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INDUCTION OVER THE CONTINUUM

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Abstract: We exploit the analogy between the well ordering principle for nonempty subsets of $\mathbb{N}$ (the set of natural numbers) and the existence of a greatest lower bound for non-empty subsets of $[a, b)$ [1] to formulate a principle of induction over the continuum for $[a, b)$ analogous to induction over $\mathbb{N}$. While the gist of the idea for this principle has been alluded to, our formulation seems novel. To demonstrate the efficiency of the approach, we use the new induction form to give a proof of the compactness of $[a, b]$. (Compactness, which plays a key role in topology, will be briefly discussed.) Although the proof is not fundamentally different from many familiar ones, it is direct and transparent. We also give other applications of the new principle.

1 INTRODUCTION

When teaching a first course in analysis recently, I formulated a proof of the Heine-Borel Theorem. Upon a search of archives, I learned that W. L. Duren Jr. [4] had a close and similar idea, except that his analogy (and formulation) is to Zorn's Lemma as applied to a chain of intervals to get a maximal element, while the present analogy (and formulation) is to ordinary induction (and so perhaps more easily accessible). When several other theorems of analysis readily found proofs through this "inductive approach", I thought to share the idea.

[1] For real numbers a and b, $[a, b)$ is the set of real numbers between a and b, including a and excluding b; $[a, b]$ is the set of real numbers between a and b, including a and b.

M. Friend, N.B. Goethe and V.S. Harizanov (eds.), Induction, Algorithmic Learning Theory, and Philosophy, 145–154.

While ordinary induction on $\mathbb{N}$ and *transfinite induction* on ordinals both hinge upon the underlying *well ordering* structures present, and while Zorn's Lemma, the Axiom of Choice, or the *well ordering principle for every set* can play interrelated roles with those induction forms, we confine the focus of this paper to *ordinary induction* and *induction on the continuum*.

In Section 2, we state and show equivalence between some principles involving induction over the set of natural numbers. The proofs are presented to help carry the analogy for the similar principles involving induction over the continuum presented in Section 3. After briefly discussing *compactness* in Section 4, we prove the Heine-Borel theorem in Section 5, and give further applications of the new principle in Section 6. Some brief historical remarks and miscellanea are presented in Sections 7 and 8.

2 ORDINARY INDUCTION

When we wish to establish the truth of an assertion $\forall n P(n)$, where n ranges over $\mathbb{N} = \{0, 1, 2, \cdots\}$, and $P(n)$ is a predicate about n, we may define $S = \{n \in \mathbb{N} : P(n)\}$ and seek to demonstrate that $S = \mathbb{N}$ by exploiting the principle of:

Induction Over $\mathbb{N}$ (**ION**). For any $S \subseteq \mathbb{N}$, if

(1) $0 \in S$, and
(2) $\forall k[\, k \in S \Rightarrow k + 1 \in S \,]$,

then $S = \mathbb{N}$.

In using **ION**, we find efficiency and satisfaction on a few counts. For example, demonstrating (1) is often a simple verification, and establishing (2) is a boot-strapping process propelled by the assumption of $k \in S$ in pursuit of concluding $k + 1 \in S$. Furthermore, when the predicate $P(n)$ is free of quantifiers, we seem to avoid the 'magic' of a proof by contradiction which often masks and mystifies some mathematically meaningful underlying processes.

The dual principle for $\mathbb{N}$, ordered under "$\leq$", sometimes used to explain the truth of **ION**, as well as used for an indirect proof of $\forall n P(n)$, is the following:

Well Ordering Principle (**WOP**). If $T \subseteq \mathbb{N}$ and $T \neq \emptyset$, then T has a least element.

Because we wish to find conceptual parallels, we restate **ION** for $S \subseteq \mathbb{N}$ in the following equivalent form (known as *strong induction*):

ION. If
(1) $\exists k[\ (k \geq 0) \wedge ([0, k) \subseteq S)\ ]$, and
(2) $\forall k[\ [0, k) \subseteq S \Rightarrow (\exists l > k)[0, l) \subseteq S\ ]$,
then $S = \mathbb{N}$.

Here we have $[0, k) =_{\text{def}} \{j \in \mathbb{N} : 0 = j \vee j < k\}$. (So $[0, 0) = \{0\}$.) At this stage, we make the following familiar observation.

Theorem 2.1 WOP *iff* **ION**.

Proof. ($\Rightarrow$) Assume **WOP**. Further suppose that S, a subset of $\mathbb{N}$, satisfies the properties (1) and (2) of **ION**. Presume $S \neq \mathbb{N}$. Then $S' = \mathbb{N} - S$, the complement of S, is a nonempty subset of $\mathbb{N}$. Applying **WOP** to S', let z be the least element of S'; note $0 < z$ by (1) of **ION**. Clearly, $[0, z) \cap S' = \emptyset$. So $[0, z) \subseteq S$, and by (2) of **ION**, there is $y > z$ such that $[0, y) \subseteq S$. Thus z is not the least element of S'; a contradiction. Hence our presumption is false and $S = \mathbb{N}$.

($\Leftarrow$) Assume **ION**. Further let $T \subseteq \mathbb{N}$ and presume T does not have a least element. We will show $T = \emptyset$. Consider $T' = \mathbb{N} - T$. Clearly $0 \in T'$ because otherwise 0 would be in T and its least element, contradicting the hypothesis. Thus T' satisfies condition (1) of **ION**. Next, for arbitrary k, assume $[0, k) \subseteq T'$. But then k cannot belong to T as otherwise it would be T's least element. So $k \in T'$. So, for some $l > k$, $[0, l) \subseteq T'$. (Here l is possibly just $k + 1$.) So T' satisfies condition (2) of **ION**. Thus, **ION** applies to T' and $T' = \mathbb{N}$. Thus $T = \emptyset$. ■

The proofs above are trivial but they are included to help carry the intended analogy that will follow next.

3 INDUCTION OVER THE CONTINUUM

We also recall a familiar property for nonempty, bounded subsets of $\mathbb{R}$ (the set of real numbers) under the ordering "$\leq$", for any reals $a, b \in \mathbb{R}$ with $a < b$:

The Greatest Lower Bound Principle (**GLBP**). If $T \subseteq [a, b)$ and $T \neq \emptyset$, then T has a greatest lower bound (in $[a, b)$).

Consider the interval $[a, b)$ with $a, b \in \mathbb{R}$, $a < b$, and $S \subseteq [a, b)$. We formulate an "induction scheme" over $[a, b)$ as follows.

Induction Over the Continuum (**IOC**). If
(1) $\exists x(x \geq a) \wedge ([a, x) \subseteq S)$, and

(2) $\forall x[\,[a,x) \subseteq S \Rightarrow (\exists y > x)[a,y) \subseteq S\,]$,
then $S = [a,b)$.

Here, we have $[a,x) =_{\text{def}} \{t \in [a,b) : a = t \vee t < x\}$. So $[a,a) = \{a\}$. Similarly to the case for **ION**, we establish the next two results.

Theorem 3.1 *If* **GLBP***, then* **IOC***.*

Proof. Assume **GLBP**. Further suppose that S, a subset of $[a,b)$, satisfies the properties (1) and (2) of **IOC**. Presume $S \subsetneq [a,b)$. Then $S' = [a,b) - S$ is a bounded, non-empty subset of $[a,b)$. Applying **GLBP**, let z be the greatest lower bound of S'; note $a < z < b$ by (1) of **IOC**. Clearly, $[a,z) \cap S' = \emptyset$. So $[a,z) \subseteq S$, and by (2) of **IOC**, there is $y > z$ such that $[a,y) \subseteq S$. Thus z is not the greatest lower bound of S'; a contradiction. Hence our presumption is false and $S = [a,b)$.[2] ■

Theorem 3.2 *If* **IOC***, then* **GLBP***.*

Proof. Assume **IOC**. Further let $T \subseteq [a,b)$ and presume T does not have a greatest lower bound. We will show $T = \emptyset$. Consider $T' = [a,b) - T$. Clearly $a \in T'$ because otherwise a would be in T and its greatest lower bound, contradicting the hypothesis. Thus T' satisfies condition (1) of **IOC**. Next, for arbitrary x, assume $[a,x) \subseteq T'$. But then x is a lower bound for T and is not, by hypothesis, its the greatest lower bound. So there exists y with $y > x$ such that y is a lower bound for T. Consequently, $[a,y) \subseteq T'$. So T' satisfies condition (2) of **IOC**. Thus, **IOC** applies to T' and $T' = [a,b)$. Thus $T = \emptyset$. ■

Theorems 3.1 & 3.2 combined and compared to Theorem 2.1 demonstrate the analogy: **WOP** is to **ION** as **GLBP** is to **IOC**. As **GLBP** is logically equivalent to the *completeness* of $[a,b]$, **IOC** could be assumed as an axiom in place of the completeness axiom.

[2] As **ION** often is metaphorically described through "domino theory", it seems that the motion of a "curling stone" can serve as a metaphorical description for **IOC**. Indeed it was surprising to excogitate the following quatrain, which seems to capture the idea of **IOC** closely:

The Curling Stone slides; and, having slid,
Passes me toward thee on this Icy Grid,
If what's reached is passed for'll Crystals amid,
Th'Stone Reaches thee in its Eternal Skid.

By Harak A'Myomy (12th century), translated by Walt Friz De Gradde (1897).

4 COMPACTNESS

The theorem of our central interest in this paper, which is about the *compactness* of $[a, b]$, helped streamline the development of analysis and topology. It could be said that *compact* is to pursuits in topology as *finite* is to pursuits in set theory. In the theory of sets, finite sets behave more tamely than infinite sets; in topology, compact sets behave more tamely than noncompact sets. For that reason, the concept was pursued mathematically in several different planes (including mathematical logic) simultaneously or independently during the last two centuries.

Definition 4.1 *A subset of $\mathbb{R}$, K, is* ***compact*** *if whenever a (possibly infinite) family $\mathcal{O}$ of open subsets of $\mathbb{R}$ covers K (that is the union of members of $\mathcal{O}$ contains K), there is a finite subfamily of $\mathcal{O}$ that covers K.*

To motivate this concept as well as the focus of this paper, the Heine-Borel Theorem (in the next section), we recall the following result:

A function which is continuous[3] *on $[a, b]$ is uniformly continuous*[4] *there.*

Clearly, continuity of a function at $x \in [a, b]$, is an ε, δ process where δ depends both on x and ε. However, uniform continuity guarantees a δ which depends on ε but applies to *any* point x in $[a, b]$.

While the proof of the above result using **IOC** is directly possible, we delay that until the subsequent section. In the following section, we prove the Heine-Borel Theorem, which is traditionally used to derive the above result and many other fundamental theorems of analysis.

5 THE HEINE-BOREL THEOREM

The fact that $[a, b]$ is a compact subset of $\mathbb{R}$ is established in what appears to be three distinct ways: through "there is a finite subcover for every infinite open cover"; through "an indirect proof to get a nested sequence of intervals leading to an application of Cantor's Nested Interval Theorem"; or through "every bounded sequence has a convergent subsequence". In the first mentioned (and the more frequently presented) type of proof, there seems to be a few "twists" necessitated by the indirect approach (see, for example,

[3] $f : \mathbb{R} \to \mathbb{R}$ is*continuous on* $L \subseteq \mathbb{R}$ if

$$(\forall x \in L)(\forall \varepsilon > 0)(\exists \delta_{(x,\varepsilon)} > 0)(\forall y \in L)[\ |x - y| < \delta_{(x,\varepsilon)} \Rightarrow |f(x) - f(y)| < \varepsilon\ ].$$

[4] $f : \mathbb{R} \to \mathbb{R}$ is *uniformly continuous on* $L \subseteq \mathbb{R}$ if

$$(\forall \varepsilon > 0)(\exists \delta_{\varepsilon} > 0)(\forall x \in L)(\forall y \in L)[\ |x - y| < \delta_{\varepsilon} \Rightarrow |f(x) - f(y)| < \varepsilon\ ].$$

Royden's *Real Analysis*, [11]); although the proof below is quite similar, twists are absent.

Theorem 5.1 (Heine-Borel) *For any $a, b \in \mathbb{R}$ with $a < b$, the interval $[a, b]$ is compact.*

Proof. Let $\mathcal{O}$ be an open cover for $[a, b]$. Set

$$S = \{t : t \in [a, b) \text{ and } [a, t] \text{ is contained in a finite cover from } \mathcal{O}\}.$$

Firstly, there exists $x \geq a$ such that $[a, x) \subseteq S$, as $a \in V_a \in \mathcal{O}$ for some open V_a. Secondly, assume $[a, x) \subseteq S$. Since, $x \in V_x$ for some open $V_x \in \mathcal{O}$, there exist $y > x$ and $x' < x$ with $x \in (x', y) \subseteq V_x \cap [a, b]$. As $x' \in S$, the finite cover for $[a, x']$ together with V_x confirm that $[a, y) \subseteq S$.

Thus the two statements in **IOC**'s hypothesis are satisfied and accordingly we have $S = [a, b)$. Finally, as $b \in V_b \in \mathcal{O}$ for some open V_b, there exists a $b' < b$ with $b' \in V_b \cap S$; so $[a, b]$ is contained in the finite cover comprised from the finite cover for $[a, b']$ together with V_b. So $[a, b]$ is compact. ■

6 OTHER APPLICATIONS

In this section, we use **IOC** to prove a sample of familiar theorems of elementary analysis.

Theorem 6.1 (Heine) *If f, a function from $[a, b]$ to $\mathbb{R}$, is continuous, then it is uniformly continuous.*

Proof. Assume f is continuous on $[a, b]$. To show f is uniformly continuous there, let $\varepsilon > 0$ be given. Also, let

$$S = \{t : t \in [a, b) \wedge (\exists \delta_\varepsilon > 0)(\forall u, v \in [a, t])[\ |u - v| < \delta_\varepsilon \Rightarrow |f(u) - f(v)| < \varepsilon\ ]\}.$$

We first use **IOC** to show $S = [a, b)$. Since f is continuous at a, $f(a)$ is defined and we have, trivially,

$$(\exists \delta_\varepsilon > 0)(\forall u, v \in [a, a))[\ |u - v| < \delta_\varepsilon \Rightarrow |f(u) - f(v)| < \varepsilon\ ].$$

So $[a, a) = \{a\} \subseteq S$ and S satisfies condition (1) of **IOC**.

Next, for an arbitrary $x \in [a, b)$, assume $[a, x) \subseteq S$. We wish to show, for some $y > x$, we have $[a, y) \subseteq S$. Consider x and $\frac{\varepsilon}{2}$. Since f is continuous at x, there exists $\delta_{(x, \frac{\varepsilon}{2})} > 0$, such that, for any x', if $|x - x'| < \delta_{(x, \frac{\varepsilon}{2})}$, then $|f(x) - f(x')| < \frac{\varepsilon}{2}$. (Assume $\delta_{(x, \frac{\varepsilon}{2})}$ is smaller than $x - a$ and $b - x$.)

Consider $t = x - \frac{1}{2}\delta_{(x,\frac{\varepsilon}{2})}$. Since $t \in S$, we have, there exists $\delta_\varepsilon > 0$ such that, for any $u, v \in [a, t]$, if $|u - v| < \delta_\varepsilon$, then $|f(u) - f(v)| < \varepsilon$.

Further let $y = x + \frac{1}{2}\delta_{(x,\frac{\varepsilon}{2})}$. We claim that $[a, y) \subseteq S$. We will actually show $[a, y] \subseteq S$. To establish this fact, we note that since $t \in S$ and $a \leq t' < t$ imply $t' \in S$, we need only show $y \in S$. To see this claim, for the given $\varepsilon > 0$, we offer δ_ε^* to be the minimum of δ_ε and $\frac{1}{2}\delta_{(x,\frac{\varepsilon}{2})}$. Next, consider arbitrary $u, v \in [a, y]$ with $|u - v| < \delta_\varepsilon^*$. If u, v are both in $[a, t] \subseteq S$, since $\delta_\varepsilon^* \leq \delta_\varepsilon$, by hypothesis we have:

$$|f(u) - f(v)| < \varepsilon.$$

If either u or v is in $(t, y]$, then both u and v are closer than $\delta_\varepsilon^* \leq \frac{1}{2}\delta_{(x,\frac{\varepsilon}{2})} < \delta_{(x,\frac{\varepsilon}{2})}$ to x, and so the continuity of f at x applies and (invoking the triangle inequality) we have:

$$|f(u) - f(v)| = |f(u) - f(x) + f(x) - f(v)| < |f(u) - f(x)| + |f(x) - f(v)| < \varepsilon.$$

This completes the proof of the claim, and so condition (2) of **IOC** is satisfied also. Thus, by **IOC**, $S = [a, b)$. Hence for any t with $a \leq t < b$ we have

$$(\exists\delta_\varepsilon > 0)(\forall u, v \in [a, t])[\ |u - v| < \delta_\varepsilon \Rightarrow |f(u) - f(v)| < \varepsilon\ ].$$

A similar argument applied to continuity of f at b shows that the same is true for $t = b$. Since this is true for any given $\varepsilon > 0$, f is uniformly continuous over $[a, b]$. ■

Theorem 6.2 (Cousin) *Let $\mathcal{C}$ be a collection of closed subintervals of* $[a, b]$ *such that for every $x \in [a, b]$ there is a corresponding 'fineness' $r_x > 0$, such that $\mathcal{C}$ contains every subintervals of* $[a, b]$ *with length smaller than or equal to r_x and containing x. Then there exist $x_0 = a < x_1 < x_2 < \cdots < x_n = b$ such that $[x_i, x_{i+1}]$ belongs to $\mathcal{C}$ for every $0 \leq i < n$; that is, $\mathcal{C}$ contains a partition of* $[a, b]$.

Proof. Let $S = \{t : t \in [a, b)$ and $\mathcal{C}$ contains a partition of $[a, t]\}$. Since r_a exists, for every nonnegative $t' < r_a$, $[a, a + t']$, which is in $\mathcal{C}$, is a partition of itself, putting each $a + t'$ in S. Thus $[a, r_a) \subseteq S$. Next, assume $[a, x) \subseteq S$ for $x \in [a, b)$. Working with r_x, and applying the induction hypothesis to $[a, x - \frac{r_x}{2}]$, find a partition of $[a, x - \frac{r_x}{2}]$ in $\mathcal{C}$, add to that partition the interval $[x - \frac{r_x}{2}, x + \varepsilon]$, for any $0 < \varepsilon \leq \frac{r_x}{2}$, and end up with a partition for $[a, x + \varepsilon]$ in $\mathcal{C}$. Thus $[a, x + \frac{r_x}{2}) \subseteq S$.

It is now clear that S satisfies the hypothesis for **IOC**, and therefore $S \supseteq [a, b) \supseteq [a, b - r_b]$. Since $(b - r_b) \in S$ and $[b - r_b, b] \in \mathcal{C}$, we have the desired partition for $[a, b]$. ■

Theorem 6.3 *(A form of the Intermediate Value Theorem) Let f be a continuous function over $[a, b]$ with no roots, and $f(a) > 0$. Then $f(x) > 0$ for all $x \in [a, b]$.*

Proof. Let $S = \{t : t \in [a, b) \text{ and } f \text{ is positive over } [a, t]\}$, and apply **IOC**.■

The reader can apply **IOC** to similar theorems to find similar proofs.

7 THE RELEVANT (TELEGRAPHIC) HISTORY

It was E. Heine [7] who first (1872) implicitly proved what is now called the 'Heine-Borel' theorem while showing *if f is continuous on $[a, b]$, then f is uniformly continuous there*. Later (1895), Cousin [3] proved similar findings and he too implicitly used the Heine-Borel result. It was Borel [1] who made this result explicit in his covering theorem that any countably infinite open cover for a bounded and closed interval of $\mathbb{R}$ can be replaced with a finite subcover. Finally, Lebesgue [8] and Lindelöf [9] independently showed Borel's result is also true in case the original cover is uncountably infinite.

O. Veblen [14] proved $[a, b]$ is compact iff $[a, b]$ is closed (he did not use the term 'compact'). Later, in [15] he defined a *linear continuum*, without the use of a metric, and observed that the same method of [14] can apply to linear continua to draw similar conclusions.

Weierstrass's theorem, *a continuous function over $[a, b]$ attains its maximum at some point of $[a, b]$*, is one of the important results of analysis and related to compactness. The term *compact* was first used by Frèchet in his thesis [6] in which, motivated to generalize Weierstrass's theorem (above) for abstract topological spaces, he described a certain phenomenon which is closely related to the modern usage of the word "compact".

Most classic textbooks that prove the Heine-Borel Theorem as a 'covering' theorem use a proof similar to the one in Royden's; I have not seen an elementary textbook that has adopted Duren's method [4]. The advanced (and comprehensive) textbook, *Real Analysis* by Bruckner-Bruckner-Thomson [2], starts with Cantor's Nested Interval Theorem & Cousin's Theorem, and considers the concepts of "full" and "additive" for a collection of closed intervals to pave the way for establishing the basic results of elementary analysis.

It should be noted that Duren attributes the origin of his approach to L.R. Ford [5] who examines proofs for "statements" that are "interval-additive"; that is, those properties that hold in the union of two overlapping intervals whenever they hold in each of the two intervals. Shanahan [12, 13] rediscovers the same "additive" approach.

Moss and Roberts [10] also isolate a theme common among elementary analysis theorems akin to the approaches by Ford, Duren, Shanahan, or through **IOC**. Namely, they establish results when they find a common theme of a transitive relation on $[a, b]$ which links the points from a to b through neighborhoods whenever the relation is such that every $x \in [a, b]$ has a neighborhood whose every point to the left of x is related to its every point to the right of x.

8 MISCELLANEA

It should be clear that **IOC**, seen as a form of *increasing* induction, can be modified to yield a form of *decreasing* induction over $(a, b]$ with equivalent results. To formulate a similar form of **IOC** for $[a, b]$, the second clause has to be altered to permit $x = b$, and all mentioned intervals of the form $[a, y)$ for $y > x$ would have to be intersected with $[a, b]$ before being required to be included in S. Furthermore the proof for the principle will have to carry the burden of two cases: when $x < b$ and when $x = b$. In that event, the proofs for some of the applications will be shorter as the last 'capping' step will be already in the principle and not needed as an additional step. However, the analogy to ordinary induction would be lost.

Our heuristic observation above has been that in the context of linear orderings on $\mathbb{N}$ and $\mathbb{R}$: **WOP** is to **ION** as **GLBP** is to **IOC**. Perhaps the reader can find other interesting analogies or other applications for the formulation of **IOC** in this paper.

9 EPILOGUE

Finally, purely for efficiency and concision, we note that we could define $[0, k) =_{\text{def}} \{j : 0 \leq j < k)\}$ (so $[0, 0) = \emptyset$), and similarly define $[a, x) =_{\text{def}} \{t : a \leq t < x)\}$ (so $[a, a) = \emptyset$), and state **ION** and **IOC** in the following condensed form: if S, a subset of the universe, has some 'inductive expansion' property, then as $\emptyset \subseteq S$, S is all of the universe. Formally:

ION. If $\forall k[\ [0, k) \subseteq S \Rightarrow (\exists l > k)[0, l) \subseteq S\ ]$, then $S = \mathbb{N}$.
IOC. If $\forall x[\ [a, x) \subseteq S \Rightarrow (\exists y > x)[a, y) \subseteq S\ ]$, then $S = [a, b)$.

ACKNOWLEDGEMENTS

I am grateful to Sara Kalantari whose curiosity prompted the approach taken here. I thank Dan Velleman, Tamara Lakins, Bahman Kalantari, Ali Enayat, and John Chisholm for encouragement, correspondence and thoughtful advice. I also thank the referee for helpful remarks.

REFERENCES

[1] Borel, E. (1895). "Sur Quelques Points de la Théorie des Fonctions", *Annales Scientifiques de l'École Normale Supérieure, Paris*, 3, 1–55.

[2] Bruckner, A.M., Bruckner, J.B. and Thomson, B.S. (1997). *Real Analysis*, Upper Saddle River (New Jersey): Prentice-Hall.

[3] Cousin, P. (1895). "Sur les Fonctions de n-Variables Complexes", *Acta Mathematica* 19, 1–61.

[4] Duren, Jr., W.L. (1957). "Mathematical Induction in Sets", *American Mathematical Monthly* 64, 19–22.

[5] Ford, L.R. (1957). "Interval-Additive Propositions", *American Mathematical Monthly* 64, 106–108.

[6] Frèchet, M. (1906). "Sur Quelques Point du Calcul Fonctionnel", *Rendiconti di Palermo* 22, 1–74.

[7] Heine E. (1872). "Die Elemente der Funktionenlehre", *Crelles Journal* 74, 172–188.

[8] Lebesgue, H. (1904). *Leçons sur l'Intégration*, Paris: Gauthier-Villars.

[9] Lindelöf, E.L. (1903). "Sur Quelques Points de la Théorie des Ensembles", *Comptes Rendus Hebdomadaire des Seances de l'Academie des Sciences, Paris*, 137, 697–700.

[10] Moss, R.M.F. and Roberts, G.T. (1968). "A Creeping Lemma", *American M athematical Monthly* 75, 649–652.

[11] Royden, H.L. (1968). *Real Analysis*, New York: MacMillan.

[12] Shanahan, P. (1972). "A Unified Proof of Several Basic Theorems of Real Analysis", *American Mathematical Monthly* 79, 895–898.

[13] Shanahan, P. (1974). "Addendum to 'A Unified Proof of Several Basic Theorems of Real Analysis'", *American Mathematical Monthly* 81, 890–891.

[14] Veblen, O. (1904). "The Heine-Borel Theorem", *Bulletin of the American Mathematical Society* 10, 436–439.

[15] Veblen, O. (1905). "Definitions in Terms of Order Alone in the Linear Continuum and in Well-Ordered Sets", *Transactions of the American Mathematical Society* 6, 165–171.