



Functions with Arbitrarily Small Periods

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CLASSROOM NOTES

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FUNCTIONS WITH ARBITRARILY SMALL PERIODS

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Recently R. Cignoli and J. Hounie [2] gave a new proof, together with applications, of Burstin's Theorem: *A Lebesgue-measurable function $f:R \rightarrow R$ having arbitrarily small periods is constant a.e.* The following is a more direct, self-contained proof.

Let I be any closed interval, and let $D = f^{-1}(I)$. Then the measure of D intersected with any interval depends linearly on the length of the interval. To see this, let $\alpha = m(D \cap [0, 1])$ and suppose we are given $\varepsilon > 0$ and $a < b$. Choose a period p of f so that $p < \varepsilon$ and $|m/n - (b - a)| < \varepsilon$, where $n = [1/p]$ and $m = [(b - a)/p]$. Since p is a period of f , the measure of D intersected with any interval of length p is the same. Thus if $d = m(D \cap [0, p])$, then $\alpha = m(D \cap [0, 1]) = nd + \varepsilon_1$ and $m(D \cap (a, b)) = md + \varepsilon_2$, with $\varepsilon_1, \varepsilon_2 < \varepsilon$. We then have:

$$\begin{aligned} |m(D \cap (a, b)) - \alpha(b - a)| &= \left| nd \left(\frac{m}{n} \right) + \varepsilon_2 - (nd + \varepsilon_1)(b - a) \right| \\ &= \left| nd \left(\frac{m}{n} - (b - a) \right) + \varepsilon_2 - \varepsilon_1(b - a) \right| < \alpha\varepsilon + \varepsilon_2 + \varepsilon_1(b - a); \end{aligned}$$

hence $m(D \cap (a, b)) = \alpha(b - a)$.

The theorem results from the following lemma.

LEMMA. *If the measure of a set D intersected with any interval depends linearly on its length, then either $m(D) = 0$ or $m(D^c) = 0$.*

Using this, for each $n > 0$, let $k_n, I_n = [k_n/n, (k_n + 1)/n]$ be such that $f^{-1}(I_n)$ is not of measure 0. By the lemma, $m(f^{-1}(I_n)^c) = 0$. Also, $\bigcap_{n < \omega} I_n$ is not empty, since

$$m \left(f^{-1} \left(\left(\bigcap_{n < \omega} I_n \right)^c \right) \right) = m \left(f^{-1} \left(\bigcup_{n < \omega} I_n^c \right) \right) = m \left(\bigcup_{n < \omega} f^{-1}(I_n^c) \right) = 0.$$

Since there can be no more than one point q in $\bigcap_{n < \omega} I_n$, $0 = m(f^{-1}(\{q\})^c)$ implies $f(x) = q$ a.e.

The lemma is proved as follows: let $\alpha = m(D \cap [0, 1])$. Given $\varepsilon > 0$, cover $D \cap [0, 1]$ with open intervals O_n so that $\sum_n m(O_n) < \alpha + \varepsilon$. Since $m(O_n \cap D) = \alpha m(O_n)$, we have

$$\alpha = m(D \cap [0, 1]) \leq \sum_n m(D \cap O_n) \leq \alpha \sum_n m(O_n) < \alpha^2 + \alpha\varepsilon.$$

As ε is arbitrary, this shows $\alpha \leq \alpha^2$, and so $\alpha = 0$ or 1. If $\alpha = 0$, $m(D) = 0$; and if $\alpha = 1$, $m(D^c) = 0$.

Another proof depends on the well-known principle that a set that covers at most a fixed fraction of every interval covers almost none of every interval.

I am informed that A. B. Novikoff has found that Burstin's original proof is incorrect.

References

1. C. Burstin, *Über eine spezielle Klasse reeller periodischer Funktionen*, *Monatsh. Math. Phys.*, 26 (1915) 229–262.
2. R. Cignoli and J. Hounie, *Functions with arbitrarily small periods*, this MONTHLY, 85 (1978) 582–584.