ABC AND THE HASSE PRINCIPLE FOR QUADRATIC TWISTS OF HYPERELLIPTIC CURVES

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Abstract. Conditionally on the ABC conjecture, we show that a hyperelliptic curve $C_{/\mathbb{Q}}$ of genus at least three has infinitely many quadratic twists that violate the Hasse Principle iff it has no $\mathbb{Q}$-rational hyperelliptic branch points.

1. Introduction

Let $C_{/\mathbb{Q}}$ be an algebraic curve. (All our curves will be *nice*: smooth, projective and geometrically integral.) An involution ι on C is an order 2 automorphism of $C_{/\mathbb{Q}}$. For any quadratic field $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$, there is a curve $\mathcal{T}_d(C,\iota)_{/\mathbb{Q}}$, the quadratic twist of C by ι and $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. After extension to $\mathbb{Q}(\sqrt{d})$ the curve $\mathcal{T}_d(C,\iota)$ is canonically isomorphic to $C_{/\mathbb{Q}(\sqrt{d})}$, but the $\operatorname{Aut}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) = \langle \sigma_d \rangle$ action on $C(\mathbb{Q}(\sqrt{d}))$ is "twisted by ι": $\sigma : P \in C(\mathbb{Q}(\sqrt{d})) \mapsto \iota(\sigma_d(P))$, and thus

$$\mathcal{T}_d(C,\iota)(\mathbb{Q}) = \{P \in C(\mathbb{Q}(\sqrt{d})) \mid \iota(P) = \sigma_d(P)\}.$$

If $d \in \mathbb{Q}^{\times 2}$ we put $\mathcal{T}_d(C,\iota) = C$, the "trivial quadratic twist."

Let $q : C \to C/\iota$ be the quotient map. Every $\mathbb{Q}$-rational point on $\mathcal{T}_d(C,\iota)$ maps via q to a $\mathbb{Q}$-rational point on C/ι. Let $\overline{P} \in C/\iota(\mathbb{Q})$. If $\overline{P}$ a branch point of ι, then the unique point $P \in C(\mathbb{Q})$ such that $q(P) = \overline{P}$ is also rational on every quadratic twist. If $\overline{P}$ is not a branch point of ι, there is a unique $d \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ such that the fiber of $q : \mathcal{T}_d(C,\iota) \to C/\iota$ consists of two $\mathbb{Q}$-rational points.

Work of Clark and Clark-Stankewicz [Cl08], [ClXX], [CS18] gives criteria on C and ι for there to be infinitely many $d \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ such that $\mathcal{T}_d(C,\iota)_{/\mathbb{Q}}$ violates the Hasse Principle: letting $\mathbf{A}_\mathbb{Q}$ be the adele ring over $\mathbb{Q}$, this means $\mathcal{T}_d(C,\iota)(\mathbf{A}_\mathbb{Q}) \neq \varnothing$ but $\mathcal{T}_d(C,\iota)(\mathbb{Q}) = \varnothing$. Here is one version.

Theorem 1. [CS18, Thm. 2] *Let $C_{/\mathbb{Q}}$ be a nice curve, and let ι be an involution on C. Suppose:*
(T1) The involution ι has no $\mathbb{Q}$-rational branch points.
(T2) The involution ι has at least one geometric branch point: $\{P \in C(\overline{\mathbb{Q}}) \mid \iota(P) = P\} \neq \varnothing$.
(T3) For some $d \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ we have $\mathcal{T}_d(C,\iota)(\mathbf{A}_\mathbb{Q}) \neq \varnothing$.
(T4) The set $(C/\iota)(\mathbb{Q})$ is finite.
Then, as $X \to \infty$, the number of squarefree d with $|d| \leq X$ such that $\mathcal{T}_d(C,\iota)_{/\mathbb{Q}}$ violates the Hasse Principle is $\gg_C \frac{X}{\log X}$.

An involution ι on a curve $C_{/\mathbb{Q}}$ is hyperelliptic if $C/\iota \cong \mathbb{P}^1$. A hyperelliptic curve is a pair (C,ι) with ι a hyperelliptic involution on C. (A curve of genus at least two admits at most one hyperelliptic involution.) A hyperelliptic curve (C,ι) of genus g has an affine model $y^2 = f(x)$ with $f(x) \in \mathbb{Q}[x]$ squarefree of degree $2g+2$ and $\iota : (x,y) \mapsto (x,-y)$. The twist $\mathcal{T}_d(C,\iota)$ has affine model $dy^2 = f(x)$. The branch points of ι are the roots of f in $\overline{\mathbb{Q}}$.[1]

If ι is a hyperelliptic involution then $(C/\iota)(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Q})$ is infinite, so (T4) is not satisfied. In this note we give a *conditional* complement to Theorem 1 that applies to hyperelliptic curves.

[1] We have chosen a model in which the point at ∞ is not a branch point; this is always possible. There is a model in which the point at ∞ is a branch point iff there is a $\mathbb{Q}$-rational branch point.

Theorem 2. *Assume the ABC conjecture. For a hyperelliptic curve (C, ι) of genus $g \geq 3$, the following are equivalent:*
(i) The hyperelliptic involution ι has no $\mathbb{Q}$-rational branch points.
(ii) As $X \to \infty$, the number of squarefree integers d with $|d| \leq X$ such that $\mathcal{T}_d(C, \iota)_{/\mathbb{Q}}$ violates the Hasse Principle is $\gg_C \frac{X}{\log X}$.
(iii) Some quadratic twist $\mathcal{T}_d(C, \iota)_{/\mathbb{Q}}$ violates the Hasse Principle.

Certainly (ii) $\implies$ (iii). As for (iii) $\implies$ (i): if ι has a $\mathbb{Q}$-rational branch point then this point stays rational on every quadratic twist.

The crux is to show that (i) $\implies$ (ii), which we will do in §2. In §3 we give upper and, in a special case, lower bounds on the number of quadratic twists having adelic points locally, leading to the conclusion that when hyperelliptic curves of genus $g \geq 3$ are ordered by height, for 100% of such curves the number of twists up to X that violate the Hasse Principle is $o(X)$, but that conditionally on ABC, there is a class of hyperelliptic curves for which the number of twists up to X that violate the Hasse Principle is $\gg X$. Some final remarks are given in §4.

2. Proof of Theorem 2

2.1. Local.

Theorem 3. *Let $(C, \iota)_{/\mathbb{Q}}$ be a hyperelliptic curve of genus $g \geq 1$. If $C(\mathbf{A}_\mathbb{Q}) \neq \varnothing$, then the set of primes $p \equiv 1 \pmod 8$ for which $\mathcal{T}_p(C, \iota)(\mathbf{A}_\mathbb{Q}) \neq \varnothing$ has positive density.*

Proof. For any place $\ell \leq \infty$ of $\mathbb{Q}$, if $p \in \mathbb{Q}_\ell^{\times 2}$ then $\mathcal{T}_p(C, \iota)_{/\mathbb{Q}_\ell} \cong C_{/\mathbb{Q}_\ell}$ and thus $\mathcal{T}_p(C, \iota)(\mathbb{Q}_\ell) \neq \varnothing$. In particular this holds for $\ell = \infty$. Henceforth ℓ denotes a prime number.

Let $M_1 \in \mathbb{Z}^+$ be such that C extends to a smooth relative curve over $\mathbb{Z}_\ell$ for all $\ell > M_1$. Such an M_1 exists for any nice curve $C_{/\mathbb{Q}}$ by openness of the smooth locus. Since C is hyperelliptic, we can take M_1 to be the largest prime dividing its minimal discriminant.

Suppose $\ell > M := \max(M_1, 4g^2 - 1)$, $\ell \neq p$ and $p \notin \mathbb{Q}_\ell^{\times 2}$. Then the minimal regular model $C_{/\mathbb{Z}_\ell}$ is smooth. We have $\mathcal{T}_p(C, \iota)_{/\mathbb{Q}_\ell(\sqrt{p})} \cong C_{/\mathbb{Q}_\ell(\sqrt{p})}$. Since $\mathbb{Q}_\ell(\sqrt{p})/\mathbb{Q}_\ell$ is unramified and formation of the minimal regular model commutes with étale base change [L, Prop. 10.1.17] it follows that the minimal regular model $\mathcal{T}_p(C, \iota)_{/\mathbb{Z}_\ell}$ is smooth. By the Riemann hypothesis for curves over a finite field, since $\ell \geq 4g^2$, we have $\mathcal{T}_p(C, \iota)(\mathbb{F}_\ell) \neq \varnothing$, so by Hensel's Lemma we have $\mathcal{T}_p(C, \iota)(\mathbb{Q}_\ell) \neq \varnothing$.

Suppose $\ell \leq M$ and $\ell \neq p$. If $\ell = 2$, then $p \in \mathbb{Q}_\ell^{\times 2}$ because $p \equiv 1 \pmod 8$. If ℓ is odd we require that p is a quadratic residue modulo ℓ, so again $p \in \mathbb{Q}_\ell^{\times 2}$. Either way, $\mathcal{T}_p(C, \iota)(\mathbb{Q}_\ell) = C(\mathbb{Q}_\ell) \neq \varnothing$.

Suppose $\ell = p$. Let $P \in C(\overline{\mathbb{Q}})$ be a hyperelliptic branch point. We assume that p splits completely in $\mathbb{Q}(P)$. Then $P \in C(\mathbb{Q}_p) \cap \mathcal{T}_p(C, \iota)(\mathbb{Q}_p)$.

All in all we have finitely many conditions on p, each of the form that p splits completely in a certain number field. Taking the compositum of these finitely many number fields and its Galois closure, say L, we see that if p splits completely in L then $\mathcal{T}_p(C, \iota)(\mathbf{A}_\mathbb{Q}) \neq \varnothing$. By (e.g.) the Chebotarev density theorem, this set of primes has positive density. $\square$

2.2. Global.

Theorem 4. *(Granville* [Gr07, Cor. 1.2]*) Assume the ABC conjecture. Let $(C, \iota)_{/\mathbb{Q}}$ be a hyperelliptic curve of genus $g \geq 3$. The number of squarefree integers d with $|d| \leq X$ such that $\mathcal{T}_d(C, \iota)(\mathbb{Q})$ has a point that is not a hyperelliptic branch point is $\ll_C X^{\frac{1}{g-1} + o(1)} \ll_C X^{2/3}$.*

2.3. **Local-global.** We now complete the proof of Theorem 2. Let (C, ι) be a hyperelliptic curve of genus $g \geq 3$ without $\mathbb{Q}$-rational hyperelliptic branch points, so C has an affine model of the form $y^2 = f(x)$ with $f(x) \in \mathbb{Z}[x]$ of degree $2g + 2$, with distinct roots in $\overline{\mathbb{Q}}$ and no roots in $\mathbb{Q}$. Put $d_0 := f(1)$. Then $(1, 1)$ is a $\mathbb{Q}$-point on $d_0 y^2 = f(x)$ and thus on $\mathcal{T}_{d_0}(C, \iota)$. The involution ι remains

$\mathbb{Q}$-rational on $\mathcal{T}_{d_0}(C,\iota)$ (cf. [CS18, §2.1]). We may thus apply Theorem 3 to the hyperelliptic curve $(\mathcal{T}_{d_0}(C,\iota),\iota)$, getting a set of primes $p \equiv 1 \pmod 8$ of density $\delta > 0$ such that

$$\mathcal{T}_{pd_0}(C,\iota)_{/\mathbb{Q}} = \mathcal{T}_p(\mathcal{T}_{d_0}(C,\iota),\iota)_{/\mathbb{Q}}$$

has points everywhere locally. By the Prime Number Theorem in Arithmetic Progressions, for at least $(\frac{\delta}{d_0} + o(1))\frac{X}{\log X}$ squarefree d with $|d| \leq X$, we have $\mathcal{T}_d(C,\iota)(\mathbf{A}_\mathbb{Q}) \neq \varnothing$. By Theorem 4, we have $\mathcal{T}_d(C,\iota)(\mathbb{Q}) \neq \varnothing$ for $\ll X^{2/3}$ squarefree d with $|d| \leq X$. So the number of squarefree d with $|d| \leq X$ such that $\mathcal{T}_d(C,\iota)_{/\mathbb{Q}}$ violates the Hasse Principle is $\gg_C \frac{X}{\log X}$.

3. Counting twists with adelic points

For a hyperelliptic curve $(C,\iota)_{/\mathbb{Q}}$, let

$$\mathfrak{U}_C = \{\text{squarefree } d \in \mathbb{Z} \mid \mathcal{T}_d(C,\iota)(\mathbf{A}_\mathbb{Q}) \neq \varnothing\}$$

be the set of twists of C having points everywhere locally. For $X \geq 1$, put

$$\mathfrak{U}_C(X) = \#(\mathfrak{U}_C \cap [-X,X]).$$

As we saw above, Theorem 3 gives $\mathfrak{U}_C(X) \gg \frac{X}{\log X}$.

Recall that a polynomial $f \in \mathbb{Z}[x]$ is **intersective** if it has roots modulo N for all $N \in \mathbb{Z}^+$, or equivalently, in $\mathbb{Z}_p$ for all primes p. We say a polynomial $f \in \mathbb{Z}[x]$ is **weakly intersective** if the set of prime numbers p such that f has a root modulo p has density 1.

Remark 5. Suppose $f = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$ has degree $n \geq 2$, is weakly intersective and has distinct roots in $\overline{\mathbb{Q}}$, with discriminant Δ. Let G be the Galois group of f.

For every prime number $p \nmid a_n\Delta$, the partition of n given by the cycle type of a Frobenius element σ_p at p coincides with the partition of n given by the degrees of the irreducible factors of the image of f in $\mathbb{Z}/p\mathbb{Z}[x]$. Since f is weakly intersective, it follows from the Frobenius Density Theorem (see e.g. [SL96, §3]) that every $\sigma \in G$ has a fixed point and thus f has a root mod p for all $p \nmid a_n\Delta$, and thus by Hensel's Lemma it has a root in $\mathbb{Z}_p$ for all but finitely many p.

Since every $\sigma \in G$ has a fixed point, it follows from the Cauchy-Frobenius(-"not Burnside") Lemma that $f \in \mathbb{Q}[x]$ is not irreducible.

Theorem 6. *Let $(C,\iota)_{/\mathbb{Q}}$ be a hyperelliptic curve. Let $y^2 = f(x)$ be an affine equation for C with $f \in \mathbb{Z}[x]$ squarefree of even degree.*
a) If f is weakly intersective then $\mathfrak{U}_C(X) \gg X$.
b) If f is not weakly intersective, let β be the density of the set of prime numbers p such that f has no root *modulo p, so $\beta \in (0,1)$.*[2] *Then $\mathfrak{U}_C(X) \ll \frac{X}{\log^\beta X}$.*

Proof. Let Δ be the discriminant of f.
Step 1: Suppose $f \in \mathbb{Z}[x]$ is weakly intersective. By Remark 5 f has a root in $\mathbb{Z}_p$ for all but finitely many p, and thus the set $\mathcal{P}$ of prime numbers p such that $C(\mathbb{Q}_p) = \varnothing$ is finite. For each $p \in \mathcal{P}$, we have $C_d(\mathbb{Q}_p) \neq \varnothing$ so long as d lies in the same $\mathbb{Q}_p$-adic square class as $f(1)$. The set of integers lying in a given $\mathbb{Q}_p$-adic square class is a nonempty union of congruence classes modulo p^2 (if $p > 2$) or modulo 16 (if $p = 2$). Applying the Chinese Remainder Theorem, there are $a, N \in \mathbb{Z}^+$ such that if $d \equiv a \pmod N$ then $\mathcal{T}_d(C,\iota)(\mathbb{Q}_p) \neq \varnothing$ for all primes p. Finally, if f has a real root then $\mathcal{T}_d(C,\iota)(\mathbb{R}) \neq \varnothing$ for all d; otherwise $\mathcal{T}_d(C,\iota)(\mathbb{R}) \neq \varnothing$ iff $df(1) > 0$. Thus $\mathfrak{U}_C(X) \gg X$. (The implied constant can be made explicit in terms of Δ.)
Step 2: Suppose f is not weakly intersective. Let E' be the set of all squarefree integers d such that for all primes $p \mid d$, either $p \mid 2\Delta$ or f has a root modulo p. Let E be the set of all squarefree integers that do not lie in E'. Thus for all $d \in E$, there is an odd prime $p \mid d$ such that the image

[2]The polynomial f has a root modulo every prime p that splits completely in the splitting field of f, so $\beta > 0$.

of f in $\mathbb{Z}/p\mathbb{Z}$ is squarefree and has no root modulo p. By a result of Sadek [Sa14, Cor. 4.2], this implies that $\mathcal{T}_d(C)(\mathbb{Q}_p) = \varnothing$. It follows that

$$\mathfrak{U}_C \subset E'.$$

Let $E'(X)$ be the number of $d \in E'$ with $|d| \leq X$. Then [Se76, Thm. 2.4] implies that if $0 < \beta < 1$ then there is $c > 0$ such that $E'(X) \sim \frac{cX}{\log^{\beta} X}$. □

We call a hyperelliptic curve $(C, \iota)_{/\mathbb{Q}}$ **weakly intersective** if it has a weakly intersective squarefree, integral, even degree defining polynomial.[3] Since no weakly intersective polynomial is irreducible, when genus g hyperelliptic curves are ordered by height, 0% of them are weakly intersective.

Theorems 2 and 6 immediately imply the following:

Corollary 7. *Let $(C, \iota)_{/\mathbb{Q}}$ be a hyperelliptic curve of genus g without $\mathbb{Q}$-rational branch points.*
a) If C is weakly intersective and $g \geq 3$, then conditionally on ABC, as $X \to \infty$ the number of quadratic twists of (C, ι) that violate the Hasse Principle is $\gg X$.
b) If C is not weakly intersective, then as $X \to \infty$, the number of quadratic twists of (C, ι) that violate the Hasse Principle is $o(X)$.

Example 8.
a) For any coprime, nonsquare integers $a, b > 1$, the polynomial $(x^2 - a)(x^2 - b)(x^2 - ab)$ is weakly intersective and without rational roots. The polynomial $(x^2 - 2)(x^2 - 3)(x^2 - 6)$ is not intersective – it has no root in $\mathbb{Q}_2$. The polynomial $(x^2 - 2)(x^2 - 17)(x^2 - 34)$ is intersective.
b) For any $g \geq 3$, let $h(x) \in \mathbb{Z}[x]$ be monic of degree $2g - 4$, without rational roots and such that $g(\pm\sqrt{2}), g(\pm\sqrt{3}), g(\pm\sqrt{6}) \neq 0$. Then $C_{/\mathbb{Q}} : y^2 = 2(x^2 - 2)(x^2 - 3)(x^2 - 6)h(x)$ is a weakly intersective hyperelliptic curve of genus $g \geq 3$ without $\mathbb{Q}$-rational branch points. So conditionally on ABC, a positive proportion of the quadratic twists of C violate the Hasse principle.
c) For every even $n \geq 2$, there is a cyclic Galois extension $F/\mathbb{Q}$ of degree n, and there is a monic polynomial $f \in \mathbb{Z}[x]$ such that $\mathbb{Q}[x]/(f) \cong F$. The hyperelliptic curve $C_{/\mathbb{Q}} : y^2 = 2f(x)$ has genus $\frac{n}{2} - 1$ and $\mathfrak{U}_C(X) \ll \frac{X}{\log^{1-\frac{1}{n}} X}$.

4. Some Remarks

In [Gr07, Conj. 1.3], Granville conjectures that for all $g \geq 2$, if $f \in \mathbb{Z}[x]$ has degree $2g+1$ or $2g+2$ and distinct roots in $\overline{\mathbb{Q}}$, then there is a constant $\kappa_f' > 0$ such that the number of squarefree d with $|d| \leq X$ such that $dy^2 = f(x)$ has a $\mathbb{Q}$-point that is not a hyperelliptic branch point is $\sim \kappa_f' X^{\frac{1}{g+1}}$. The above arguments apply verbatim to show that conditionally on Granville's conjecture, for all $g \geq 2$, a hyperelliptic curve $C_{/\mathbb{Q}}$ has $\gg_C \frac{X}{\log X}$ twists that violate the Hasse principle iff C has no $\mathbb{Q}$-rational branch points. On the other hand, Vatsal has exhibited a genus one hyperelliptic curve $(C, \iota)_{/\mathbb{Q}}$ for which a positive proportion of the quadratic twists have infinitely many rational points [Va98]. Still, it may be true that every hyperelliptic curve of genus 1 without $\mathbb{Q}$-rational branch points has infinitely many twists that violate the Hasse Principle.

The present work should be compared to two other works that apply Theorem 1 (or its predecessor [Cl08, Thm. 2]) and Faltings' Theorem to get (unconditional) Hasse Principle violations. Namely, Ozman [Oz12] works with the Atkin-Lehner involution w_N on a modular curve $X_0(N)$ for a prime $N \equiv 1 \pmod 4$ and Clark-Stankewicz [CS18] works with the Atkin-Lehner involution w_D on a Shimura curve X^D for a squarefree $D > 1$. Taking $N > 131$ (resp. $D > 546$) ensures that

[3]By Theorem 6, if one defining polynomial is weakly intersective, then all are.

$X_0(N)/\langle w_N\rangle$ (resp. $X^D/\langle w_D\rangle$) has genus at least 2 and thus finitely many $\mathbb{Q}$-points. In each work there is an analysis of $\mathfrak{U}_C(X)$, the number of twists up to X with adelic points. For modular curves $X_0(N)$, Ozman shows that $\mathfrak{U}_C(X) \sim \frac{CX}{\log^\gamma X}$ for a positive constant C and a $\gamma \in [0,1]$ determined in terms of the class group of $\mathbb{Q}(\sqrt{-N})$ [Oz12, Thm. 5.4] (and cf. [CS18, p. 2841, footnote 5]). In the case of Shimura curves X^D, Clark-Stankewicz show [CS18, Thm. 8] that

$$\frac{X}{\log^{\alpha_D} X} \ll \mathfrak{U}_{X^D}(X) \ll \frac{X}{\log^{\beta_D} X}$$

for constants $0 < \beta_D < \alpha_D < 1$ determined in terms of D, such that $\lim_{D\to\infty} \alpha_D - \beta_D = 0$.

There is some overlap: for a finite nonempty set of N (resp. of D), the pair $(X_0(N), w_N)$ (resp. (X^D, w_D)) is hyperelliptic. E.g. the pair $(X_0(41), w_{41})$ is hyperelliptic of genus 3 and [Oz12, *loc. cit.*] gives $\mathfrak{U}_{X_0(41)}(X) \sim \frac{CX}{\log^{\frac{11}{16}} X}$. Similarly, the pair (X^{35}, w_{35}) is hyperelliptic of genus 3 and [CS18, *loc. cit.*] gives $\frac{X}{\log^{\frac{15}{16}} X} \ll \mathfrak{U}_{X^{35}}(X) \ll \frac{X}{\log^{\frac{11}{16}} X}$.

It can be shown that for all hyperelliptic curves $(C,\iota)_{/\mathbb{Q}}$, there is $\alpha = \alpha(C) < 1$ such that $\mathfrak{U}_C(X) \gg \frac{X}{\log^\alpha X}$. In fact the same conclusion should hold for any $(C,\iota)_{/\mathbb{Q}}$ satisfying (T1), (T2) and (T3) in Theorem 1, which amounts to a quantitative strengthening of the local part of this result. We hope to return to this in a future work.

Recent work of Bhargava-Gross-Wang [BGW17] shows that for each fixed $g \geq 1$, when genus g hyperelliptic curves $(C,\iota)_{/\mathbb{Q}}$ are ordered by height, a positive proportion violate the Hasse Principle. This work is unconditional; moreover, the positive proportion result should be contrasted with Corollary 7b). On the other hand, since all quadratic twists of a hyperelliptic curve induce the same point of the moduli space $\mathcal{H}_g$ of hyperelliptic curves of genus g, our result gives (conditionally on ABC) Hasse Principle violations on the largest possible subset of $\mathcal{H}_g$.

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