A Note on Zariski's Lemma

Author(s): John McCabe

Source: *The American Mathematical Monthly*, Vol. 83, No. 7 (Aug. - Sep., 1976), pp. 560-561
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/2319358
Accessed: 09-04-2016 20:09 UTC

4. L. E. El'sgol'ts, Differential Equations, Hindustan Publishing, Delhi, 1961.
5. W. F. Osgood, Beweis der Existenz einer Lösung der Differentialgleichung $y' = f(x, y)$ ohne Hinzunahme der Cauchy-Lipschitzschen Bedingung, Monatsh. Math., 9 (1898) 331–345.
6. G. Peano, Sull'integrabilita delle equazioni differenziali del primo ordine, Atti Accad. Sci. Torino, 21 (1886) 677–685.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT AUSTIN, AUSTIN, TX 78712.

A NOTE ON ZARISKI'S LEMMA

JOHN McCABE

Various elementary proofs of the Hilbert *Nullstellensatz* are to be found in the literature. It is well known that the *Nullstellensatz* may be derived from the statement known as the weak *Nullstellensatz*; this is done by the celebrated device called "Rabinowitsch's trick" [**4**]. Zariski [**5**] introduced a lemma from which the weak *Nullstellenatz* follows easily, and he proved this lemma using remarkably little algebraic machinery. Our purpose in this note is to give an alternative proof of Zariski's Lemma, a proof which will also use little in the way of algebraic preliminaries.

Since the Hilbert *Nullstellensatz* lies at the foundations of algebraic geometry, it is a matter of considerable interest to be able to prove this result as expeditiously as possible. In this regard, the reader might like to consult [**1**], [**3**], and [**6**], Volume II, pp. 164–167.

LEMMA (Zariski's lemma). *If k is a field, if $R = k[x_1,\ldots,x_n]$ is a finite integral domain over k, which is a field, then R is an algebraic extension of k.*

Proof. Assume that R is not algebraic over k. We may assume the x's have been so indexed that $x_1,\ldots,x_t$ are independent transcendentals over k, and that $x_{t+1},\ldots,x_n$ are algebraic over $k(x_1,\ldots,x_t)$. Put

$$S = k[x_1,\ldots,x_t].$$

For an appropriate element $y \neq 0$ in S, the elements $yx_{t+1},\ldots,yx_n$ will all be integral over S. It follows that $x_{t+1},\ldots,x_n$ are all integral over $S[1/y]$. Since $k \subseteq S[1/y]$ and $x_1,\ldots,x_t \in S[1/y]$, we conclude that $R = k[x_1,\ldots,x_n]$ is integral over $S[1/y]$. Since R is a field, $S[1/y]$ must be a field ([**2**], Proposition 5.7, p. 61).

Let m be any maximal ideal of S. Since $S = k[x_1,\ldots,x_t]$ is not a field, $m \neq \{0\}$. Choose a non-zero element f of m. In $S[1/y]$, f is invertible so

$$1/f = g/y^N,$$

where $g \in S$ and where we may assume N is a positive integer. We than have

$$y^N = fg.$$

Since $f \in m$, $y^N \in m$. This implies that $y \in m$.

Thus y is in every maximal ideal of S. It follows that $1 + y$ is in no maximal ideal of S, and hence $1 + y$ is a unit of S ([**2**], Corollary 1.5, p. 4). It's well known that the units of the polynomial ring $S = k[x_1,\ldots x_t]$ are exactly the non-zero elements of k, so

$$1 + y = c,$$

where $c \neq 0$ is in k. Thus $y = c - 1$, and y is in k. Since $y \neq 0$, y is a unit of k and therefore a unit of S also. It follows that $S[1/y] = S$, hence S is a field.

But $S = k[x_1,\ldots,x_t]$ is certainly not a field. This contradiction establishes that R must be algebraic over k.

References

1. E. Artin and J. Tate, A note on finite ring extensions, J. Math. Soc. Japan, 3 (1951) 74–77.
2. M. F. Atiyah and I. G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley, Reading, Mass., 1969.
3. R. Brauer, A note on Hilbert's Nullstellensatz, Bull. Amer. Math. Soc., 54 (1948) 894–896.
4. J. L. Rabinowitsch, Zum Hilbertschen Nullstellensatz, Math. Ann., 102 (1929) 520.
5. O. Zariski, A new proof of Hilbert's Nullstellensatz, Bull. Amer. Math. Soc., 53 (1947) 362–368.
6. O. Zariski and P. Samuel, Commutative Algebra, Volumes I and II, Van Nostrand, Princeton, N. J., 1958 and 1960.

DEPARTMENT OF MATHEMATICS, MANHATTAN COLLEGE, NEW YORK, NY 10471.

MATHEMATICAL EDUCATION

EDITED BY PAUL T. MIELKE AND SHIRLEY HILL

Material for this Department should be sent to Paul T. Mielke, Department of Mathematics, Wabash College, Crawfordsville, IN 47933, or to Shirley Hill, Department of Mathematics, University of Missouri, Kansas City, MO 64110.

Your attention is drawn to the existence of the new quarterly "Journal of Personalized Instruction," 29 Loyola Hall, Georgetown University, Washington, D.C. 20057. Our continuing backlog prompts us to ask potential contributors of articles on personalized systems of instruction to keep the above journal in mind as a possible publishing alternative. — *The Editors.*

IMPLICIT AXIOMS, ω·RULE AND THE AXIOM OF INDUCTION IN HIGH SCHOOL MATHEMATICS

SHLOMO VINNER

1. Implicit Axiom Systems. The question of whether or not the axiomatic method should be taught to high school students is an extremely controversial one. One way of dealing with it is to describe the various aspects in detail and to argue the case for each of them. To a certain extent this has been done in [1] (especially in the papers of R. C. Buck, A. M. Gleason and L. Henkin). We shall confine ourselves here to only one aspect of the axiomatic method which we can call the naive aspect. In this, the axioms are taken as clear and well-known facts (or obvious and well-known statements)[1] of a known structure. These facts seem to be more fundamental than others and they have a special role in the mathematical activity of the high school student. To be more specific, certain facts[2] which often appear in the algebraic activity of the high school student obtain a special status after a certain period of time. The student is not only familiar with them but he also realizes that they are important and widely used in proofs and calculations. This process is implicit and the student is usually unaware of it. In most cases it is a long process. At the beginning of the process the facts are introduced to the student, and at the end they obtain their special status. For instance, at a certain age it takes some time to realize that $a + b = b + a$ for all numbers a, b. It takes perhaps more time to find out that this fact is extremely important in calculations and other arithmetical activities. However, although teachers tell their students that various facts are important, they do not call them axioms. So, the status of the

[1] We say "facts" instead of "statements," because the high school student usually does not make a distinction between semantics and syntactics. For example, to such a student it is a fact that $a + b = b + a$ for all numbers a, b.

[2] We shall confine ourselves here only to algebraic facts.