

EUCLIDEAN QUADRATIC FORMS ARE ADC FORMS: A SHORT PROOF

FRANCE DACAR

ABSTRACT. This note presents a short, transparent proof of the theorem that every Euclidean quadratic form over a normed integral domain is an ADC form. The theorem, as formulated in the note, allows additional linear terms, imposes no restrictions on the characteristic of the integral domain, and makes no unnecessary assumptions about the norm.

1. INTRODUCTION

In this note we state and prove a sharpened version of Theorem 8 in [1], which says that every Euclidean quadratic form over a certain kind of a normed integral domain is an ADC form; this theorem is a generalization of the Davenport-Cassels Lemma. The sharpened theorem presented in this note allows linear terms (so that there is a quadratic polynomial instead of a quadratic form, albeit with a Euclidean homogenous quadratic part), imposes no restrictions on the characteristic of the integral domain, and makes no unnecessary assumptions about the norm. The theorem and its proof are modeled on the lemma (the only one) in [2], which has an impressive provenance: its central idea was generalized and clarified, in stages, from Aubry through Cassels, Davenport, Weil, and Deligne to Serre; the author of this note just ironed out a couple of wrinkles, so that the lemma could be further generalized in the spirit of Theorem 8 in [1].

2. DEFINITIONS

Let R be a commutative ring with unity $1 \neq 0$.

A *discrete multiplicative norm on R* (shorter, a *norm on R*) is a mapping $|\cdot| : R \rightarrow \mathbb{N}$ that satisfies the following two conditions:

(N0) for every $x \in R$, $|x| = 0$ if and only if $x = 0$;

(N1) for all $x, y \in R$, $|xy| = |x| |y|$.

2010 *Mathematics Subject Classification*. Primary 11E20, 13F07.

Key words and phrases. Discrete multiplicative norm, integral domain, quadratic form, Euclidean form, ADC form.

Since $|1||1| = |1 \cdot 1| = |1| \neq 0$, we have $|1| = 1$, thus $|-|$ is a homomorphism of multiplicative monoids, and as such it maps every unit of R to the only invertible element 1 of the multiplicative monoid \mathbb{N} .

Let $|-|$ be a discrete multiplicative norm on R .

If x and y are any non-zero elements of R , then $|xy| = |x||y| \neq 0$, thus $xy \neq 0$; the ring R is an integral domain. Let K be the field of fractions of R . The given norm $|-|$ on R extends in a unique way to a mapping $|-| : K \rightarrow \mathbb{Q}^{\geq 0}$ satisfying the condition (N1) (which then also satisfies the condition (N0)); if $x = a/b$ with $a, b \in R$ and $b \neq 0$, then $|x| = |a|/|b|$.

A form over a ring R is a homogenous polynomial in $R[X_1, \dots, X_d]$ ($d > 0$), where X_i are formal variables; we shall write $X = (X_1, \dots, X_d)$ etc.

Let $q \in R[X_1, \dots, X_d]$ be a quadratic form. The polynomial $\langle X, Y \rangle_q := q(X + Y) - q(X) - q(Y) \in R[X_1, \dots, X_d, Y_1, \dots, Y_d]$ is a bilinear form said to be *associated with* q . If T is a formal variable different from the formal variables X_i and Y_i , then

$$q(X + TY) = q(X) + \langle X, Y \rangle_q T + q(Y) T^2 ;$$

it is the fact that the coefficient at T^2 is $q(Y)$ which we will find useful in the proof of the theorem below.

Let R be an integral domain with the field of fractions K .

A form g over R , in d variables, is said to be an *ADC form*¹ if for every $x \in K^d$ at which $g(x) \in R$ there exists $y \in R^d$ such that $g(y) = g(x)$.

Let $|-|$ be a discrete multiplicative norm on R , uniquely extended to a multiplicative norm on K (still written $|-|$). A form g over R , of any degree, is said to be *Euclidean with respect to* $|-|$ if for every $x \in K^d \setminus R^d$ there exists $y \in R^d$ such that $0 < |g(x - y)| < 1$.

3. THE THEOREM

Theorem 1. *Let R be an integral domain, with the field of fractions K , and let $|-|$ be a discrete multiplicative norm on R . Let $f = f_2 + f_1 + f_0 \in R[X_1, \dots, X_d]$, where f_i is homogenous of degree i , and f_2 is Euclidean with respect to $|-|$. If f has a zero in K^d , then it has a zero in R^d .*

Proof. Let $x \in K^d$ be a zero of f ; if $x \in R^d$, we are done, so we can assume that $x \notin R^d$.

We have $x = a/b$ for some $a \in R^d$ and $b \in R \setminus \{0\}$, and there exists $y \in R^d$ such that $0 < |f_2(x - y)| < 1$. We have $x - y = v/b$ with $v = a - by \in R^d$. For any $t \in K$ set $F(t) := f(y + tv) = At^2 + Bt + C$, where the coefficients

¹ADC stands for Aubry-Davenport-Cassels.

$A = f_2(v) = f_2(x - y)b^2 \neq 0$, $C = f(y)$, and $B = f(y + v) - A - C$ are in R ; $\tau := 1/b$ is a zero of F because $x = y + v/b$. Let τ' be the other zero of F . Since $\tau\tau' = C/A$, we have $\tau' = C/\tau A = C/(A/b)$, where $A/b = -B - Cb$ is in R ; since also $A/b = f_2(x - y)b$, we have $|A/b| = |f_2(x - y)||b| < |b|$. The point $x' := y + \tau'v$ is a zero of f , and it can be represented as $x' = a'/b'$, where $b' = A/b \in R \setminus \{0\}$, $a' = b'y + Cv \in R^d$, and $|b'| < |b|$.

If the zero x' of f is not yet in R^d , we repeat the procedure and construct another zero $x'' = a''/b''$ of f , where $a'' \in R^d$, $b'' \in R \setminus \{0\}$, and $|b''| < |b'|$. And so on. The sequence x, x', x'', \dots of zeros of f eventually ends with a zero $x^* \in R^d$ of f . \square

Looking at the constructed sequence $x = a/b, x' = a'/b', x'' = a''/b'', \dots$ of zeros of f , where $b' = f_2(x - y) \cdot b, b'' = f_2(x' - y') \cdot b', \dots$, we can clearly see how the Euclidean form f_2 forces termination of the sequence through the finiteness of the descent of the norms $|b| > |b'| > |b''| > \dots$.

Corollary 2. *Let R be an integral domain with a discrete multiplicative norm $|-|$. If a quadratic form q over R is Euclidean with respect to $|-|$ then it is an ADC form.*

Proof. Given an arbitrary $r \in R$, apply Theorem 1 to $q - r$. \square

4. A REMARK ON A CERTAIN PROPERTY OF THE NORM

Let R be an integral domain with a discrete multiplicative norm $|-|$. Suppose there exists a Euclidean form g over R , in any number $d > 0$ of variables and of any degree $m > 0$. Then the norm has the following property:

(N2) if $x \in R$ has $|x| = 1$, then x is a unit of R .

Indeed, let a be a non-zero non-unit of R ; we shall show that $|a| > 1$. The point $x := (a^{-1}, 0, \dots, 0) = a^{-1}e_1$ lies in $K^d \setminus R^d$, thus there exists $y \in R^d$ such that $0 < |g(x - y)| < 1$, that is, $0 < |g(e_1 - ay)| < |a|^m$; since $|g(e_1 - ay)|$ is an integer, it follows that $|a|^m > 1$, whence $|a| > 1$.

Though in Theorem 1 we do not explicitly assume the property (N2), the presence of the Euclidean quadratic form f_2 implies it. Theorem 8 in [1] thus unnecessarily assumes the property (N2).²

²The proof of this theorem refers to the property (N1), which is a typo: property (N2), as defined in [1], was property (N1) in an earlier draft version of the paper.

REFERENCES

- [1] Pete L. Clark, *Euclidean quadratic forms and ADC forms: I*, Acta Arithmetica 154 (2012), 137–159.
- [2] Jean-Pierre Serre communicated to Bjorn Poonen posted to MathOverflow.net on December 31, 2009: mathoverflow.net/questions/3269.

DEPARTMENT OF KNOWLEDGE TECHNOLOGIES, JOŽEF STEFAN INSTITUTE, JAMOVA
CESTA 39, 1000 LJUBLJANA, SLOVENIA
E-mail address: `france.dacar@ijs.si`