ r = (r_1, r_2) \in R$ such that $(0,x) = q(0,y) + r = (r_1, q_2 y + r_2)$ and either $r = 0$ or $\varphi_R(r) < \varphi_R((0,y))$. Thus $r_1 = 0$ and $x = q_2 y + r_2$. Since $y \nmid x$ we have $r_2 \neq 0$, so $r \neq 0$ and thus $\varphi_R((0,r_2)) < \varphi_R((0,y))$. By Lemma 2.38b) we may subtract $\varphi_R(0,1)$ – on the left! – from both sides to get

$$\psi(r_2) = -\varphi_R((0,l)) - \varphi_R((0,r_2)) < -\varphi_R((0,1)) - \varphi_R((0,y)) = \psi(y).$$

Step 2: By Theorem 2.13a), $\varphi_{R_1} \oplus \varphi_{R_2}$ is a Euclidean function on R, so $e(R) = \varphi_R(0) \leq \varphi_{R_1}(0) \oplus \varphi_{R_2}(0) = e(R_1) \oplus e(R_2)$. □

We deduce the following structural result on Euclidean order types.

Theorem 2.41. *Let R be a Euclidean ring. As in (3) we write $R = \prod_{i=1}^{n} R_i \times A$, with each R_i a PID and A an Artinian principal ring. Put $R' = \prod_{i=1}^{n} R_i$.*
a) We have $e(R) = e(R') + e(A)$.
b) $e(R')$ is a limit ordinal (possibly zero).
c) $e(R') \geq n\omega$.

Proof. a) By Theorem 2.20a), $e(A) < \omega$, so by Theorem 2.40,

$$e(R') + e(A) \leq e(R' \times A) \leq e(R') \oplus e(A) = e(R') + e(A).$$

The remaining assertions hold trivially if $n = 0$, so we assume $n \geq 1$.
b) Since R' is a product of domains, the set of nonzero ideals of R has no minimal element, so $e(R')$ is a nonzero limit ordinal.
c) By part b) and Theorem 2.40, $n\omega \leq e(R_1) + \ldots + e(R_n) \leq e(R')$. □

3. ℓ-Euclidean Rings and Small Euclidean Rings

3.1. ℓ-Euclidean Rings.

In Examples 2.5 and 2.6, the given Euclidean function was the length function ℓ_R. By Theorem 2.17 we must have $\ell_R = \varphi_R$ in both cases.

Let us say that a Noetherian ring is **ℓ-Euclidean** when its transfinite length function ℓ_R is a Euclidean function: as above, we must then have $\varphi_R = \ell_R$ and thus $e(R) = \operatorname{len} R$. Such rings have other pleasant properties.

Theorem 3.1. *An Artinian principal ring is ℓ-Euclidean.*

Proof. This was shown directly for *local* Artinian principal rings in Example 2.5. By Theorem 2.1c), an Artinian principal ring R is a finite direct product $\prod_{i=1}^r R_i$ of local Artinian principal rings. By Theorem 1.10c), $\ell_R = \bigoplus_{i=1}^r \ell_{R_i}$. Since each ℓ_{R_i} is a Euclidean function on R_i, by Theorem 2.13, $\ell_R \in \operatorname{Euc} R$. $\square$

Much of Theorem 3.1 seems implicit in [Fl71]. For instance, he writes down the bottom Euclidean function on $\mathbb{Z}/12\mathbb{Z}$. The following result goes beyond this by using the formalism of transfinite length functions.

Theorem 3.2. *Let R be a semilocal principal ring. Then:*
a) R is ℓ-Euclidean.
b) If $R \cong \prod_{i=1}^n R_i \times A$ as in (3), then $e(R) = n\omega + \operatorname{len} A < \omega^2$.
c) For every $\alpha < \omega^2$, there is a semilocal Euclidean ring R with $e(R) = \operatorname{len} R = \alpha$.

Proof. a) Since R is semilocal and principal, by Theorem 2.1 we may write $R = \prod_{i=1}^n R_i \times \prod_{j=1}^r A_j$ with each R_i a semilocal PID and each A_j a local Artinian principal ring. In Examples 2.5 and 2.6 we showed that each A_j and R_i is ℓ-Euclidean. The rest of the argument is the same as the proof of Theorem 3.1.
b) Since for every PID R we have $\operatorname{len} R = \omega$, this follows immediately.
c) The ordinals less than ω^2 are of the form $a\omega + b$ for $a, b < \omega$. We may take e.g. $R = \bigoplus_{i=1}^a \mathbb{C}[[t]] \oplus \mathbb{C}[t]/(t^b)$. $\square$

If R is ℓ-Euclidean, then R_1 must be the set of all maximal elements of R. Thus:

Example 3.3. *$\mathbb{Z}$ is not ℓ-Euclidean. For a field k, the polynomial ring $k[t]$ is ℓ-Euclidean iff every irreducible polynomial has degree 1 iff k is algebraically closed.*

Theorem 3.4. *For a principal ring R with finite unit group, the following are equivalent:*
(i) R is finite.
(ii) R is semilocal.
(iii) R is ℓ-Euclidean.

Proof. (i) $\implies$ (ii) is immediate, and (ii) $\implies$ (iii) by Theorem 3.2.
(iii) $\implies$ (ii): By Proposition 2.27, $R_{[1]}$ is finite, hence R is semilocal.
(ii) $\implies$ (i): A variant of the Euclidean(!) proof of the infinitude of the primes shows: an infinite ring with finite unit group has infinitely many maximal ideals. $\square$

Proposition 3.5. *A finite product, localization or quotient of ℓ-Euclidean rings is ℓ-Euclidean.*

Proof. Left to the reader. $\square$

3.2. **Small Euclidean Rings.**

Let R be a Euclidean ring with decomposition $R = \prod_{i=1}^n R_i \times A$ as in (3). We say R is **small** if $e(R_i) = \omega$ for all i; otherwise R is **large**. In this terminology, Theorem 2.21 precisely asserts the existence of large Euclidean rings.

Observe that an ℓ-Euclidean ring is small. The following result shows a closer relationship between the two properties.

Theorem 3.6. *Let $R = \prod_{i=1}^n R_i \times A$ be Euclidean. The following are equivalent:*
(i) R is small.
(ii) $e(R) = \operatorname{len} R$.

Proof. (i) $\Longrightarrow$ (ii): We have

$$n\omega + \ell(A) = \operatorname{len} R = \ell_R(0) \leq \varphi_R(0) \leq \bigoplus_{i=1}^{n} \varphi_{R_i}(0) \oplus \varphi_A(0) = n\omega + \ell(A).$$

(ii) $\Longrightarrow$ (i): By contraposition: suppose R is large. We may assume $e(R_1) \geq \ldots \geq e(R_n)$ and $e(R_1) > \omega$. By Theorem 2.41b), $e(R_1) \geq 2\omega$. By Theorem 2.40,

$$e(R) \geq e(R_1) + \ldots + e(R_n) + e(A) \geq 2\omega + (n-1)\omega + \ell(A) > n\omega + \ell(A) = \ell R. \qquad \square$$

Theorem 3.7. *Let $R = \prod_{i=1}^{n} R_i \times A$ be a small Euclidean ring. Then*

$$\varphi_R = \varphi_{R_1} \oplus \ldots \oplus \varphi_{R_n} \oplus \varphi_A.$$

Proof. For notational simplicity we will assume $A = 0$. The reader will have no trouble recovering the general case.

Let $x = (x_1, \ldots, x_n) \in R$. Again to simplify the notation, we will assume that $x_1 = \ldots = x_I = 0$ and $x_{I+1} = \ldots = x_n \neq 0$: permuting the factors to achieve this is harmless because of the commutativity of the Hessenberg-Brookfield sum. Write $S_1 = \prod_{i=1}^{I} R_i$ and $S_2 = \prod_{i=I+1}^{n} R_i$, so $R = S_1 \times S_2$. Also write $s_1 = (x_1, \ldots, x_I) = 0 \in S_1$ and $s_2 = (x_{I+1}, \ldots, x_n) \in S_2$. We have $\varphi_{S_2}(s_2) \leq \bigoplus_{i=I+1}^{n} \varphi_{R_i}(x_i) < \omega$, so applying Theorem 2.31c) and Theorem 3.6 we get

$$\varphi_R(x) = \varphi_{S_1}(s_1) + \varphi_{S_2}(s_2) = e(S_1) + \varphi_{S_2}(s_2) = I\omega + \varphi_{S_2}(s_2).$$

Since for all $I+1 \geq i \geq n$, $\varphi_i(x_i) < \omega$, we may apply Theorem 2.31c) $n - I - 1$ more times to get $\varphi_{S_2}(s_2) = \sum_{i=I+1}^{n} \varphi_{R_i}(x_i)$ and thus

$$\varphi_R(x) = I\omega + \sum_{i=I+1}^{n} \varphi_{R_i}(x_i) = \bigoplus_{i=1}^{n} \varphi_{R_i}(x_i). \qquad \square$$

Proposition 3.8. *Any finite product, localization or quotient of small Euclidean rings is small Euclidean.*

Proof. Left to the reader. $\square$

4. Some Questions

Question 4.1. *Let R be a Euclidean ring with bottom Euclidean function φ_R.*
a) Let S be a multiplicative subset of R. Can we give an explicit description of $\varphi_{S^{-1}R}$ in terms of S and the bottom Euclidean function φ_R of R?
b) Let $b \in R^{\bullet}$. Can we give an explicit description of $\varphi_{R/(b)}$ in terms of φ_R and b?

Question 4.2. *Let R be a principal ring such that in the Motzkin-Samuel process, R_1 consists of all maximal elements of R.*
a) Must R be Euclidean?
b) If R is Euclidean, must it be ℓ-Euclidean?

Let R be an S-integer ring in a global field. C. Queen [Qu74] and H.W. Lenstra [Le77] showed that – assuming a Generalized Riemann Hypothesis in the number field case – if $\#S \geq 2$, then R is Euclidean iff it is principal. Moreover, when these conditions hold the bottom Euclidean function evaluated at x is

$$\varphi(x) = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}}(x)$$

where the sum ranges over all nonzero primes $\mathfrak{p} = (\pi_{\mathfrak{p}})$ of R, $\operatorname{ord}_{\mathfrak{p}}$ is the corresponding discrete valuation, and $n_{\mathfrak{p}}$ is equal to 1 if $\pi_{\mathfrak{p}} \in R_1$ and 2 otherwise [Le77, Thm. 9.1]. This certainly addresses Question 4.1a) and shows that both parts of Question 4.2 have an affirmative answer in this case.

Question 4.3. *Let R_1 and R_2 be Euclidean rings, and put $R = R_1 \times R_2$.*
a) We know that $\varphi_R \leq \varphi_{R_1} \oplus \varphi_{R_2}$. Must we have equality?
b) Must we have $e(R) = e(R_1) \oplus e(R_2)$?

Question 4.4. *Are there Euclidean domains R_1 and R_2 with $e(R_1) \geq e(R_2)$ such that $e(R_1) + e(R_2) < e(R_1) \oplus e(R_2)$?*

Of course an affirmative answer to Question 4.3a) implies an affirmative answer to Question 4.3b). Though we were not able to give such an affirmative answer in general, nevertheless we feel that this is a promising line of attack on Question 4.4.

On the other hand, a *negative* answer to Question 4.4 gives, via the Product Theorem, an affirmative answer to Question 4.3b). Let us explore the underlying ordinal arithmetic in more detail:

If $\alpha_1 \geq \ldots \geq \alpha_n$ are ordinal numbers, then $\alpha_1 + \ldots + \alpha_n = \alpha_1 \oplus \ldots \alpha_n$ iff the α_i's are "unlaced" in the sense that for all $1 \leq i \leq n-1$, the least exponent γ of a term ω^γ appearing in the Cantor normal form of α_i is at least as large as the greatest exponent γ' of a term appearing in the Cantor normal form of α_{i+1}. In particular this relation holds whenever each α_i is "homogeneous", i.e., of the form $m\omega^\gamma$ for some $m < \omega$ and $\gamma \in \mathbf{Ord}$. Thus Question 4.3b) would have an affirmative answer if every Euclidean domain has homogeneous Euclidean order type.

In this regard Theorem 2.41 gives a (very) partial result: the order type of a Euclidean domain can have no finite part. Thus the smallest conceivable "inhomogeneous" Euclidean order type of a domain is $\omega^2 + \omega$. For such a domain, the Product Theorem leaves ambiguous whether $e(R \times R)$ is $e(R) + e(R) = 2\omega^2 + \omega$ or $e(R) \oplus e(R) = 2\omega^2 + 2\omega$. These types of questions seem difficult to address directly.

Question 4.5. *What are the Euclidean order types of the large Euclidean domains of Theorem 2.21?*

Question 4.6. *In Lemma 2.23 we associated an ordinal invariant α_0 to any commutative ring R, namely the least ordinal such that $R_{[\alpha_0]} = R_{[\alpha_0+1]}$? What values can this invariant take on the class of all commutative rings?*

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