AUTOMORPHISMS OF THE FIELD OF COMPLEX NUMBERS

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1. *Introduction.* A function $\phi(z)$ defined for all complex numbers z so that for every z_1 and z_2

(1) $$\phi(z_1+z_2) = \phi(z_1)+\phi(z_2)$$

and

(2) $$\phi(z_1 z_2) = \phi(z_1)\phi(z_2),$$

and so that

(3) the equation $\phi(z) = \zeta$ has a solution for every complex number ζ

is said to define an *automorphism* of the field of complex numbers. Obvious functions satisfying these conditions are $\phi(z) = z$, $\phi(z) = \bar{z}$. (It is plain that if $\phi(u) = \phi(v)$, then $\phi(u-v) = 0$, and consequently, by (2), any ϕ satisfying (1) and (2) which is not identically zero cannot assume any value twice.) The problem of the existence of non-trivial solutions was propounded by C. Segre in 1889 (**1**, p. 288) in connexion with the problem of existence of non-projective collineations in a bicomplex plane. Coolidge (**2**, p. 38) referred to it as an unsolved problem in 1921, though in 1945 he mentioned (**3**, p. 63) reported solutions of which he had not seen proofs. The corresponding problem in the field of *real* numbers had been solved by Darboux (**4**) in 1880; he proved that $\phi(x) = x$ is the only solution, thereby completing the proof of von Staudt's theorem that a *real* collineation is necessarily projective.

The existence of non-trivial solutions of the functional equation (1) was demonstrated by Hamel (**5**) in 1905, but no progress seems to have been made with Segre's problem until Lebesgue's paper (**6**), in the form of a letter to Segre, appeared in 1907. This brilliant contribution to the problem ignores condition (3) above and is in places obscure, but it contains the essentials of a complete solution; in a subsequent article (**7**) Lebesgue refers to the 1907 note as a "réponse partielle" to Segre's problem.

The original version of this paper had for its main aim the completion of Lebesgue's argument by such methods as would occur to one inexpert in algebra. It has since been pointed out by the referees that the existence of non-trivial automorphisms of the complex numbers is deducible from general theorems due to Steinitz. I am, above all, indebted to Dr. R. Rado who has summarized the argument in a form that will be at once comprehensible and satisfying to an algebraist. I give Dr. Rado's account, in his own words, in § 3. The theorems quoted from Steinitz and van der Waerden are, however, not needed in their full generality as far as our problem is concerned. Accordingly, in § 4, the argument is formulated in a more concrete way; this is less revealing than the general approach, but it is quite elementary and, except for an appeal to a few basic results in the theory of extension of number fields, quite self-contained.

The arguments used by Lebesgue and by Steinitz rely on Zermelo's theorem that the continuum can be well ordered, and the non-trivial automorphisms which are subsequently "constructed" are defined by means of transfinite induction. Both Hamel and Lebesgue had proved that every non-trivial solution of (1) alone must be non-measurable, and Ostrowski's researches on the same equation (8) have since proved that every solution of (1) in the field of real numbers which is not trivial must fluctuate wildly in every linear set of positive measure. In the second part of this paper we prove, among other results, that every function which defines a non-trivial automorphism of the complex numbers transforms every bounded set (in the Argand plane) into a set of Lebesgue measure zero or else into a non-measurable set. It is therefore not surprising that a complete solution of Segre's problem should be virtual.

From a geometric point of view, the function $\phi(z)$ defines an interesting transformation of the bicomplex plane (the set of all points (x, y) where x and y are complex numbers). If the point $\{\phi(x), \phi(y)\}$ is assigned to the point (x, y), the transformation is a collineation (in the simple sense that it is 1–1 and that collinear points are transformed into collinear points, both ways) since the algebraic condition for three points to be in line will, by (1) and (2), ensure the collinearity of the transforms. Further, if four ordered collinear points have a cross-ratio ρ, their transforms have cross-ratio $\phi(\rho)$ by (1) and (2). The collineation is neither projective nor anti-projective, though it will of course, as a collineation, preserve harmonic relations (in other words, $\phi(\rho) = \rho$ whenever ρ is rational). The transformation can be extended to cover the points of a line at infinity by assigning the point $\{\phi(x_1), \phi(x_2), \phi(x_3)\}$ to the point (x_1, x_2, x_3); the line at infinity, $x_3 = 0$, is then a fixed one. In spite of the fact that the transformation is discontinuous at *every* point of the plane, it has a number of unexpectedly normal properties. It transforms every conic into a conic, every circle into a circle; this follows from (1) and (2) since the algebraic curve with equation $\sum a_{rs}x^r y^s = 0$ is transformed into the curve $\sum \phi(a_{rs})x^r y^s = 0$. In particular, if the a_{rs} are all integers, the curve will be *fixed* since $\phi(n) = n$ for every integer n. The circular lines $x^2+y^2 = 0$, for example, are fixed or permuted. Parallelism and perpendicularity of lines are both invariant, and so in fact is any geometric property of an algebraic curve which can be expressed by the vanishing of a polynomial in the a_{rs} with integer coefficients: for instance, a parabola ($h^2 = ab$, $\Delta \neq 0$) is transformed into a parabola.

The question of what restriction need be imposed on ϕ, in addition to (1), (2), and (3), to *exclude* the non-trivial solutions is considered in § 7. Coolidge (**2**) assumes continuity, but much weaker restrictions suffice, in view of the extreme fluctuation, already referred to, of any non-trivial ϕ.

2. *Notation.* Z is the set of all complex numbers; it also sometimes denotes the Argand plane (a *real* plane in which the complex number $x+iy$ is represented by the point with real coordinates (x,y)). In § 7 we deal with a bicomplex plane $\mathfrak{Z}$ which consists of all (x,y), x and y being any *complex* numbers. The set of all z in Z which satisfy $|z-\gamma| < \delta$ is called the circle in Z with centre γ and radius δ.

R is the set of all rational and $\mathscr{R}$ the set of all real numbers. If $f(z)$ is a function defined in a set E, $f(E)$ is the set of all numbers $f(z)$ for which $z \in E$; $f^{-1}(S)$ is the set of all z for which $f(z) \in S$, S being any set of complex numbers.

$m_i E$ is the interior Lebesgue measure of E (upper bound of Lebesgue measures of closed subsets of E), $|E|$ is the exterior Lebesgue measure of E (whether the measure is linear or plane will be made clear by the context). If $|E| = 0$, we say E is *null*.

The algebraic terminology follows (**9**). In particular, if $E \subset Z$, $R[E]$ consists of all numbers which can be expressed as finite sums of products of a finite number of terms in $E+R$, and $R(E)$ is the field of quotients of such numbers. A number z is algebraic in a field K if z is a zero of a not identically vanishing polynomial over K. A function $\phi(z)$ (not identically zero) which satisfies (1) and (2) for all z_1 and z_2 in a field K will be called a Segre function, or an isomorphism, on K, a trivial one if $\phi(z) = z$ or $\phi(z) = \bar{z}$. If $\psi(z)$ is an isomorphism on $H \supset K$ and $\psi(z) = \phi(z)$ in K, we say ψ extends ϕ. If K is algebraically closed, i.e. every polynomial over K is a product of linear factors, then it is clear that $\phi(K)$ is also algebraically closed if ϕ is a Segre function on K.

The cross-ratio $\dfrac{z_3-z_1}{z_3-z_2}\Big/\dfrac{z_4-z_1}{z_4-z_2}$ is denoted by $\{z_1, z_2; z_3, z_4\}$.

3. The existence of non-trivial automorphisms of Z is contained implicitly in Steinitz's classical paper (**10**). The deduction has been summarized by Dr. R. Rado as follows:

Z is an extension of R; hence (**10**, p. 293, 2) there is a set $T \subset Z$ such that

(4) any finite number of elements of T are algebraically independent over R, and

(5) Z is algebraic over $R(T)$.

T includes more than two elements, for otherwise $R(T)$ and Z would be enumerable. Let $x_0 \in T$; then T includes an x_1 such that $x_1 \neq x_0, x_1 \neq \bar{x}_0$. Let $f(x)$ be a permutation of the numbers of T such that $f(x_0) = x_1$. By (4) there is a unique extension of the definition of f into $R(T)$ such that $f(x)$ defines an automorphism of $R(T)$. By a general principle in algebra (e.g. **12**, vol. 1, p. 42) there exists a field $Y \ni R(T)$ and an extension of the definition of $f(x)$ into Z such that f defines an isomorphism from Z to Y. Since Z is algebraic over $R(T)$ and algebraically closed, it follows that $f(Z) = Y$ is algebraic over $F\{R(T)\} = R(T)$ and algebraically closed. Hence (**10**, p. 287, 9) Y and Z are equivalent extensions of $R(T)$, i.e. there exists an isomorphism $g(y)$ from Y to Z such that $g(t) = t$ for $t \in R(T)$. Then $h(x) = g\{f(x)\}$ defines an automorphism of Z, and

$$h(x_0) = g\{f(x_0)\} = g(x_1) = x_1.$$

Hence $h(x)$ is a non-trivial automorphism of Z.

4. In this paragraph we give a more concrete and self-contained version of the argument contained in § 3.

4·1. The construction of the set T, which is due to Lebesgue and to Steinitz, is easily described. Let $z_1, z_2, \ldots, z_\omega, \ldots$ be a well-ordered series, denoted by (S), of all the complex numbers, and let P_α be the set of all z_r with $r < \alpha$. T is defined to consist of all those z_α which are not algebraic in the corresponding field $R(P_\alpha)$. If there exist complex numbers not algebraic over $R(T)$, let z_σ be the first such number to occur in (S). Since $z_\sigma \bar{\in} T$, z_σ is algebraic over $R(P_\sigma)$. Since every z in P_σ is algebraic over $R(T)$ so is z_σ (**9**, p. 389), and this is a contradiction. This proves (4) and (5) of § 3.

4·2. Let f be any permutation of T satisfying $f(x_0) = x_1$ (§ 3). By (4) of § 3 it follows that if θ and ψ are rational functions (over R) of indeterminates $x_1, x_2, \ldots, x_n$, and they are equal for one set $z_1, z_2, \ldots, z_n$ in T, then they are also equal for the set $f(z_1), f(z_2), \ldots, f(z_n)$. Hence, if

$$z = \theta(z_1, z_2, \ldots, z_n),$$

where $z_1, z_2, \dots, z_n$ are in T, it follows that the definition

$$f(z) = \theta\{f(z_1), f(z_2), \dots, f(z_n)\}$$

defines a (one-valued) function over $R(T)$. If $x = \theta(z_1, z_2, \dots, z_n)$ and $y = \psi(z_1, z_2, \dots, z_n)$ are in $R(T)$, then $x+y$ is the value of $\theta+\psi$ at $(z_1, z_2, \dots, z_n)$ and consequently $f(x+y) = f(x)+f(y)$. Similarly $f(xy) = f(x)f(y)$. Moreover, $f(z) = x$ if $z = \theta\{f^{-1}(z_1), \dots, f^{-1}(z_n)\}$, and so $f\{R(T)\} = R(T)$; hence $f(z)$ defines an *automorphism* of $R(T)$.

4·3. We now extend f, defined on $R(T)$, to a Segre function ϕ on Z; if this is done, $\phi(Z)$ will be algebraically closed (like Z) and therefore include all numbers algebraic in $\phi\{R(T)\}$. Since $\phi\{R(T)\} = R(T)$, this means $\phi(Z) \supset Z$ and so ϕ is an automorphism of Z with $\phi(x_0) = x_1$. Suppose $\zeta_1, \zeta_2, \dots,$ is a well-ordered series of all numbers in $Z - R(T)$, and suppose ϕ has been defined as a Segre function on F_α, the extension of $R(T)$ by all ζ_r having $r < \alpha$. If $\zeta_\alpha \in F_\alpha$, $\phi(\zeta_\alpha)$ will have been defined. Suppose now that $\zeta_\alpha \bar{\in} F_\alpha$. Since ζ_α is algebraic in F_α (being algebraic in $R(T)$), there is a unique irreducible polynomial

$$P(t) = \sum_{r=0}^{m} a_r t^r \quad (a_r \in F_\alpha,\ a_m = 1)$$

such that $P(\zeta_\alpha) = 0$. Let X be chosen to satisfy

$$\sum_{r=0}^{m} \phi(a_r) X^r = 0.$$

If $p(t)$ and $q(t)$ are polynomials over F_α with $p(\zeta_\alpha) = q(\zeta_\alpha)$, then $p \equiv q \pmod{P}$; hence, if

$$x = \sum_{r=0}^{n} c_r \zeta_\alpha^r \ (c_r \in F_\alpha) \quad \text{and we define} \quad \phi(x) = \sum_{r=0}^{n} \phi(c_r) X^r,$$

it follows, since ϕ is a Segre function on F_α, that ϕ is one-valued in $F_\alpha(\zeta_\alpha)$, and arguments parallel to those used in 4·2 show that ϕ is a Segre function on $F_\alpha(\zeta_\alpha)$.

By the principle of transfinite induction, the above process defines ϕ in Z so that ϕ is a Segre function on every F_α. Since every pair of numbers in Z belong to F_α for some α, ϕ is an isomorphism defined on Z which extends f, and this completes the proof.

4·4. I am indebted to a referee for the suggestion to use Zorn's principle, and also to Dr. J. L. B. Cooper who made some very helpful suggestions in the same sense. It may be of interest to show how the argument of 4·3 is transformed when Zorn's principle (**14**) replaces the well-ordering theorem of Zermelo.

Let f_0 be the automorphism of $R(\sqrt{2})$ defined by $f_0(\sqrt{2}) = -\sqrt{2}$ (any non-trivial automorphism of a subfield of Z would do instead). We show that there is an automorphism of Z which extends f_0.

Suppose B is a set of isomorphisms such that any element of B either extends or is extended by any other element of B. It is clear that, if E is the union of the fields of all the elements of B, then (i) E is a field, (ii) if $F(z)$ is defined in E as $\theta(z)$, where θ is any element of B whose field of definition includes z, then F is an isomorphism on E, and (iii) F is an automorphism if every element of B is one.

Now consider the class $\mathscr{A}$ of all *automorphisms* which extend f_0. By Zorn's principle, $\mathscr{A}$ has a maximal element μ which cannot be extended by any element of $\mathscr{A}$. Our object is attained if $H_\mu = Z$, H_μ denoting the field on which μ is defined. Suppose that $G \equiv Z - H_\mu \neq 0$. If $\zeta \in G$, then ζ is algebraic over H_μ, for if this were false we could, following the argument of 4·2, extend μ by an *automorphism* over $H_\mu(\zeta)$. Now consider the class $\mathscr{I}$ of all isomorphisms which extend μ. By Zorn's principle $\mathscr{I}$ has a maximal element ν. The arguments of 4·2 and 4·3 show that a maximal element of $\mathscr{I}$ must have Z for its field. Hence ν is an isomorphism on Z which extends f_0. Our object is attained if $\nu(Z) = Z$, and since ν extends μ it is enough to show that $\nu(Z) \supset G$. This follows at once from the fact that $\nu(Z)$, like Z, is algebraically closed and the fact that G is algebraic in H_μ.

5. It was proved by Hamel (5) that if $\phi(x)$ is real and satisfies (1) for all *real* z_1 and z_2, then the values assumed by $\phi(x)$ in any interval of $\mathscr{R}$, however small, are everywhere dense in $(-\infty, \infty)$, except in the trivial case $\phi(x) = x\phi(1)$. This was extended by Ostrowski (8), who showed that *interval* may be replaced by *set having positive interior measure.* In this section we prove a series of similar results for non-trivial Segre functions on Z; they include and extend results of E. Noether (15). These results have a bearing on the problem of finding conditions which are sufficient to restrict a Segre function to the trivial cases. From the geometrical point of view, there is some interest in considering the character of $\phi(z)$ in so far as it is determined by its values for real z.

LEMMA 1. *Suppose that $\phi(z)$ is a Segre function in Z and $\phi(z)$ is continuous relative to $\mathscr{R}$ at one point of $\mathscr{R}$, then ϕ is trivial.*

Proof. By (1), $\phi(z)$ is continuous relative to $\mathscr{R}$ at all points of $\mathscr{R}$. Since $\phi(x) = x$ whenever x is rational, this means that $\phi(x) = x$ whenever x is real. By (2), $\{\phi(i)\}^2 = \phi(-1) = -1$, and so $\phi(i) = \pm i$. By (1) and (2) we now conclude that

$$\phi(x+iy) = \phi(x)+\phi(i)\phi(y) = x+\phi(i)y,$$

i.e. ϕ is trivial.

THEOREM 1. *Suppose that* $\phi(z)$ *is a Segre function in* Z. *If* $E \subset \mathscr{R}$ *and* $m_i E > 0$, *and* $\phi(z)$ *is either* (i) *bounded in* E *or* (ii) *real for all* z *in* E, *then* ϕ *is trivial.*

Proof. By a theorem of Steinhaus (**11**), there is a positive number δ such that every x satisfying $-\delta < x < \delta$ can be written $x = a - b$, where a and b belong to E.

If (i) holds, let M be the upper bound of $|\phi(z)|$ in E; then $-\delta/n < x < \delta/n$ implies $|\phi(x)| \leqslant 2M/n$, and so $\phi(z)$ is continuous relative to $\mathscr{R}$ at $z = 0$. Lemma 1 completes the proof.

If (ii) holds, then $\phi(x)$ is real whenever $-\delta < x < \delta$, and, since $\phi(rx) = r\phi(x)$ for rational r, $\phi(x)$ must be real throughout $\mathscr{R}$. If now x and h are real, $h \neq 0$, then

$$\phi(x+h^2)-\phi(x) = \phi(h^2) = \{\phi(h)\}^2 > 0.$$

Thus $\phi(x)$ is monotone in $\mathscr{R}$ and $\phi(x) = x$ whenever $x \in R$. This implies $\phi(x) = x$ throughout $\mathscr{R}$; lemma 1 completes the proof.

THEOREM 2. *If* $\phi(z)$ *is a non-trivial Segre function in* Z, *then* $\phi(\mathscr{R})$ *is everywhere dense in* Z.

Proof. By theorem 1 (ii), there is a real ξ with $\phi(\xi)$ unreal. Hence, if r_1 and r_2 range through R, the numbers $r_1\phi(\xi)+r_2$, i.e. the numbers $\phi(r_1\xi+r_2)$ are everywhere dense in Z.

THEOREM 3. *If* $\phi(z)$ *is a non-trivial Segre function in* Z, *and* U *is any circle in* Z, *then the set of* real x *with* $\phi(x) \in U$ *is non-measurable and has zero interior measure* (*in* $\mathscr{R}$).

Proof. Let $H = \mathscr{R}\phi^{-1}(U)$. Take ξ as in the proof of theorem 2 and then r_1 and r_2, $r_1 \neq 0$, so that $\phi(r_1\xi+r_2) \in U$, i.e. $r_1\xi+r_2 \in H$. If $|H| = 0$ it follows, since R is enumerable, that the set of all ξ is null and consequently that $\phi(x)$ is real almost everywhere in $\mathscr{R}$, which contradicts theorem 1 (ii). Hence $|H| > 0$. Since $\phi(H)$ is bounded, $m_i H = 0$ by theorem 1 (i).

THEOREM 4. *If* $\phi(z)$ *is a non-trivial Segre function in* Z, *and* S *is any set of* real *numbers with* $m_i S > 0$, *then* $\phi(S)$ *is everywhere dense in* Z.

Proof. There is no loss of generality if we assume S to be closed and bounded and that the lower bound of $|x|$ for x in S is positive. Let ν be the greatest value of $|x|$ for x in S.

Suppose first that there is a positive number ρ such that $|\phi(x)| > \rho$ for all x in S. Let T be the set of numbers $1/x$ as x ranges through S. Plainly T is closed, and $|\phi(t)| < 1/\rho$ for all t in T; this contradicts theorem 3 if $mT > 0$. Now if T is covered by intervals (x_r, y_r) $(r = 1, 2, \ldots)$ having

$|x_r|$ and $|y_r|$ both greater than $\frac{1}{2}\nu^{-1}$ and $x_r y_r > 0$, then S is covered by intervals the sum of whose lengths is $\sum_r |x_r^{-1} - y_r^{-1}|$, which is less than $(2\nu)^2 \sum (y_r - x_r)$, and so $mT = 0$ implies $mS = 0$. Thus $mT > 0$, and it follows that $\phi(S)$ has points in every circle centred on the origin.

Now suppose that $\phi(S)$ has no points in a circle $|z-\gamma| < \rho$. By theorem 2 there is a real number ξ with $|\phi(\xi)-\gamma| < \frac{1}{2}\rho$. Let E be the set of real x for which $(x+\xi) \in S$. Then $mE > 0$, and if $x \in E$ we have $|\phi(x+\xi)-\gamma| \geqslant \rho$, i.e. $|\phi(x)+\phi(\xi)-\gamma| \geqslant \rho$, as well as $|\phi(\xi)-\gamma| < \frac{1}{2}\rho$. Hence $|\phi(x)| > \frac{1}{2}\rho$ for all x in E, and this contradicts the case first considered.

THEOREM 5. *Suppose that ϕ is a non-trivial Segre function in Z and E is any subset of Z with both interior and exterior points.* Let l be any line in the Argand plane; then $l\phi^{-1}(E)$ is* (i) *everywhere dense in l,* (ii) *non-measurable and of zero interior measure. The same is true of $l\phi(E)$ if ϕ is an automorphism.*

Proof. First suppose that l is the real axis. If $m_i\{l\phi^{-1}(E)\} > 0$, then by theorem 4 and the hypothesis on E, $\phi\{l\phi^{-1}(E)\}$ includes points of $Z-E$, which is impossible. Hence $m_i l\phi^{-1}(E) = 0$. Replace E by $Z-E$ to get $m_i l\phi^{-1}(Z-E) = 0$. This implies (i) and (ii).

If l is the imaginary axis, then $\phi(iy) \in E$ means that $\phi(i)\phi(y) \in E$, i.e. $\phi(y) \in f(E)$, where $f(E)$ is a linear transformation of E. This reduces to the first case. Finally, if l has the equation $y = ax+b$ (a, b real), then the projection on the real axis of $l\phi^{-1}(E)$ is the set of real x for which $\phi\{x+i(ax+b)\}$, i.e. $\{\phi(x)\phi(1+ia)+\phi(ib)\} \in E$; this again restricts $\phi(x)$ to a linear transformation of E and so reduces to the first case.

If $\phi(z)$ is an automorphism, so is $\phi^{-1}(z)$, and we may replace ϕ by ϕ^{-1} in (i) and (ii).

We now consider properties of Segre functions on plane sets in Z.

THEOREM 6. *Suppose that $\phi(z)$ is a non-trivial Segre function in Z; then* (i) *$\phi(S)$ is everywhere dense in Z if $m_i S > 0$;* (ii) *if E is a set with both interior and exterior points, then $\phi^{-1}(E)$ has zero interior measure and is non-measurable. (The same is true of $\phi(E)$ if ϕ is an automorphism.)*

Proof. If $\phi(S)$ has no points in some circle U, then $S \subset \phi^{-1}(Z-U)$. But (putting $E = Z-U$ in theorem 5) this implies that every line cuts S in a linear set with zero interior measure. This, by Fubini's theorem, implies $m_i S = 0$ and proves (i). If $m_i \phi^{-1}(E) > 0$, then by (i) E is everywhere dense, and this is impossible since E has exterior points. Hence $m_i \phi^{-1}(E) = 0$. Since E may be replaced by $Z-E$, we conclude that

* i.e. E and $Z-E$ both possess interior points.

$\phi^{-1}(E)$ has zero interior measure and is non-measurable. Finally, if ϕ is an automorphism we have the desired conclusion as in theorem 5.

Theorem 6 leads to the paradoxical conclusion that, if M is a measurable subset of Z and E is a set which has both interior and exterior points, then $|M\phi^{-1}(E)| = |M\phi(E)| = mM$ if ϕ is a non-trivial automorphism of Z; this is because $\phi^{-1}(Z-E)$ and $\phi(Z-E)$ have zero interior measure. For example, if $\{U_n\}$ is a sequence of mutually exclusive circles, the sets $M\phi^{-1}(U_r)$ will be a sequence of *exclusive* subsets of M each having the same exterior measure as M.

6. In this section we discuss the geometric aspect of the transformation of the Argand plane Z (*not* the bicomplex plane $\mathfrak{Z}$) by a function ϕ which defines an automorphism of Z.

LEMMA 2. *Suppose that S is a subset of Z which includes unreal numbers and which has the properties*

(i) *if $a \in S$ and $b \in S$, then $(a-b) \in S$,*

(ii) *if $a \in S$ and r is rational, then $ar \in S$,*

((i) *and* (ii) *hold, for example, if S is a field in Z); then there are three possibilities, just one of which must be realized*:

$$\textit{either} \quad S = Z, \quad \textit{or} \quad |S| = 0, \quad \textit{or} \quad m_i S = m_i(Z-S) = 0.$$

Proof. Suppose that $m_i S > 0$. By Steinhaus's theorem and (i), S contains a circle with its centre at O. Hence, by (ii) $S = Z$.

Now suppose $0 = m_i S < |S|$. We have only to prove that $m_i(Z-S) = 0$. The points of S do not lie on a single line through O; hence, by (i) and (ii), S is everywhere dense in Z. Suppose now that $m_i(Z-S) > 0$, and let ζ be a point of $Z-S$ which is the centre of arbitrarily small circles U satisfying
$$m_i\{U(Z-S)\} > \tfrac{3}{4}|U|, \quad \text{i.e. } |US| < \tfrac{1}{4}|U|.$$
Since ζ is a limit point of S it follows that there are arbitrarily small circles V which have their centres in S and satisfy $|VS| < \frac{1}{2}|V|$. Now it follows easily from (i) that, for any circle V which has its centre in S, $|VS|$ depends only on the radius of V. Since we have just shown that for such circles $|VS| < \frac{1}{2}|V|$ for arbitrarily small V, it follows that the exterior density of S is never unity at any point of S. This contradicts $|S| > 0$ and so establishes $m_i(Z-S) = 0$.

THEOREM 7.* *If $\phi(z)$ is a non-trivial Segre function in Z, then* (i) *$\phi(\mathscr{R})$ is a proper subfield of Z with cardinal number c,* (ii) *either $\phi(\mathscr{R})$ is null, or else $m_i\phi(\mathscr{R}) = m_i\phi(Z-\mathscr{R}) = 0$.*

Proof. (i) That $\phi(\mathscr{R})$ is a field follows from (1) and (2), the rest from the fact that the mapping of Z on $\phi(Z)$ is 1–1. (ii) Put $S = \phi(\mathscr{R})$ in

* In connexion with (i), see also M. Souslin (**13**).

lemma 2 to get $m_i\phi(\mathscr{R}) = 0$. Since $\phi(Z-\mathscr{R}) \subset Z-\phi(\mathscr{R})$, $m_i\phi(Z-\mathscr{R}) = 0$ by the same lemma.

We can now give a picture of the geometric transformation of the Argand plane Z into $\phi(Z)$ when ϕ defines an automorphism of Z. We show that the $\phi(Z)$ plane is covered by each of two "orthogonal" families of sets (corresponding to the lines in the Z-plane parallel to the real and imaginary axes). The set $\phi(\mathscr{R})$ may be used as a basis for the description.

Let Y_η and X_ξ denote the transforms by $\phi(z)$ of the lines $y = \eta$ and $x = \xi$ respectively in Z. Plainly $Y_0 = \phi(\mathscr{R})$ and, since $\phi(iz) = \pm i\phi(z)$, X_0 is the result of rotating Y_0 about the origin through an angle $\pm\frac{1}{2}\pi$. Y_η is the result of translating Y_0 by the vector $\phi(i\eta)$ and X_ξ the result of translating X_0 by the vector $\phi(\xi)$.

Thus $\phi(Z)$ is covered by the system of X_ξ and also by the system of Y_η; through any point w in $\phi(Z)$ there passes just one Y_η set and one X_ξ set: the former is obtained by translating $\phi(\mathscr{R})$ so that one of its points falls on w, and the latter by rotating the Y_η set about w through an angle $\pm\frac{1}{2}\pi$.

If $\phi(\mathscr{R})$ is not null, it follows by theorem 7 (ii) that the transform by $\phi(z)$ of the line $y = \eta$ cuts every measurable set E in the $\phi(Z)$ plane in a set whose exterior measure is mE.

It may be noted in passing that, if $\phi^{-1}(w) = X(w)+iY(w)$, where X and Y are real functions, then these functions are additive (i.e. they satisfy equations such as (1)) and that, since $\phi^{-1}(Y_0) = \mathscr{R}$, $Y(w) = 0$ means $w \in Y_0$. Hence $\phi(\mathscr{R})$ is the set of all the periods of the function $Y(w)$.

7. Segre's geometrical problem may be stated as follows.

Suppose that $T(P)$ is a 1–1 transformation of a bicomplex plane $\mathfrak{Z}$ into itself, which transforms every three collinear points into collinear points: what is the relation between the cross-ratio of four collinear points and that of their transforms by T?

Both Segre and Coolidge conjectured that the two cross-ratios are either always equal or always conjugate complex numbers. If this were so, it would mean that T is either a projectivity or else the product of a projectivity and the transformation which assigns $(\bar{z}_1, \bar{z}_2, \bar{z}_3)$ to the point with homogeneous coordinates (z_1, z_2, z_3). If a collineation T satisfies one of these alternatives, we shall write $T \in S$. It was pointed out in § 1 that every non-trivial automorphism of Z could be used to define a T such that $T \notin S$. In Coolidge (2) it is usual to impose a condition of continuity on T in order to make $T \in S$, but we shall see that weaker restrictions suffice.

A collineation T will be said to have the Segre property in a set E if the cross-ratio of every four collinear points of E is either always equal to, or is always the conjugate-complex of, that of their transforms under

T. We observe first that if T has the Segre property in a line l of $\mathfrak{Z}$, then it has it in the whole of $\mathfrak{Z}$: for, if Q_r ($r = 1, 2, 3, 4$) are collinear points not on l, project them on to l from some point V, Q_r going into Q'_r. The line joining $T(V)$ to $T(Q_r)$ cuts $T(l)$ in $T(Q'_r)$, and since $\{Q_1, Q_2; Q_3, Q_4\}$ and $\{Q'_1, Q'_2; Q'_3, Q'_4\}$ are equal, it follows easily that T has the Segre property in the line $Q_1 Q_2$ if it has it in l. It is therefore sufficient, as well as obviously necessary, to show that T has the Segre property in l_x, the "x-axis" of $\mathfrak{Z}$, if we wish to conclude $T \in S$. We suppose, for convenience, that T is extended in the obvious way so as to apply to the points at infinity of $\mathfrak{Z}$. Let P_z denote the point $(z, 0)$ and P_∞ the point at infinity on l_x. Suppose that $T(P_0) = A$, $T(P_1) = U$, and $T(P_\infty) = I$. For every finite z we define $\phi(z) = \{I, A; U, T(P_z)\}$. [The condition that T shall have the Segre property on l_x is then the condition that $\phi(z) = z$ or $\phi(z) = \bar{z}$ in Z, i.e. that $\phi(z)$ is a *trivial* Segre function in Z.] We first show that $\phi(z)$ is a Segre function in Z. If z_1 and z_2 are in Z, then $(P_{z_1+z_2}; P_0)$, $(P_{z_1}; P_{z_2})$, $(P_\infty; P_\infty)$ are mates in an involution on l_x. Using Desargues's theorem on the involution properties of a quadrangle and the fact that incidence of lines is invariant under T, we deduce that the transforms of these elements are likewise mates in an involution on $T(l_x)$. Consequently, since $\{a, b; c, t\}$ is, for fixed a, b, c, a linear fractional function of t, it follows that $\{\phi(z_1+z_2); 0\}$, $\{\phi(z_1); \phi(z_2)\}$, $(\infty; \infty)$ are mates in an involution of complex numbers, i.e. that (1) holds. Similarly, since $(z_1 z_2; 1)$, $(z_1; z_2)$, $(0; \infty)$ are mates in an involution, we deduce that (2) holds, and so $\phi(z)$ is a Segre function in Z.

From what has just been proved, it follows that any set of conditions which is sufficient to make a Segre function trivial will at the same time give a geometrical condition sufficient to make $T \in S$. The properties of non-trivial Segre functions considered in § 5 make it easy to state such conditions. For example, if we restrict our attention to the real points of l_x, it follows from theorem 4 that if l_x contains a real interval, or even a real set E with positive interior measure, such that $T(E)$ is not everywhere dense in the complex line $T(l_x)$, then $T \in S$. In particular, if $T(P)$ is real or if $T(P)$ is bounded in E, or if $T(P)$ is continuous relative to E at one point of E, then $T \in S$.

Addendum. Dr. L. Roth has kindly shown me a paper by B. Segre entitled "Gli automorfismi del corpo complesso, ed un problema di Corrado Segre" which has appeared in the *Atti della Acc. dei Lincei* (Fasc. 5–6, novembre–dicembre 1947, p. 414). The existence of non-trivial automorphisms is established there as a deduction from Steinitz's theorems in much the same way as in § 3 above.

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