# FUNCTIONAL DEGREES AND ARITHMETIC APPLICATIONS, I: THE SET OF FUNCTONAL DEGREES

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Abstract. We give a further development of the Aichinger-Moosbauer calculus of functional degrees of maps between commutative groups. For any fixed given commutative groups $A$ and $B$, we compute the largest possible finite functional degree that a map $f : A \to B$ can have. We also determine the set of all possible degrees of such maps. This also yields a solution to Aichinger and Moosbauer's problem of finding the nilpotency index of the augmentation ideal of group rings of the form $Z_{p^\beta}[Z_{p^{\alpha_1}} \times Z_{p^{\alpha_2}} \times \cdots \times Z_{p^{\alpha_n}}]$ with $p, \beta, n, \alpha_1, \ldots, \alpha_n \in \mathbb{Z}^+$, $p$ prime.

## Contents



## 1. Introduction

1.1. **Polynomial Maps.** This is the first of two papers in which we attempt a synthesis and development of two prior works: a 2006 paper of R. Wilson [Wi06] and a 2021 paper of E. Aichinger and J. Moosbauer [AM21]. These works revisit and generalize number-theoretic results in connection with polynomials, using methods that seem to belong to pure algebra.

The major algebraic theme is the notion of a "polynomial map" $f : A \to B$ between commutative groups $A$ and $B$. It is an important concept that arises in many contexts, for instance in the notion of a quadratic map between commutative groups: see e.g. [Za74] and [S, p. 34-35]. Over commutative groups $A$ and $B$, however, there is not a set standard definition for "polynomial maps and their degree," but various different definitions. Indeed, many concurrent definitions are compared in a work of Laczkovich [La04].

One frequently used, simple and successful way of viewing polynomial maps comes from the calculus of finite differences. In 1909 Fréchet showed [Fr09] that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is a polynomial if and only if for some $d \in \mathbb{N}$ we have $\Delta^d f = 0$, where $\Delta$ is the forward **difference operator** with the defining equation

$$(\Delta f)(x) = f(x+1) - f(x)$$

and $\Delta^d$ is its $d$-fold iterate. Moreover, for a nonzero polynomial $f$ over $\mathbb{R}$, the least $d$ such that $\Delta^d f = 0$ is equal to $\deg(f) + 1$. It is this property that can be used to generalize the degree of polynomials to more general functions between arbitrary commutative groups. One simply has to extend the calculus of finite differences to such functions. Various authors have taken on aspects of this task. A particularly systematic and penetrating take was recently given in work of Aichinger and Moosbauer [AM21]. Using finite differences over groups, the authors associate, to any map $f : A \to B$, a **functional degree** $\operatorname{fdeg}(f)$ that is either a non-negative integer or $\infty$. They simply set

$$\operatorname{fdeg}(f) := \min\{d \in \mathbb{Z}^+ \mid \Delta^d f = 0\} - 1\,,$$

a definition that we follow, except that, for the zero function, we set

$$\operatorname{fdeg}(0) := -\infty\,.$$

1.2. **Main Results.** Basic properties of the functional degree, such as its behavior under compositions and under formation of pointwise sums and products, are thoroughly analyzed in the first part of [AM21]. The next important problem is the determination of $\mathcal{D}(A, B)$, the set of all functional degrees of maps from $A$ to $B$. In particular we wish to determine $\delta(A, B)$, the supremum of all functional degrees of maps from $A$ to $B$.

In the middle part of their work, in [AM21, §7-9], Aichinger and Moosbauer make progress on this question. They showed that the determination of $\delta(A, B)$ for all finite commutative $p$-groups $A$ and $B$ will give the determination of $\mathcal{D}(A, B)$ for all finite commutative groups $A$ and $B$. Moreover they recast the determination of $\delta(A, B)$ as an open problem in commutative algebra [AM21, Problem 8.3]. In the special case of finite *cyclic* $p$-groups, a solution can then be derived from work of Wilson [Wi06], as Aichinger and Moosbauer point out. As we will see in §4.3, this case is also a consequence of earlier work of Weisman [We77]. In order to complete the determination of $\mathcal{D}(A, B)$ when $A$ and $B$ are both finite, it remains to determine $\delta(A, B)$ when $A$ and $B$ are both finite commutative $p$-groups and $A$ is not cyclic.

In this paper, we compute $\delta(A, B)$ for *all* commutative groups $A$ and $B$: Theorem 4.9. This yields, in particular, an answer to [AM21, Problem 8.3]. We also determine whether $\infty$ lies in $\mathcal{D}(A, B)$: also in Theorem 4.9. As we will see, there are cases in which $\delta(A, B) = \infty$ but every single map $f : A \to B$ has finite functional degree.

For many pairs $(A, B)$ of commutative groups, it is much easier to see that there is a function of infinite functional degree than to determine the entire set $\mathcal{D}(A, B)$. However we will compute the full set $\mathcal{D}(A, B)$ for a class of pairs $(A, B)$ that includes all pairs of finitely generated commutative groups: Proposition 4.10 and Theorem 4.11.

1.3. **The Contents of the Paper.** In §2 and 3 we give a self-contained exposition of the aspects of the functional degree that we will need here. We do so mainly in order to establish several new intermediate results, some of which will be used later on.[1]

Aichinger and Moosbauer place their functional calculus in the setting of maps between *aribitrary* commutative groups, and throughout this work we try to follow this lead by working without finiteness restrictions when we can do so. For this we need a few tenets of the theory of commutative groups, e.g. the structure of groups of finite exponent. This is recalled in §2 2. In that section we also introduce the finite difference calculus and give the definition of functional degree in those terms.

Exploiting the fact that the set $B^A$ of all maps from $A$ to $B$ is naturally a module under the group ring $\mathbb{Z}[A]$, Aichinger and Moosbauer gave an elegant module-theoretic interpretation of the functional degree. In §3 we further develop the calculus of functional degrees from this module-theoretic point of view. One unifying feature of our approach is an attention to the effect of the functional degree of maps $f : A \to B$ under composition with group homomorphisms $\varepsilon : A' \to A$ and $\mu : B \to B'$: see Lemma 3.9 and Corollary 3.10. We also examine, in §3.3, what happens with the functional degree when $B$ is a direct product and when $A$ is a direct sum. In particular we establish the Diagonalization Theorem (Theorem 3.13), which generalizes Lemma 9.3 and Theorem 9.4 of [AM21].

In §4 we recall the ideal-theoretic interpretation of $\delta(A, B)$ due to Aichinger and Moosbauer. We then establish a key result, Theorem 4.8. It tells us how the largest possible degree depends on the direct summands of $A$, if the $p$-group $A$ can be written as a direct sum. Using this result and the earlier results, we prove the main results of the paper.

1.4. **In the Sequel.** Aichinger and Moosbauer also gave some striking Diophantine applications of their functional calculus. Namely, they obtained group-theoretic generalizations [AM21, Thm. 11.1, Thm. 12.2] of the theorems of Chevalley [Ch35] and Warning [Wa35] on systems of polynomials over a finite field $\mathbb{F}_{p^a}$: the latter result says that for a system of polynomials $P_1, \ldots, P_r \in \mathbb{F}_{p^a}[t_1, \ldots, t_n]$ with $\sum_{j=1}^r \deg(P_j) < n$, the number $\#Z$ of simultaneous solutions is divisible by the prime $p$. In the sequel [CSII] we will use results from the present paper and a generalization of the main result of [Wi06] to give group- and ring-theoretic generalizations of the results of Ax [Ax64], Katz [Ka71] and Moreno-Moreno [MM95] on higher $p$-adic congruences for $\#Z$.

## 2. Finite Differences and the Functional Degree

2.1. **Notation.** We denote by $\mathcal{P}$ the set of (positive) prime numbers. We denote by $\mathbb{N}$ the non-negative integers and put $\mathbb{Z}^+ := \mathbb{N} \setminus \{0\}$. For $n \in \mathbb{N}$, we set $Z_n := \mathbb{Z}/n\mathbb{Z}$. In

[1] We were very taken with the results of [AM21] and their elegant exposition of them. When we can push these results further or contribute a useful new way to look at them, we do so. There is more in [AM21] than is revisited here, and we will make use of some of their other results in the sequel [CSII].

particular, $Z_0 = \mathbb{Z}$. Moreover, we endow the set

$$\widetilde{\mathbb{N}} := \mathbb{N} \cup \{-\infty, \infty\}$$

with the most evident total ordering, in which $-\infty$ is the least element and $\infty$ is the greatest element.

2.2. **Preliminaries on Commutative Groups.** Let $(A, +)$ be a commutative group. For $n \in \mathbb{Z}^+$ we define

$$A[n] := \{x \in A \mid nx = 0\}$$

and

$$A[n^\infty] := \{x \in A \mid n^k x = 0 \text{ for some } k \in \mathbb{Z}^+\}.$$

For a prime number $p$, we say that $A$ is a **p-group** if $A = A[p^\infty]$.

The **torsion subgroup** of $A$ is

$$A[\text{tors}] := \{x \in A \mid nx = 0 \text{ for some } n \in \mathbb{Z}^+\} = \bigcup_{n \in \mathbb{Z}^+} A[n].$$

We say that $A$ is **torsion** if $A = A[\text{tors}]$. We say that $A$ has **finite exponent** if $A = A[n]$ for some $n \in \mathbb{Z}^+$, in which case the least such $n$ is called the **exponent** of $A$ and is denoted by $\exp(A)$. In the other case, we may write $\exp(A) = \infty$. We also set

$$e(A) := \begin{cases} \exp(A) & \text{if } \exp(A) < \infty, \\ 0 & \text{if } \exp(A) = \infty. \end{cases}$$

For a finitely generated commutative group $A$, we denote by $\operatorname{rank}(A)$ the least $n$ such that $A$ is a direct sum of $n$ cyclic groups.

A commutative group $A$ is **torsion-split** if $A[\text{tors}]$ is a direct summand of $A$. Every finitely generated commutative group is torsion-split. More generally, Baer showed [Ba36] that if $A[\text{tors}]$ is the direct sum of a group of finite exponent and a divisible group (i.e., one in which the map $x \mapsto nx$ is surjective for all $n \in \mathbb{Z}^+$), then $A$ is torsion-split.

**Theorem 2.1** (Prüfer-Baer)**.** *Let $G$ be a commutative group of finite exponent $N$. Then there is a family $(N_\gamma)_{\gamma \in \Gamma}$ of numbers $N_\gamma \in \mathbb{Z}^+$ with $N_\gamma \mid N$ for all $\gamma$ and $\max_{\gamma \in \Gamma} N_\gamma = N$, such that $G$ is isomorphic to the direct sum $\bigoplus_{\gamma \in \Gamma} Z_{N_\gamma}$.*

*Proof.* See e.g. [K, Thm. 6]. □

2.3. **The Functional Degree.** For commutative groups $A$ and $B$, let $B^A$ be the set of all functions $f : A \to B$. The set $B^A$ is a commutative group under pointwise addition. For $f \in B^A$ and $a \in A$, we define the shift operator $\tau_a \in \operatorname{End} B^A$ by

$$(\tau_a f)(x) := f(x + a),$$

and the difference operator $\Delta_a \in \operatorname{End} B^A$ by

$$(\Delta_a f) := \tau_a f - f.$$

In other words, for all $f : A \to B$ and all $x \in A$ we have

$$(\Delta_a f)(x) = f(x + a) - f(x).$$

The following is a well-known formula from the calculus of finite differences carried over to the present context. The proof is straightforward using induction on $n$.

**Lemma 2.2.** *Let $a \in A$, $n \in \mathbb{N}$ and let $\Delta_a^n$ be the $n$-fold product $\Delta_a \cdots \Delta_a \in \operatorname{End} B^A$. For all $f \in B^A$ and all $x \in A$,*

$$(\Delta_a^n f)(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (n-i)a) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(x + ja).$$

**Definition 2.3.** For $f \in B^A$ we define the **functional degree** $\operatorname{fdeg}(f) \in \widetilde{\mathbb{N}}$ as follows:

- We put $\operatorname{fdeg}(f) = -\infty$ if and only if $f = 0$.
- For $n \in \mathbb{N}$, we say that a nonzero function $f$ has $\operatorname{fdeg}(f) \leq n$ if and only if $\Delta_{a_1} \cdots \Delta_{a_{n+1}} f = 0$ for all $a_1, \ldots, a_{n+1} \in A$.
- If there is an $n \in \mathbb{N}$ such that $\operatorname{fdeg}(f) \leq n$ and $f \neq 0$, then we define the functional degree $\operatorname{fdeg}(f)$ as the least $n \in \mathbb{N}$ such that $\operatorname{fdeg}(f) \leq n$.
- If there is no $n \in \mathbb{N}$ such that $\operatorname{fdeg}(f) \leq n$, then we put $\operatorname{fdeg}(f) = \infty$.

Our definition of functional degree differs from the one given in [AM21, §2] precisely in that we set $\operatorname{fdeg}(0) = -\infty$, whereas Aichinger-Moosbauer take $\operatorname{fdeg}(0) = 0$. Our choice is motivated by the corresponding convention for the degree of polynomials, which ensures that the identity $\deg(f \cdot g) = \deg(f) + \deg(g)$ for polynomials $f$ and $g$ over an integral domain holds even when $f = 0$ or $g = 0$.

**Remark 2.4.** *Let $f \in B^A$.*

a) *(cf.* [AM21, Lemma 3.1(4)]*) We have* $\operatorname{fdeg}(f) \leq 0$ *if and only if $f(x+a) = f(x)$ for all $a, x \in A$, that is, if and only if $f$ is constant.*

b) [AM21, Lemma 3.1(5)] *If $f$ is a nonzero group homomorphism, then for all $a \in A$ the function $\Delta_a f$ is constant, as for all $x \in A$*

$$(\Delta_a f)(x) = f(x+a) - f(x) = f(a) = const.$$

*Hence,* $\operatorname{fdeg}(f) \leq 1$*, and since $f$ is nonconstant,* $\operatorname{fdeg}(f) = 1$*.*

c) *Suppose $f \in B^A$ has functional degree 1. Then for all $a_1, a_2 \in A$ we have*

$$0 = \Delta_{a_1} \Delta_{a_2} f(x) = f(a_1 + a_2 + x) - f(a_2 + x) - f(a_1 + x) + f(x).$$

*Taking $x = 0$ we get*

$$f(a_1 + a_2) - f(0) = (f(a_1) - f(0)) + (f(a_2) - f(0)).$$

*So, $x \mapsto f(x) - f(0)$ is a nonzero group homomorphism.*

**Lemma 2.5.** *Let $f \in B^A$. If* $\operatorname{fdeg}(f) = n \in \mathbb{Z}^+$*, then for every $a \in A$ we have* $\operatorname{fdeg}(\Delta_a f) \leq n - 1$ *and for some $a \in A$ we have* $\operatorname{fdeg}(\Delta_a f) = n - 1$*.*

*Proof.* Since $\operatorname{fdeg}(f) \leq n$, for all $a, a_1, \ldots, a_n \in A$ we have

$$\Delta_{a_1} \cdots \Delta_{a_n} (\Delta_a f) = \Delta_{a_1} \cdots \Delta_{a_n} \Delta_a f = 0,$$

so $\operatorname{fdeg}(\Delta_a f) \leq n - 1$. Since $\operatorname{fdeg}(f)$ is *not* less than or equal to $n - 1$, there are $a_1, \ldots, a_n \in A$ such that

$$\Delta_{a_1} \cdots \Delta_{a_{n-1}} (\Delta_{a_n} f) = \Delta_{a_1} \cdots \Delta_{a_n} f \neq 0,$$

so $\operatorname{fdeg}(\Delta_{a_n} f)$ is *not* less than or equal to $n - 2$. Thus $\operatorname{fdeg}(\Delta_{a_n} f) = n - 1$. $\square$

For commutative groups $A$ and $B$, let

$$\mathcal{D}(A,B) := \{\mathrm{fdeg}(f) \mid f \in B^A\} \subset \widetilde{\mathbb{N}}$$

be the set of all functional degrees of maps between $A$ and $B$. The main result of §3 is the complete determination of $\mathcal{D}(A,B)$ for certain groups $A$ and $B$, including all finitely generated commutative groups. The set $\mathcal{D}(A,B)$ is completely determined by two pieces of information: the invariant

$$\delta^\circ(A,B) := \sup\{\mathrm{fdeg}(f) \mid f \in B^A,\ \mathrm{fdeg}(f) < \infty\},$$

and whether there is a function $f \in B^A$ with $\mathrm{fdeg}(f) = \infty$. Indeed:

**Lemma 2.6.** *Let $A$ and $B$ be commutative groups.*

a) *If $B$ is trivial, then $B^A = \{0\}$, so $\mathcal{D}(A,B) = \{-\infty\}$.*
b) *If $A$ is trivial and $B$ is nontrivial, then $\mathcal{D}(A,B) = \{-\infty, 0\}$.*
c) *If $A$ and $B$ are both nontrivial, then*

$$\mathcal{D}(A,B) \setminus \{\infty\} = \{n \in -\infty \cup \mathbb{N} \mid n \leq \delta^\circ(A,B)\}.$$

*Proof.* a) This is trivial.

b) In this case, every function is constant (as its domain has only one element), and some are nonzero.

c) This follows from Lemma 2.5. □

The following consequence of Remark 2.4 and Lemma 2.6 shows a connection between functional degrees and the existence of nontrivial group homomorphisms from $A$ to $B$.

**Corollary 2.7.** *For commutative groups $A$ and $B$, the following are equivalent:*

(i) $\mathcal{D}(A,B) \subset \{-\infty, 0, \infty\}$.
(ii) $\mathrm{Hom}(A,B) = \{0\}$.
(iii) $\delta^\circ(A,B) = 0$.

Next we introduce a quantity that is closely related to $\delta^\circ(A,B)$ but easier to compute:

$$\delta(A,B) := \sup\{\mathrm{fdeg}(f) \mid f \in B^A\}.$$

Unlike $\delta^\circ(A,B)$, we can compute $\delta(A,B)$ for all commutative groups $A$ and $B$: this is because, for large classes of groups $A$ and $B$, we can simply write down an $f \in B^A$ with $\mathrm{fdeg}(f) = \infty$.

If $\delta(A,B) < \infty$, then $\delta^\circ(A,B) = \delta(A,B)$. So it follows from Lemma 2.6 that

$$\mathcal{D}(A,B) = \{n \in \widetilde{\mathbb{N}} \mid n \leq \delta(A,B)\} \ \text{ if } \ \delta(A,B) < \infty.$$

## 3. Module-theoretic Interpretation of the Functional Degree

3.1. **Preliminaries on Group Rings.** For a group $(A,+)$ and a ring $R$, let $R[A]$ be the corresponding **group ring** [Cl-CA, §5.6]. Its elements are formal linear combinations $\sum_{a\in A} r_a[a]$ with $r_a \in R$ and $r_a = 0$ for all but finitely many $a \in A$. The ring homomorphism

$$\epsilon : R[A] \to R,\ \sum_{a\in A} r_a[a] \mapsto \sum_{a\in A} r_a.$$

is called **augmentation map**, and its kernel is the **augmentation ideal** $I$.

**Remark 3.1** (Reminders on Group Rings)**.** *Let $R$ be a nonzero commutative ring.*

a) *The $R$-algebra homomorphism $R[\mathbb{Z}] \to R[t, t^{-1}]$ that sends $[1]$ to $t$ is an isomorphism. Under this map the augmentation ideal maps to the principal ideal $\langle t-1\rangle$.*
b) *Let $N \geq 2$. The $R$-algebra homomorphism $R[Z_N] \to R[t]/\langle t^N - 1\rangle$ that sends $[1]$ to $t + \langle t^N - 1\rangle$ is an isomorphism. Under this map the augmentation ideal maps to the principal ideal $\langle t - 1 + \langle t^N - 1\rangle\rangle = \langle t-1\rangle/\langle t^N - 1\rangle$ (which we identify with $\langle t-1\rangle \subset R[t]$).*
c) *For groups $G_1$ and $G_2$, the natural map*
$$R[G_1] \otimes_R R[G_2] \to R[G_1 \times G_2],\ [g_1] \otimes [g_2] \mapsto [(g_1, g_2)]$$
*is an $R$-algebra isomorphism.*
d) *Combining parts b) and c), we get that if $A \cong \bigoplus_{i=1}^r Z_{N_i}$, then*
$$R[A] \cong R[t_1, \ldots, t_r]/\langle t_1^{N_1} - 1, \ldots, t_r^{N_r} - 1\rangle.$$
*The augmentation ideal is mapped to $\langle t_1 - 1, \ldots, t_r - 1\rangle/\langle t_1^{N_1} - 1, \ldots, t_r^{N_r} - 1\rangle$ (which we identify with $\langle t_1 - 1, \ldots, t_r - 1\rangle$) under this isomorphism.*

We call $\mathbb{Z}[A]$ the **integral group ring** of $A$. The commutative group $B^A$ has the structure of a $\mathbb{Z}[A]$-module via
$$\big(\sum_{a\in A} n_a[a]\big) f := \sum_{a \in A} n_a \tau_a(f).$$
Moreover, if $B$ has finite exponent $n$, then $B$, hence also $B^A$, is a $Z_n$-module, and the $\mathbb{Z}[A]$-module structure on $B^A$ factors through $Z_n[A]$. We see that $B^A$ is always a $Z_{e(B)}[A]$-module, even if $\exp(B) = \infty$.

**Lemma 3.2.** *$B^A$ is a faithful $Z_{e(B)}[A]$-module.*

*Proof.* Let $0 \neq r = \sum_{a\in A} n_a[a] \in Z_{e(B)}[A]$ be arbitrary. Choose $a_0 \in A$ such that $n_{a_0} \neq 0$, and let $\hat{n}_{a_0} \in \mathbb{Z}$ be a least nonnegative representative of $n_{a_0} \in Z_{e(B)}$. As $0 < \hat{n}_{a_0} < \exp(B)$, we have $B[\hat{n}_{a_0}] \neq B$ and there exists an element $b \in B$ with $n_{a_0} b := \hat{n}_{a_0} b \neq 0$. From this property it follows easily that if $\delta_{0,b} \in B^A$ is the function that maps $0$ to $b$ and every other element of $A$ to $0$, then $r \cdot \delta_{0,b} \neq 0$, which proves the claimed faithfulness. Indeed, if we evaluate the function
$$r \cdot \delta_{0,b} : x \longmapsto \sum_{a\in A} n_a \delta_{0,b}(a + x)$$
at $x = -a_0$, we see that $[r \cdot \delta_{0,b}](-a_0) = n_{a_0} b \neq 0$. □

**Remark 3.3** (Reminders on Nil Ideals)**.** *Let $R$ be a commutative ring, and let $I$ be an ideal of $R$.*

a) *We say that $I$ is **nil** if every $x \in I$ is nilpotent: i.e., there is $n = n(x) \in \mathbb{Z}^+$ such that $x^n = 0$. We say that $I$ is **nilpotent** if $I^n = 0$ for some $n \in \mathbb{Z}^+$. The **nilpotency index** $\nu(I)$ of an ideal $I$ is the least $n \in \mathbb{N}$ such that $I^n = 0$ or $\infty$ if there is no such $n$. Thus $\nu(I) < \infty$ if and only if $I$ is nilpotent.*
b) *Let $x_1, \ldots, x_r \in S$ be nilpotent elements; more precisely, let $a_1, \ldots, a_r \in \mathbb{Z}^+$ be such that $x_i^{a_i} = 0$ for all $1 \leq i \leq r$. Then $\langle x_1, \ldots, x_r\rangle^{a_1 + \ldots + a_r - (r-1)} = 0$: indeed, if $N \geq a_1 + \ldots + a_r - (r-1)$ then in any expression $y = x_1^{b_1} \cdots x_r^{b_r}$ with $b_1 + \ldots + b_r = N$ we have $b_i \geq a_i$ for some $i$ and thus $y = 0$. It follows from this*

*that an ideal is nil if and only if it is generated by nilpotent elements and also that every finitely generated nil ideal is nilpotent.*

For a nonzero commutative group $A$ and a nonzero commutative ring $R$, we denote by $\nu(R[A])$ the nilpotency index of the augmentation ideal $I$ in $R[A]$.

**Example 3.4.** *Let $p \in \mathcal{P}$ and let $n \in \mathbb{Z}^+$. According to Remark 3.1, $\nu(Z_p[Z_{p^n}])$ is the nilpotency index of the ideal $\langle t-1\rangle$ in the ring $Z_p[t]/\langle t^{p^n}-1\rangle$. Since $(t-1)^{p^n} = t^{p^n}-1$ in $Z_p[t]$, evidently we have*

$$\nu(Z_p[Z_{p^n}]) = p^n.$$

**Lemma 3.5.** *Let $A$ be a nontrivial commutative group, and let $R$ be a nonzero commutative ring.*

a) *If $A$ has an element of infinite order, then the augmentation ideal of $R[A]$ is not nilpotent.*
b) *The augmentation ideal of $\mathbb{Z}[A]$ is not nilpotent.*
c) *Let $m, N \geq 2$. Suppose that there is $p \in \mathcal{P}$ such that $p \mid N$ and $p \nmid m$ and that $A$ has an element of order $m$. Then $\nu(Z_N[A]) = \infty$.*
d) *If $A$ is finite, then $\nu(R[A]) \geq \max(\exp(A), \operatorname{rank}(A))$.*

*Proof.* Let $H$ be a subgroup of $A$. There is a natural injective ring homomorphism $\iota : R[H] \hookrightarrow R[A]$. Moreover, if $I_H$ (resp. $I_A$) is the augmentation ideal of $R[H]$ (resp. of $R[A]$), then for all $n \in \mathbb{Z}^+$ we have $\iota(I_H^n) \subset I_A^n$, so $\nu(R[H]) \leq \nu(R[A])$.

a) Let $H$ be the subgroup generated by an element of $A$ of infinite order. So, $H \cong (\mathbb{Z}, +)$ and thus $\nu(R[A]) \geq \nu(R[\mathbb{Z}])$. It suffices to show $\nu(R[\mathbb{Z}]) = \infty$. The group ring $R[\mathbb{Z}]$ is isomorphic to the Laurent polynomial ring $R[t, t^{-1}]$, and under this isomorphism the augmentation ideal maps to the principal ideal $\langle t-1\rangle$. The element $t-1$ is not nilpotent in $R[t, t^{-1}]$.

b) In view of part a) we may assume that $A$ has an element $x$ of finite order $m \geq 2$. Taking $H$ to be the subgroup generated by $x$, we have $I_H \subset I_A$. Thus, it is enough to show that the augmentation ideal of $\mathbb{Z}[H] \cong \mathbb{Z}[Z_m]$ is not nilpotent. We have

$$\mathbb{Z}[Z_m] \cong \mathbb{Z}[t]/\langle t^m - 1\rangle,$$

and we claim that the ring $\mathbb{Z}[t]/\langle t^m-1\rangle$ is reduced, i.e., has no nonzero nilpotent elements, which suffices. Since $t^m - 1$ is monic, we have an injective ring homomorphism

$$\mathbb{Z}[t]/\langle t^m - 1\rangle \hookrightarrow \mathbb{Q}[t]/\langle t^m - 1\rangle \cong \prod_{d \mid m} \mathbb{Q}[t]/\langle \Phi_d\rangle,$$

where $\Phi_d$ is the $d^{\text{th}}$ cyclotomic polynomial. The ring $\prod_{d\mid m} \mathbb{Q}[t]/\langle \Phi_d\rangle$ is a product of fields, hence reduced, hence its subring $\mathbb{Z}[t]/\langle t^m - 1\rangle$ is also reduced.

c) Arguing as above, it suffices to show that $\nu(Z_N[Z_m]) = \infty$. Let $p \in \mathcal{P}$ be such that $p \mid N$ and $p \nmid m$. Then the surjective ring homomorphism $Z_N \to Z_p$ induces a surjective ring homomorphism

$$q : Z_N[Z_m] \to Z_p[Z_m],$$

and the map $q$ induces for all $n \in \mathbb{Z}^+$ a surjection between the $n$th powers of the augmentation ideals, so it suffices to show that $\nu(Z_p[Z_m]) = \infty$. This follows from a similar argument to that of part b): we have

$$Z_p[Z_m] \cong Z_p[t]/\langle t^m - 1\rangle,$$

and $t^m - 1 \in \mathbb{F}_p[t]$ is separable since $p \nmid m$, so $Z_p[t]/\langle t^m - 1\rangle$ is reduced.
d) If an $a \in A$ has order $n$ then $([a]-[0])^{n-1} \neq 0$. This shows that $\nu(R[A]) \geq \exp(A)$. If $A$ has rank $n$ and $x_1, \ldots, x_n \in A$ are such that $A \cong \bigoplus_{i=1}^n \langle x_i \rangle$, then $\prod_{i=1}^n([x_i]-[0]) \neq 0$. This shows that $\nu(R[A]) \geq \operatorname{rank}(A)$. □

3.2. **Group Ring Interpretation of the Functional Degree.** With the described $Z_{e(B)}[A]$-module structure on $B^A$, we may view the operators $\tau_a = [a]$ and $\Delta_a = [a]-[0]$ as elements of the group ring $Z_{e(B)}[A]$, and observe that the elements $\Delta_a$ lie in the augmentation ideal $I$. Alternatively, we may also view $B^A$ as $\mathbb{Z}[A]$-module and $I$ as the augmentation ideal of $\mathbb{Z}[A]$. The latter choice allows us to simultaneously consider different codomains $B$. For a fixed $A$ all possible $B^A$ are modules over the same ring $\mathbb{Z}[A]$. Generalizing and unifying both choices, we may also view $B^A$ as $Z_{ke(B)}[A]$-module, for any $k \in \mathbb{N}$. No matter which ring we choose, the operators $\Delta_a$ always lie in the corresponding augmentation ideal $I$, and they generate that ideal:

**Lemma 3.6.** *Let $R$ be a commutative ring and let $A$ be a commutative group.*

a) *The augmentation ideal $I$ of $R[A]$ is generated as an $R$-module by $\{\Delta_a \mid a \in A\}$.*
b) *If $S \subset A$ generates $A$ as a commutative group, then for each $n \in \mathbb{Z}^+$, the set of $n$-fold products $\{\Delta_{s_1} \cdots \Delta_{s_n} \mid s_1, \ldots, s_n \in S\}$ generates $I^n$ as an ideal.*

*Proof.* a) Assume $x = n_1[a_1] + n_2[a_2] + \cdots + n_N[a_N]$ lies in $I$. We show that $x$ lies in the $R$-submodule $M$ of $I$ spanned by the elements $\Delta_a$. If $N = 1$ then $n_1 = 0$ by the definition of $I$, so $x = 0 \in M$. If $N \geq 2$, then the element

$$\begin{aligned}(n_1 + \cdots + n_N)[a_1] &= x - n_2([a_2]-[a_1]) - n_3([a_3]-[a_1]) - \cdots - n_N([a_N]-[a_1]) \\ &= x - n_2(\Delta_{a_2} - \Delta_{a_1}) - n_3(\Delta_{a_3} - \Delta_{a_1}) - \cdots - n_N(\Delta_{a_N} - \Delta_{a_1})\end{aligned}$$

lies in $I$, since $x$ and the $\Delta_{a_i}$ lie in $I$. Hence, $n_1 + n_2 + \cdots + n_N = 0$, as in the case $N = 1$. It follows that

$$x \;=\; n_2(\Delta_{a_2} - \Delta_{a_1}) + n_3(\Delta_{a_3} - \Delta_{a_1}) + \cdots + n_N(\Delta_{a_N} - \Delta_{a_1}) \;\in\; M.$$

b) We observe that for all $s_1, s_2 \in S$ we have

$$\Delta_{-s} = [-s]-[0] = -[-s]\cdot([s]-[0]) = -[-s]\cdot\Delta_s \in \langle \Delta_s \mid s \in S\rangle,$$

$$\Delta_{s_1+s_2} = [s_1+s_2]-[0] = [s_1]([s_2]-[0]) + ([s_1]-[0]) = [s_1]\Delta_{s_2} + \Delta_{s_1} \in \langle \Delta_s \mid s \in S\rangle.$$

This shows that $\langle \Delta_s \mid s \in S\rangle = \langle \Delta_a \mid a \in A\rangle = I$; the result for $I^n$ follows immediately. □

From this Lemma, with $R := \mathbb{Z}$ or with $R := Z_{e(B)}$, we obtain Aichinger and Moosbauer's module-theoretic interpretation of the functional degree:

**Lemma 3.7.** *Let $A$ and $B$ be commutative groups, and let $I$ be the augmentation ideal of $\mathbb{Z}[A]$ or of $Z_{e(B)}[A]$. For each $f \in B^A$ and $n \in \mathbb{N}$, the following statements are equivalent:*

(i) *$I^{n+1}$ kills $f$, i.e., $\theta f = 0$ for every $\theta \in I^{n+1}$.*
(ii) $\operatorname{fdeg}(f) \leq n$.

This equivalence can also be expressed by saying that the elements of $B^A$ with functional degree at most $n$ form the set

$$B^A[I^{n+1}] := \{f \in B^A \mid I^{n+1}f = 0\} = \{f \in B^A \mid \forall\theta \in I^{n+1},\ \theta f = 0\}.$$

The module-theoretic interpretation of the functional degree was introduced in [AM21, Def. 2.1]. It is a useful perspective, because it allows us to apply the language and the tools of commutative algebra to the calculus of finite differences. Here is one simple example.

**Lemma 3.8.** *Let $A$ and $B$ be commutative groups.*

a) *For $f, g \in B^A$ we have* $\operatorname{fdeg}(f+g) \leq \max(\operatorname{fdeg}(f), \operatorname{fdeg}(g))$.

b) *For all $n \in \widetilde{\mathbb{N}}$ the set*

$$\mathcal{F}_n(A,B) := \{f \in B^A \mid \operatorname{fdeg}(f) \leq n\}$$

*is a subgroup of $B^A$, as is*

$$\mathcal{F}(A,B) := \bigcup_{n<\infty} \mathcal{F}_n(A,B).$$

*Proof.* a) We may assume without loss of generality that $\operatorname{fdeg}(g) \leq \operatorname{fdeg}(f) =: n \in \mathbb{N}$. Then $I^{n+1}$ kills both $f$ and $g$, so it kills the $\mathbb{Z}[A]$-submodule of $B^A$ generated by $f$ and $g$, so it kills $f+g$.

b) This follows immediately. □

Let $\varepsilon : A' \to A$ and $\mu : B \to B'$ be homomorphisms of commutative groups. This yields group homomorphisms

$$\varepsilon^* : B^A \to B^{A'},\ f \mapsto \varepsilon^* f := f \circ \varepsilon$$

and

$$\mu_* : B^A \to (B')^A,\ f \mapsto \mu_* f := \mu \circ f.$$

If $\varepsilon$ is injective (resp. surjective), then $\varepsilon^*$ is surjective (resp. injective), while if $\mu$ is injective (resp. surjective), then $\mu_*$ is injective (resp. surjective).

**Lemma 3.9** (Homomorphic Functoriality I)**.** *Let $A, A', B, B'$ be commutative groups and let $f \in B^A$. Let $\varepsilon : A' \to A$, $\mu : B \to B'$ be group homomorphisms. Then:*

a) *We have* $\operatorname{fdeg} \varepsilon^* f \leq \operatorname{fdeg} f$*, with equality if $\varepsilon$ is surjective.*

b) *We have* $\operatorname{fdeg} \mu_* f \leq \operatorname{fdeg} f$*, with equality if $\mu$ is injective.*

*Proof.* A homomorphism $\varphi : G \to H$ of commutative groups induces a homomorphism of group rings $\mathbb{Z}[\varphi] : \mathbb{Z}[G] \to \mathbb{Z}[H]$. If $I_G$ (resp. $I_H$) is the augmentation ideal of $\mathbb{Z}[G]$ (resp. of $\mathbb{Z}[H]$) then, for all $n \in \mathbb{N}$, we have

$$\mathbb{Z}[\varphi](I_G^n) \subset I_H^n,$$

with equality if $\varphi$ is surjective.

a) Using the ring homomorphism $\mathbb{Z}[\varepsilon]$, we may view $B^A$ as a $\mathbb{Z}[A']$-module, and then $\varepsilon^* : B^A \to B^{A'}$ is a $\mathbb{Z}[A']$-module homomorphism. For each $n \in \mathbb{N}$, we have $\mathbb{Z}[\varepsilon](I_{A'}^{n+1}) \subset I_A^{n+1}$ and can conclude as follows: if $I_A^{n+1}$ kills $f \in B^A$, then in particular $I_{A'}^{n+1}$ kills $f$, hence $I_{A'}^{n+1}$ also kills $\varepsilon^* f$. This shows that $\operatorname{fdeg} \varepsilon^* f \leq \operatorname{fdeg} f$. In the case that $\varepsilon$ is surjective, we have that $\varepsilon^*$ is injective, and we can also use that $\mathbb{Z}[\varepsilon](I_{A'}^{n+1}) = I_A^{n+1}$. So, $I_{A'}^{n+1}$ kills $\varepsilon^* f$ if and only if $I_{A'}^{n+1}$ kills $f$ if and only if $I_A^{n+1}$ kills $f$. Thus $\operatorname{fdeg}(\varepsilon^* f) = \operatorname{fdeg}(f)$, if $\varepsilon$ is surjective.

b) The pushforward map $\mu_* : B^A \to (B')^A$ is a $\mathbb{Z}[A]$-module homomorphism, so if $\operatorname{fdeg}(f) \leq n$ then $I_A^{n+1}$ kills $f$ and thus also kills $\mu_* f$, so $\operatorname{fdeg}(\mu_* f) \leq n$; we deduce that $\operatorname{fdeg}(\mu_* f) \leq \operatorname{fdeg}(f)$. If $\mu$ is injective then $\mu_*$ maps $B^A$ onto a $\mathbb{Z}[A]$-submodule of $(B')^A$, and the annihilator ideal of an element $x$ of a submodule $N$ of a module $M$ is the same as the annihilator ideal of $x$ viewed as an element of $M$. $\square$

The following useful result is an immediate consequence of Lemma 3.9.

**Corollary 3.10.** *Let $f : A \to B$ be a map between commutative groups.*

a) *(Domain Restriction) Let $\underline{A}$ be a subgroup of $A$, and let $f|_{\underline{A}} : \underline{A} \to B$, $x \mapsto f(x)$ be the **restriction** of $f$ to $\underline{A}$. Then*
$$\operatorname{fdeg}(f|_{\underline{A}}) \leq \operatorname{fdeg}(f).$$

b) *(Codomain Restriction) Let $\underline{B}$ be a subgroup of $B$ such that $f(A) \subset \underline{B}$, and let $f|^{\underline{B}} : A \to \underline{B}$ be given by $x \mapsto f(x)$. Then*
$$\operatorname{fdeg}(f|^{\underline{B}}) = \operatorname{fdeg}(f).$$

Lemma 3.9 also implies that if $A, A', B, B'$ are commutative groups such that $A \cong A'$ and $B \cong B'$ then $\delta(A, B) = \delta(A', B')$.

3.3. **Between Sums and Products.** Next we study maps $f : A \to B$ when $B$ is a direct product and also when $A$ is a direct sum. Since the set-theoretic Cartesian product is also the direct product in the category of commutative groups but the direct sum in the category of commutative groups is not the set-theoretic coproduct, one expects the case of maps into a product to be simpler than the case of maps out of a sum. This turns out to be true, but we still have a result for maps out of a sum that is satisfactory for our purposes.

**Lemma 3.11** (Mappings into Products)**.** *Let $A$ be a commutative group, and let $B = \prod_{\gamma \in \Gamma} B_\gamma$ be the direct product of commutative groups $B_\gamma$ over a nonempty index set $\Gamma$. For each $\gamma \in \Gamma$, let $\pi_\gamma : B \to B_\gamma$ be the canonical projection. For all $f \in B^A$, we have*
$$\operatorname{fdeg}(f) = \sup_{\gamma \in \Gamma} \operatorname{fdeg}(\pi_\gamma \circ f).$$

*Proof.* The set-theoretic identity $B^A = \prod_{\gamma \in \Gamma} B_\gamma^A$ is also a $\mathbb{Z}[A]$-module isomorphism. For a commutative ring $R$ and a family $(M_\gamma)_{\gamma \in \Gamma}$ of $R$-modules, if $f = (f_\gamma) \in \prod_{\gamma \in \Gamma} M_\gamma$ is an element of the product $R$-module and $J$ is an ideal of $R$, then $Jf = 0$ if and only if $Jf_\gamma = 0$ for all $\gamma \in \Gamma$. Thus the least $n \in \mathbb{N}$ such that $I^n$ kills $f$ is the least $n$ such that $I^n$ kills all $f_\gamma$. $\square$

Now suppose $(A_\gamma)_{\gamma \in \Gamma}$ is a nonempty family of nontrivial commutative groups $A_\gamma$. We put $A := \bigoplus_{\gamma \in \Gamma} A_\gamma$ and view $A_\gamma$ as a subgroup of $A$ via the canonical injection $\iota_\gamma : A_\gamma \hookrightarrow A$. Hence, if $a_\gamma \in A_\gamma$, it makes sense to write $\Delta_{a_\gamma}$ for $\Delta_{\iota_\gamma(a_\gamma)}$. If each $A_\gamma$ is generated by a set $S_\gamma$, we may also write $A = \bigoplus_{\gamma \in \Gamma} \langle S_\gamma \rangle$. With this setup, we can formulate the following result:

**Lemma 3.12** (Mappings out of Sums)**.** *Assume $A = \bigoplus_{\gamma \in \Gamma} A_\gamma = \bigoplus_{\gamma \in \Gamma} \langle S_\gamma \rangle$, as described above, and let $B$ be another commutative group. For each $f \in B^A$ and $d \in \mathbb{N}$, the following statements are equivalent:*

(i) *There are $r \in \mathbb{N}$, elements $\gamma_1, \gamma_2, \dots, \gamma_r \in \Gamma$ and $d_1, d_2, \dots, d_r \in \mathbb{Z}^+$ with $d_1 + d_2 + \cdots + d_r = d$, and finite sequences*
$$a_{1,1}, \dots, a_{1,d_1} \in S_{\gamma_1},$$

$$\vdots$$

$$a_{r,1}, \ldots, a_{r,d_r} \in S_{\gamma_r}$$

*such that*

$$\Big(\prod_{i=1}^{r} \prod_{j=1}^{d_i} \Delta_{a_{i,j}}\Big) f \neq 0.$$

(ii) *We have* $\operatorname{fdeg}(f) \geq d$.

*Proof.* If $d = 0$, the theorem holds, as $\big(\prod_{i=1}^{r} \prod_{j=1}^{d_i} \Delta_{a_{i,j}}\big) f = f$ if $r = 0$. So, assume $d \geq 1$, and let $I$ be the augmentation ideal of $\mathbb{Z}[A]$. As $S := \bigcup_{\gamma \in \Gamma} S_\gamma$ is a set of generators of $A$, we have that $\{\Delta_{s_1} \cdots \Delta_{s_d} \mid s_1, \ldots, s_d \in S\}$ is a set of generators for the ideal $I^d$, by Lemma 3.6. With that, the result follows from the contrapositive of Lemma 3.7 with $n := d - 1$. □

Our main application of Lemma 3.12 will be Theorem 4.8, which reduces the determination of functional degrees of maps between finite commutative $p$-groups to the cyclic case.

The next result is a discrete analogue of the fact that a smooth function $f : \mathbb{R}^n \to \mathbb{R}^n$ with diagonal Jacobian matrix decomposes as $f = (f_1, \ldots, f_n)$ with $f_i : \mathbb{R} \to \mathbb{R}$. Let $\Gamma$ be a nonempty index set, for each $\gamma \in \Gamma$ let $A_\gamma$ and $B_\gamma$ be commutative groups, and put

$$A := \bigoplus_{\gamma \in \Gamma} A_\gamma \quad \text{and} \quad B := \prod_{\gamma \in \Gamma} B_\gamma.$$

Then we may naturally view each $f_\bullet = (f_\gamma)_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} B_\gamma^{A_\gamma}$ as a function in $B^A$, and $\prod_{\gamma \in \Gamma} B_\gamma^{A_\gamma}$ as a subgroup of $B^A$. We just need to define

$$f_\bullet(x) := (f_\gamma(x_\gamma))_{\gamma \in \Gamma} \quad \text{for all} \;\; x = (x_\gamma)_{\gamma \in \Gamma} \in A.$$

These $f_\bullet$ are exactly the "diagonal" functions in $B^A$, in the sense that for each $\gamma \in \Gamma$ the $\gamma$-component of the output depends only on the $\gamma$-component of the input.

**Theorem 3.13** (Diagonalization Theorem). *With $A := \bigoplus_{\gamma \in \Gamma} A_\gamma$ and $B := \prod_{\gamma \in \Gamma} B_\gamma$, as above, suppose that for all $\gamma \in \Gamma$ we have $\operatorname{Hom}\big(\bigoplus_{\lambda \neq \gamma} A_\lambda, B_\gamma\big) = \{0\}$. Then*

a) $\operatorname{fdeg}(f_\bullet) = \sup_{\gamma \in \Gamma} \operatorname{fdeg}(f_\gamma)$ *for all* $f_\bullet = (f_\gamma)_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} B_\gamma^{A_\gamma}$.

b) $\mathcal{F}_n(A, B) = \prod_{\gamma \in \Gamma} \mathcal{F}_n(A_\gamma, B_\gamma)$ *for all* $n < \infty$.

c) $\mathcal{F}(A, B) = \bigcup_{n < \infty} \prod_{\gamma \in \Gamma} \mathcal{F}_n(A_\gamma, B_\gamma) \subset \prod_{\gamma \in \Gamma} \mathcal{F}(A_\gamma, B_\gamma)$.

*Proof.* a) For $\gamma \in \Gamma$, let $\iota_\gamma : A_\gamma \hookrightarrow A$ and $\pi_\gamma : B \to B_\gamma$ be the canonical maps. Whenever $\lambda \neq \gamma$ Corollary 2.7 implies that $\pi_\gamma \circ f \circ \iota_\lambda$ is constant, as $\operatorname{Hom}(A_\lambda, B_\gamma) = \{0\}$. It follows (e.g. by Lemma 3.12 with $S_\gamma := A_\gamma$) that

$$\operatorname{fdeg}(\pi_\gamma \circ f_\bullet) = \operatorname{fdeg}(\pi_\gamma \circ f_\bullet \circ \iota_\gamma) = \operatorname{fdeg}(f_\gamma),$$

for all $\gamma \in \Gamma$. So, by Lemma 3.11 we have

$$\operatorname{fdeg}(f_\bullet) = \sup_{\gamma \in \Gamma} \operatorname{fdeg}(\pi_\gamma \circ f_\bullet) = \sup_{\gamma \in \Gamma} \operatorname{fdeg}(f_\gamma).$$

b) The inclusion $\mathcal{F}_n(A,B) \supset \prod_{\gamma\in\Gamma} \mathcal{F}_n(A_\gamma, B_\gamma)$ follows directly from part a). To prove the opposite inclusion, we first introduce some notation. For each $\gamma \in \Gamma$, we identify $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ with $A_\gamma \times \bigoplus_{\lambda\neq\gamma} A_\lambda$. Having done so, each $x_\gamma \in A_\gamma$ and $x^\gamma \in \bigoplus_{\lambda\neq\gamma} A_\lambda$ yields an element $(x_\gamma, x^\gamma)$ of $A$. Now, let $f \in \mathcal{F}_n(A,B)$. We need to show that $f \in \prod_{\gamma\in\Gamma} \mathcal{F}_n(A_\gamma, B_\gamma)$. Consider first just one fixed $\gamma \in \Gamma$ and one fixed $x_\gamma \in A_\gamma$. Let $f^\gamma : A \to B_\gamma$ be the $\gamma$-component of $f$. As $f$ has finite functional degree, so does the function

$$g_{\gamma,x_\gamma} : \bigoplus_{\lambda\neq\gamma} A_\lambda \to B_\gamma \ , \quad g_{\gamma,x_\gamma}(x^\gamma) := f^\gamma(x_\gamma, x^\gamma).$$

Since $\mathrm{Hom}(\bigoplus_{\lambda\neq\gamma} A_\lambda, B_\gamma) = \{0\}$, Corollary 2.7 implies that $g_{\gamma,x_\gamma}$ is constant, and we may write $f_\gamma(x_\gamma)$ for that constant value. This applies to all $x_\gamma \in A_\gamma$ and all $\gamma \in \Gamma$. Now, because $f$ lies in $\mathcal{F}_n(A,B)$, the functions $f_\gamma : x_\gamma \mapsto f_\gamma(x_\gamma)$ lie in $\mathcal{F}_n(A_\gamma, B_\gamma)$, and $f = (f_\gamma) \in \prod_{\gamma\in\Gamma} \mathcal{F}_n(A_\gamma, B_\gamma)$, indeed.

c) This follows from part b), as $\prod_{\gamma\in\Gamma} \mathcal{F}_n(A_\gamma, B_\gamma) \subset \prod_{\gamma\in\Gamma} \mathcal{F}(A_\gamma, B_\gamma)$ for all $n < \infty$. $\square$

**Remark 3.14.** *If we define $f_\bullet(x)$ only for $x \in \bigoplus_{\gamma\in\Gamma} A_\gamma$, as we did, Theorem 3.13 holds also with the direct sum $\bigoplus_{\gamma\in\Gamma} B_\gamma$ in the place of the direct product $B := \prod_{\gamma\in\Gamma} B_\gamma$. One can also formulate that theorem for the direct product $\prod_{\gamma\in\Gamma} A_\gamma$ instead of the direct sum $\bigoplus_{\gamma\in\Gamma} A_\gamma$, but this is not possible in Lemma 3.12, and the direct sum is also more important in the study of torsion groups.*

Our main application of Theorem 3.13 is to the case when $A$ and $B$ are both torsion group. We take $\Gamma$ to be the set of prime numbers $\mathcal{P}$, and decompose $A$ and $B$ as

$$A = \bigoplus_{p\in\mathcal{P}} A[p^\infty] \ \text{ and } \ B = \bigoplus_{p\in\mathcal{P}} B[p^\infty] \subset \prod_{p\in\mathcal{P}} B[p^\infty] =: \tilde{B}.$$

We may apply Theorem 3.13 in this setting. Moreover, if $B[p^\infty] = \{0\}$ for all but finitely many $p \in \mathcal{P}$, e.g. if $B$ has finite exponent, then $\tilde{B} = B$ and we can write all the direct products in the theorem as direct sums. In that situation, we can then also use that

$$\bigcup_{n<\infty} \bigoplus_{p\in\mathcal{P}} \mathcal{F}_n(A[p^\infty], B[p^\infty]) = \bigoplus_{p\in\mathcal{P}} \mathcal{F}(A[p^\infty], B[p^\infty])$$

and obtain the following corollary:

**Corollary 3.15.** *If $A$ and $B$ are torsion groups, and if $B[p^\infty] = \{0\}$ for all but finitely many $p \in \mathcal{P}$, then*

a) $\mathrm{fdeg}(f_\bullet) = \max_{p\in\mathcal{P}} \mathrm{fdeg}(f_p)$ *for all* $f_\bullet = (f_p)_{p\in\mathcal{P}} \in \bigoplus_{p\in\mathcal{P}} B[p^\infty]^{A[p^\infty]}$,

b) $\mathcal{F}_n(A,B) = \bigoplus_{p\in\mathcal{P}} \mathcal{F}_n(A[p^\infty], B[p^\infty])$ *for all* $n < \infty$,

c) $\mathcal{F}(A,B) = \bigoplus_{p\in\mathcal{P}} \mathcal{F}(A[p^\infty], B[p^\infty])$.

We will use the previous results to reduce the study of maps of finite functional degree between torsion groups to the $p$-primary case.

## 4. The set $\mathcal{D}(A,B)$ of functional degrees

4.1. **Ideal-theoretic Interpretation of $\delta(A,B)$.** Recall that for commutative groups $A$ and $B$, we have defined $\delta(A,B)$ to be the supremum of all functional degrees of maps $f \in B^A$ and $\mathcal{D}(A,B)$ to be the set of all functional degrees of maps $f \in B^A$. In this section we will compute $\delta(A,B)$ for all $A$ and $B$ and $\mathcal{D}(A,B)$ for a class of commutative groups that includes all finitely generated groups $A$ and $B$.

For $f \in B^A$ and $n \in \mathbb{N}$, by Lemma 3.7, we have

$$\operatorname{fdeg}(f) = n \iff f \in B^A[I^{n+1}] \setminus B^A[I^n].$$

Considering this connection between the degree of $f$ and the powers of the augmentation ideal $I$ of $Z_{e(B)}[A]$, it is no surprise that the quantity $\delta(A,B)$ has an ideal-theoretic interpretation:

**Theorem 4.1.** *Let $A$ and $B$ be nontrivial commutative groups. Then*

$$\delta(A,B) = \nu(Z_{e(B)}[A]) - 1.$$

*Proof.* Let $I$ be the augmentation ideal of $Z_{e(B)}[A]$, and put $\nu := \nu(Z_{e(B)}[A])$. Lemma 3.2 tells us that $B^A$ is a faithful $Z_{e(B)}[A]$-module. If $\nu = \infty$, then for all $n \in \mathbb{Z}^+$ there is $\eta \in I^n \setminus \{0\}$. By faithfulness, there is $f \in B^A$ such that $\eta f \neq 0$, and it follows that $\delta(A,B) = \infty$. If $\nu < \infty$ then $I^\nu = 0$, so $B^A = B^A[I^{(\nu-1)+1}]$, so $\delta(A,B) \leq \nu - 1$. The converse is the same as above: there is $\eta \in I^{\nu-1} \setminus \{0\}$, hence by faithfulness there is $f \in B^A$ such that $I^{\nu-1} f \neq 0$, so $\delta(A,B) \geq \nu - 1$. □

We deduce:

**Corollary 4.2.** *The following are equivalent:*

(i) $\delta(A,B) < \infty$.
(ii) *The augmentation ideal of $Z_{e(B)}[A]$ is nilpotent.*

*Proof.* This is immediate from the theorem and Remark 3.3a). □

We also see that:

**Corollary 4.3.** *For commutative groups $A$ and $B$, the quantity $\delta(A,B)$ depends only on $A$ and on the exponent $\exp(B)$ of $B$.*

Thus we have an interplay between the ideal theory of the group ring $Z_{e(B)}[A]$ and the structure of its faithful module $B^A$. Notice that the nilpotency index of any ideal $J$ in a commutative ring $R$ can be computed using any faithful $R$-module $M$: it is the least $n \in \mathbb{Z}^+$ such that $M = M[J^n]$ or $\infty$ if no such $n$ exists. Because of this, having a "concretely given" faithful $R$-module can be useful for studying the ideal theory of $R$. Following Aichinger-Moosbauer [AM21, §7] we introduce a natural family of elements of $B^A$, the **delta functions**, that in many cases can be shown (via their generation properties) to be elements of $B^A$ of maximal functional degree. They provide a convenient tool for giving lower bounds on $\delta(A,B)$.

4.2. **Delta Functions.** Let $A$ and $B$ be commutative groups. For $a \in A$ and $b \in B$, we define the **delta function** $\delta_{a,b} \in B^A$ via

$$\delta_{a,b}(x) := \begin{cases} b & \text{if } x = a, \\ 0 & \text{if } x \neq a. \end{cases}$$

**Proposition 4.4.** *Let $A$ and $B$ be nontrivial commutative groups.*

a) *The $\mathbb{Z}[A]$-submodule of $B^A$ generated by $\mathcal{D} := \{\delta_{0,b} \mid b \in B\}$ is the set of all functions $f : A \to B$ such that $f(a) = 0$ for all but finitely many $a \in A$. Thus the subset $\mathcal{D}$ generates $B^A$ as a $\mathbb{Z}[A]$-module if and only if $A$ is finite.*

b) *If $A$ is finite and $B = Z_b$, for some $b \geq 2$, then the $Z_b[A]$-module $B^A$ is free of rank 1 with $\delta_{a,u}$ as generator, for any choice of $a \in A$ and $u \in Z_b^\times$.*

c) *If $A$ is finite and $B$ has finite exponent then, for each $a \in A$ and each $b \in B$ of order* $\exp(B)$,

$$\delta(A, B) = \operatorname{fdeg}(\delta_{a,b}).$$

*Proof.* a) For $a \in A$ we have that $\tau_a \delta_{0,b} = \delta_{a,b}$, from which the result follows easily.

b) For $B := Z_b^\times$, the $\mathbb{Z}$-module generated by $\delta_{0,u}$ contains $\mathcal{D}$, so by part a) the $\mathbb{Z}[A]$-module generated by $\delta_{0,u}$ is $Z_b^A$. Being a faithful $Z_b[A]$-module that is generated by $\delta_{0,u}$, the module $Z_b^A$ is therefore free of rank 1.

c) Let $U$ be the set of elements of $B$ of order $\exp(B)$. We claim that $U$ generates $B$ as a group. To see this, let $\underline{B}$ be the subgroup generated by $U$. We first suppose that $B = B[p^\infty]$ for some $p \in \mathcal{P}$. In this case, if $x \in U$ and $y \in B \setminus U$ then $x + y \in U$. It follows that $y \in \underline{B}$, i.e. $\underline{B} = B$. In the general case, for $u \in U$ we write $u = \sum_{p \in \mathcal{P}} u_p$ with $u_p \in B[p^\infty]$. For each $p \in \mathcal{P}$ the element $u_p$ lies in the cyclic subgroup generated by $u$, hence also in $\underline{B}$, and moreover $u_p$ has maximal order $\exp(B[p^\infty])$. Conversely, every element $v$ of $B[p^\infty]$ of maximal order is of the form $u_p$ for some $u \in U$. This shows that $\underline{B}$ contains $B[p^\infty]$ for all $p \in \mathcal{P}$, so $\underline{B} = B$, indeed.

Since $\delta_{0,u_1} + \delta_{0,u_2} = \delta_{0,u_1+u_2}$, it follows that

$$\mathcal{U} := \{\delta_{0,u} \mid u \text{ has order } \exp(B)\}$$

generates $B^A$ as a $Z_{\exp(B)}[A]$-module. Moreover, all $\delta_{0,u}$ have the same degree. This is because for each $u \in U$ the functional degree of $\delta_{0,u} : A \to B$ is equal to the functional degree of $\delta_{0,u} : A \to \langle u \rangle$, by Corollary 3.10, and thus also equal to the functional degree of $\delta_{0,1} : A \to Z_{\exp(B)}$, by Lemma 3.9, which does not depend on $u$. It follows that the least power of $I$ (if any) that kills $\delta_{0,b}$ is the least power of $I$ that kills $\mathcal{U}$, which is the least power of $I$ that kills $B^A$. Hence, $\delta(A, B) = \operatorname{fdeg}(\delta_{0,b}) = \operatorname{fdeg}(\delta_{a,b})$. $\square$

4.3. **Computing $\delta(Z_{p^\alpha}, Z_{p^\beta})$.** We will now compute $\delta(Z_{p^\alpha}, Z_{p^\beta})$ by two different arguments. The first uses delta functions, via a 1977 result of Weisman. The second uses group rings, via a 2006 result of Wilson.

For $n \in \mathbb{Z}^+$ and $j, k \in \mathbb{Z}$, we put

$$M_k(j, n) := \sum_{\substack{0 \leq i \leq n \\ i \equiv j \pmod{k}}} (-1)^i \binom{n}{i}.$$

**Theorem 4.5** (Weisman)**.** *Let $p \in \mathcal{P}$, $\alpha, \beta \in \mathbb{Z}^+$ and $j \in \mathbb{Z}$.*

a) *If* $n \geq \beta((p-1)+1)p^{\alpha-1}$*, then* $M_{p^\alpha}(j,n) \equiv 0 \pmod{p^\beta}$.
b) $M_{p^\alpha}(j, \beta((p-1)+1)p^{\alpha-1}-1) \equiv (-p)^{\beta-1} \pmod{p^\beta}$.

*Proof.* This is the main result of [We77]. □

**Lemma 4.6** (Wilson)**.** *Let* $p \in \mathcal{P}$ *and let* $\alpha, \beta \in \mathbb{Z}^+$*. Then*

$$(t-1)^{(\beta(p-1)+1)p^{\alpha-1}-1} \equiv (-p)^{\beta-1} \sum_{i=0}^{p^\alpha-1} t^i \pmod{t^{p^\alpha}-1, p^\beta}.$$

*Proof.* This is [Wi06, (22)][2]. □

**Theorem 4.7.** *Let* $p \in \mathcal{P}$*, let* $\alpha, \beta \in \mathbb{Z}^+$ *and let* $B$ *be a commutative group of exponent* $p^\beta$*. Then*

$$\delta(Z_{p^\alpha}, B) = (\beta(p-1)+1)p^{\alpha-1} - 1.$$

*Proof.* (Via Weisman) By Corollary 4.3, we may assume that $B = Z_{p^\beta}$. By Proposition 4.4b), then $\delta(Z_{p^\alpha}, B) = \operatorname{fdeg}(\delta_{0,1})$. So, by Lemma 3.6 and Lemma 3.7, we need to determine the least $n \in \mathbb{N}$ such that $\Delta^n \delta_{0,1} = 0$, where $\Delta := \Delta_1$. If we view $\delta_{0,1}$ as map from $Z_{p^\alpha}$ into $\mathbb{Z}$, this condition is met if and only if

$$\forall x \in Z_{p^\alpha},\ (\Delta^n \delta_{0,1})(x) \equiv 0 \pmod{p^\beta}.$$

However, Lemma 2.2 tells us that

$$(\Delta^n \delta_{0,1})(x) = \sum_{\substack{0 \leq j \leq n \\ j \equiv -x \pmod{p^\alpha}}} (-1)^{n-j} \binom{n}{j} = \pm M_{p^\alpha}(-x, n),$$

and by Theorem 4.5 we know exactly when this is zero modulo $p^\beta$. We see that

$$\delta(Z_{p^\alpha}, B) = (\beta(p-1)+1)p^{\alpha-1} - 1.$$ □

*Proof.* (Via Wilson) Theorem 4.1 gives

$$\delta(Z_{p^\alpha}, B) = \nu(Z_{p^\beta})[Z_{p^\alpha}]) - 1.$$

Therefore, by Remark 3.1b), it suffices to show that the nilpotency index of the augmentation ideal $\langle t-1 \rangle$ in the ring $Z_{p^\beta}[t]/\langle t^{p^\alpha} - 1\rangle$ is $(\beta(p-1)+1)p^{\alpha-1}$. Phrasing this in terms of the ring $\mathbb{Z}[t]$, we wish to show that the least $N \in \mathbb{Z}^+$ such that $(t-1)^N$ lies in the ideal $J := \langle p^\beta, t^{p^\alpha} - 1\rangle$ is $(\beta(p-1)+1)p^{\alpha-1}$. This follows from Lemma 4.6. Indeed, that congruence directly entails $(t-1)^{(\beta(p-1)+1)p^{\alpha-1}-1} \notin J$ , and indirectly (after multiplying both sides by $t-1$) implies $(t-1)^{(\beta(p-1)+1)p^{\alpha-1}} \equiv 0 \pmod{t^{p^\alpha}-1, p^\beta}$ and thus $(t-1)^{(\beta(p-1)+1)p^{\alpha-1}} \in J$. □

### 4.4. **The $p$-Primary Sum Theorem.**

**Theorem 4.8** ($p$-Primary Sum Theorem)**.** *Let* $p \in \mathcal{P}$*. For* $1 \leq i \leq r$*, let* $A_i$ *be a nonzero finite commutative* $p$*-group, let* $A := \bigoplus_{i=1}^r A_i$*, and let* $B$ *be a commutative group of exponent* $p^\beta$*. Then*

$$\delta(A, B) = \max_{\underline{\beta}} \sum_{i=1}^r \delta(A_i, Z_{p^{\beta_i+1}}),$$

*where the maximum extends over all* $\underline{\beta} = (\beta_1, \ldots, \beta_r) \in \mathbb{N}^r$ *with* $\beta_1 + \ldots + \beta_r = \beta - 1$.

[2] In [Wi06] Wilson speaks of a "sketch of a derivation," but in fact he provides a complete proof.

*Proof.* By Corollary 4.3 we may assume that $B = Z_{p^\beta}$. To be able to apply Proposition 4.4c), we further define for $1 \le i \le r$ and $\ell \in \mathbb{Z}^+$ the delta functions

$$\delta^i_\ell := \delta_{0,1} \in Z_{p^\ell}^{A_i}, \quad \delta^i := \delta_{0,1} \in \mathbb{Z}^{A_i} \quad \text{and} \quad \delta_\ell := \delta_{0,1} \in Z_{p^\ell}^{A}.$$

To use Lemma 3.12 (with the $A_i$ in the place of the generating sets $S_i$), we write $\underline{a} \in \prod_{i=1}^r A_i^{d_i}$ to say that $\underline{a}$ is a family $(a_{i,j})$ with $d_i$ entries $a_{i,1}, a_{i,2}, \dots, a_{i,d_i}$ in $A_i$, for each $1 \le i \le r$. Similarly, $\underline{x} \in \prod_{i=1}^r A_i$ means that $\underline{x}$ is an $r$-tuple with $i^{\text{th}}$ entry $x_i$ in $A_i$, etc. Using the map $\sigma_r : \mathbb{N}^r \to \mathbb{N}$, $\underline{d} \mapsto \sum_{i=1}^r d_i$ we will also write $\sigma_r^{-1}(d)$ to denote the set of all $\underline{d} \in \mathbb{N}^r$ whose sum of entries is $d$, for a given $d \in \mathbb{N}$. Moreover, we use the $p$-adic valuation $v_p : \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ with $v_p(0) = \infty$ and $v_p(n) = \max\{a \in \mathbb{Z}^+ \mid p^a \text{ divides } n\}$ for $n \in \mathbb{Z} \setminus \{0\}$. Now let $d \in \mathbb{N}$. It is enough to prove the equivalence

$$\delta(A, Z_{p^\beta}) \ge d \quad \Longleftrightarrow \quad \max_{\underline{\beta}} \sum_{i=1}^r \delta(A_i, Z_{p^{\beta_i+1}}) \ge d.$$

We show this through the following chain of equivalences (where the range of the variables is specified the first time they appear but not thereafter):

$$\begin{aligned}
& & & \delta(A, Z_{p^\beta}) \ge d \\
\Leftrightarrow & & & \mathrm{fdeg}(\delta_\beta) \ge d \\
\Leftrightarrow & & \exists \underline{d} \in \sigma_r^{-1}(d) : \exists \underline{a} \in \prod_{i=1}^r A_i^{d_i} : \exists x \in A : \ & \Big(\prod_{i=1}^r \prod_{j=1}^{d_i} \Delta_{a_{i,j}} \delta_\beta\Big)(x) \ne 0 \in Z_{p^\beta} \\
\Leftrightarrow & & \exists \underline{d} : \exists \underline{a} : \exists \underline{x} \in \prod_{i=1}^r A_i : \ & \prod_{i=1}^r \Big( (\prod_{j=1}^{d_i} \Delta_{a_{i,j}} \delta^i_\beta)(x_i)\Big) \ne 0 \in Z_{p^\beta} \\
\Leftrightarrow & & \exists \underline{d} : \exists \underline{a} : \exists \underline{x} : \ & \sum_{i=1}^r v_p\Big( (\prod_{j=1}^{d_i} \Delta_{a_{i,j}} \delta^i)(x_i)\Big) \le \beta - 1 \\
\Leftrightarrow & & \exists \underline{d} : \exists \underline{a} : \exists \underline{x} : \exists \underline{\beta} \in \sigma_r^{-1}(\beta - 1) : \forall 1 \le i \le r : \ & v_p\Big( (\prod_{j=1}^{d_i} \Delta_{a_{i,j}} \delta^i)(x_i)\Big) \le \beta_i \\
\Leftrightarrow & & \exists \underline{d} : \exists \underline{\beta} : \forall i : \exists \underline{a_i} \in A_i^{d_i} : \exists x_i \in A_i : \ & (\prod_{j=1}^{d_i} \Delta_{a_{i,j}} \delta^i_{\beta_i+1})(x_i) \ne 0 \in Z_{p^{\beta_i+1}} \\
\Leftrightarrow & & \exists \underline{\beta} : \exists \underline{d} : \forall i : \ & \mathrm{fdeg}(\delta^i_{\beta_i+1}) \ge d_i \\
\Leftrightarrow & & \exists \underline{\beta} : \ & \sum_{i=1}^r \mathrm{fdeg}(\delta^i_{\beta_i+1}) \ge d \\
\Leftrightarrow & & & \max_{\underline{\beta}} \sum_{i=1}^r \delta(A_i, Z_{p^{\beta_i+1}}) \ge d. \qquad \square
\end{aligned}$$

4.5. **Computation of $\delta(A, B)$.** The following result computes $\delta(A, B)$ for all nontrivial commutative groups $A$ and $B$, answering a question of Aichinger-Moosbauer. When $\delta(A, B) = \infty$ we specify whether every $f \in B^A$ has finite functional degree or whether there are functions of degree $\infty$.

**Theorem 4.9.** *Let $A$ and $B$ be nontrivial commutative groups.*

a) *If $A$ is infinite, then $\operatorname{fdeg}(\delta_{0,b}) = \infty$ for all $b \in B \backslash \{0\}$, in particular, $\delta(A,B) = \infty$.*

b) *If there is no $p \in \mathcal{P}$ such that $A$ is a finite $p$-group and $B$ is a $p$-group, then $\operatorname{fdeg}(\delta_{0,b}) = \infty$ for some $b \in B$, in particular, $\delta(A,B) = \infty$.*

c) *If, for some $p \in \mathcal{P}$, $A$ is a finite $p$-group, say $A \cong \bigoplus_{i=1}^{r} Z_{p^{\alpha_i}}$ with $\alpha_1 \geq \ldots \geq \alpha_r$, and $B$ is a $p$-group of finite exponent $p^\beta$, then*

$$\delta(A,B) = \sum_{j=1}^{r} (p^{\alpha_j} - 1) + (\beta - 1)(p-1)p^{\alpha_1 - 1}.$$

d) *If, for some $p \in \mathcal{P}$, $A$ is a finite $p$-group and $B$ is a $p$-group of infinite exponent, then every $f \in B^A$ has finite functional degree but $\delta(A,B) = \infty$.*

*Proof.* a) Let $b \in B \setminus \{0\}$. Since $\delta_{0,b}(A) = \{0, b\} \subset \langle b \rangle$, Corollary 3.10b) reduces us to the case $B = \langle b \rangle$. We may even assume $B = Z_p$ and $b = 1 \in Z_p$, for a suitable $p \in \mathcal{P}$. This is because there is always a prime $p$ such that a homomorphism $\mu : \langle b \rangle \to Z_q$ with $\mu(b) = 1$ exists, and then Lemma 3.9b) tells us that $\delta_{0,b} : A \to \langle b \rangle$ has infinite degree if $\delta_{0,1} = \mu \circ \delta_{0,b} : A \to Z_q$ has infinite degree. So, we only need to show that $\delta_{0,1} : A \to Z_q$ has infinite degree, if $A$ is infinite.

*Case 1, $A$ has infinite exponent:* In this case, $A$ contains an element $a_n$ of order greater than $n$, for every $n \in \mathbb{Z}^+$. It follows that the elements $0a_n, 1a_n, \ldots, na_n$ are pairwise different, so that

$$[\Delta_{a_n}^n \delta_{0,b}](0) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \delta_{0,b}(0 + ja_n) = (-1)^n \delta_{0,b}(0) = (-1)^n b \neq 0,$$

as in Lemma 2.2. Hence, $\operatorname{fdeg}(\delta_{0,1}) = \infty$, indeed.

*Case 2, $A$ is infinite and has finite exponent:* In this case, $\operatorname{rank}(A) = \infty$. So, for every $n \in \mathbb{Z}^+$ there exists a subgroup $\underline{A}$ of $A$ with $\operatorname{rank}(\underline{A}) = n$. Applying first Proposition 4.4b), then Theorem 4.1 and then Lemma 3.5d), we see that

$$\operatorname{fdeg}(\delta_{0,1}) = \delta(\underline{A}, Z_p) = \nu(Z_p[\underline{A}]) - 1 \geq \operatorname{rank}(\underline{A}) = n.$$

Hence, $\operatorname{fdeg}(\delta_{0,1}) = \infty$, as desired. Alternatively, this also follows from the insight that, if $a_1, a_2, \ldots, a_n$ generate $\underline{A}$, then the sum $a_I := \sum_{i \in I} a_i$ over a subset $I \subset \{1, 2, \ldots, n\}$ is zero only if $I = \emptyset$, so that

$$[\Delta_{a_n} \Delta_{a_{n-1}} \cdots \Delta_{a_1} \delta_{0,1}](0) f = \sum_{I \subset \{1,\ldots,n\}} (-1)^{n-|I|} \delta_{a_I,1}(0) = (-1)^n \delta_{0,1}(0) = (-1)^n \neq 0.$$

b) We may assume that $A$ is finite, as otherwise part a) applies. So, there exists an element $a \in A$ of order $p \in \mathcal{P}$ such that $B$ is not a $p$-group. Corollary 3.10a) reduces us now further to the case $A = \langle a \rangle \cong Z_p$. Moreover, $B$ contains an element $b$ of order $q$ coprime to $p$. To show than $\operatorname{fdeg}(\delta_{0,b}) = \infty$, we may also assume $B = Z_q$ and $b = 1 \in Z_q$, exactly as in the proof of part a). In this setting, $\operatorname{Hom}(A,B) = \{0\}$ so that Corollary 2.7 implies $\operatorname{fdeg}(\delta_{0,1}) = \infty$, as desired.

c) First applying Theorem 4.8 with $A_i = Z_{p_i^{\alpha_i}}$, for $i = 1, \ldots, r$, and then applying Theorem 4.7, we get

$$\delta(A,B) = \max_{\underline{\beta}} \sum_{i=1}^{r} \delta(Z_{p^{\alpha_i}}, Z_{p^{\beta_i+1}}) = \sum_{i=1}^{r} (p^{\alpha_i} - 1) + \max_{\underline{\beta}} \sum_{i=1}^{r} \beta_i (p-1) p^{\alpha_i - 1},$$

where the maximum ranges over all $\underline{\beta} \in \mathbb{N}^r$ with $|\underline{\beta}| := \beta_1 + \ldots + \beta_r = \beta - 1$. However, for all $\underline{\beta} \in \mathbb{N}^r$ with $|\underline{\beta}| = \beta - 1$,

$$\sum_{i=1}^{r} \beta_i (p-1) p^{\alpha_i - 1} \le \sum_{i=1}^{r} \beta_i (p-1) p^{\alpha_1 - 1} = (\beta - 1)(p-1) p^{\alpha_1 - 1},$$

with equality for $\underline{\beta} = (\beta - 1, 0, \ldots, 0)$. So, $\delta(A, B) = \sum_{i=1}^{r} (p^{\alpha_i} - 1) + (\beta - 1)(p-1)p^{\alpha_1 - 1}$.
d) Let $f \in B^A$. Since $A$ is finite and $B$ is a torsion group, the subgroup $\underline{B}$ generated by $f(A)$ is finite. By part c) the function $f|^{\underline{B}} : A \to \underline{B}$ has finite degree, so that by Corollary 3.10b), we have $\operatorname{fdeg}(f) = \operatorname{fdeg}(f|^{\underline{B}}) < \infty$. Finally, since $B$ is a $p$-group of infinite exponent, $\exp(B[p^\beta]) = p^\beta$ for all $\beta \in \mathbb{Z}^+$. Thus, Theorem 4.9 c) implies

$$\delta(A, B) \ge \sup_{\beta \in \mathbb{Z}^+} \operatorname{fdeg}(A, B[p^\beta]) = \infty.$$

□

4.6. **Computation of $\mathcal{D}(A, B)$.** In this section we will compute $\mathcal{D}(A, B)$ for a class of commutative groups including all finitely generated groups $A$ and $B$. By Theorem 4.9 and Lemma 2.6, to compute $\mathcal{D}(A, B)$ for any $A$ and $B$ it remains to determine $\delta^\circ(A, B)$.

**Proposition 4.10.** *Let $A$ and $B$ be nontrivial commutative groups.*

a) *If $A$ is a torsion group and $B$ is torsion free, then $\mathcal{D}(A, B) = \{-\infty, 0, \infty\}$.*
b) *If $A$ is a torsion group, $B$ is torsion-split with $B[\mathrm{tors}] \neq \{0\}$ and $B/B[\mathrm{tors}] \neq \{0\}$, then*
$$\mathcal{D}(A, B) = \mathcal{D}(A, B[\mathrm{tors}]) \cup \{\infty\}.$$
c) *If there is a surjective homomorphism $\varepsilon : A \to \mathbb{Z}$, then $\mathcal{D}(A, B) = \widetilde{\mathbb{N}}$.*

*Proof.* a) Since $\operatorname{Hom}(A, B) = \{0\}$, this follows from Corollary 2.7.
b) We may write $B = B[\mathrm{tors}] \times B'$. Let $\pi_1 : B \to B[\mathrm{tors}]$ and $\pi_2 : B \to B'$ be the two projection maps. Then for $f \in B^A$, Lemma 3.11 tells us that

$$\operatorname{fdeg}(f) = \max(\operatorname{fdeg}(\pi_1 \circ f), \operatorname{fdeg}(\pi_2 \circ f)).$$

With this equation it is easy to verify that $\mathcal{D}(A, B) = \mathcal{D}(A, B[\mathrm{tors}]) \cup \{\infty\}$ by considering all possible degrees of functions $f_1 : A \to B[\mathrm{tors}]$ and $f_2 : A \to B'$, independently. By part a), $\mathcal{D}(A, B') = \{-\infty, 0, \infty\}$, as $B' \neq \{0\}$. So, $\mathcal{D}(A, B') \cap \mathbb{N} = \{0\}$ and this already shows that $\mathcal{D}(A, B) \cap \mathbb{N} = \mathcal{D}(A, B[\mathrm{tors}]) \cap \mathbb{N}$, as $\mathcal{D}(A, B[\mathrm{tors}]) \cap \mathbb{N} \neq \emptyset$. That $\mathcal{D}(A, B) \cap \{-\infty, +\infty\} = \mathcal{D}(A, B[\mathrm{tors}]) \cap \{-\infty, +\infty\} \cup \{\infty\}$ follows from $\infty \in \mathcal{D}(A, B')$.
c) Evidently $-\infty, 0 \in \mathcal{D}(A, B)$, and by Theorem 4.9a) we have $\infty \in \mathcal{D}(A, B)$. Let $d \in \mathbb{Z}^+$. Consider first the case $A = B = \mathbb{Z}$. We claim that the map $x \mapsto \binom{x}{d}$ from $\mathbb{Z}$ to $\mathbb{Z}$ has functional degree $d$. Indeed, by "Pascal's Identity" we have, for all $x \in \mathbb{Z}$, that

$$\Delta_1 \binom{x}{d} = \binom{x}{d-1}.$$

Hence, for a fixed $b_0 \in B \setminus \{0\}$, the map $g_d : \mathbb{Z} \to B$ given by $x \mapsto \binom{x}{d} b_0$ has functional degree $d$. So, by Lemma 3.9a), the map $\varepsilon^* g : A \to B$ also has functional degree $d$. □

Proposition 4.10 reduces the computation of $\mathcal{D}(A, B)$ for $A$ and $B$ finitely generated to the case in which $A$ and $B$ are finite. The next result accomplishes this for all torsion groups $A$ and $B$ such that $\exp(A[p^\infty])$ is finite for all $p \in \mathcal{P}$, hence in particular for all finite groups $A$.

**Theorem 4.11.** *Let $A$ and $B$ be nontrivial torsion groups such that $\exp(A[p^\infty])$ is finite for all $p \in \mathcal{P}$.*

a) *We have*
$$\delta^\circ(A,B) = \sup_{p\in\mathcal{P}} \delta^\circ(A[p^\infty], B[p^\infty]).$$

b) *If there are infinitely many $p \in \mathcal{P}$ such that $A[p^\infty]$ and $B[p^\infty]$ are both nonzero, then*
$$\delta^\circ(A,B) = \infty.$$

c) *If there is $p \in \mathcal{P}$ such that $A[p^\infty] \neq 0$ and $\exp(B[p^\infty]) = \infty$, then $\delta^\circ(A,B) = \infty$.*

d) *If there is $p \in \mathcal{P}$ such that $A[p^\infty]$ is infinite and $B[p^\infty] \neq 0$, then $\delta^\circ(A,B) = \infty$.*

e) *In the remaining case – that is, the set of $p \in \mathcal{P}$ such that $A[p^\infty]$ and $B[p^\infty]$ are both nonzero is finite and for each such $p$, $A[p^\infty]$ is finite and $B[p^\infty]$ has finite exponent – the parameter $\delta^\circ(A,B) = \sup_{p\in\mathcal{P}} \delta^\circ(A[p^\infty], B[p^\infty])$ is finite, and it can be computed using Theorem 4.9c).*

*Proof.* a) This follows from Theorem 3.13.

b) If for $p \in \mathcal{P}$ we have that $A[p^\infty]$ and $B[p^\infty]$ are both nonzero, then there is a surjective group homomorphism $\varepsilon : A \to Z_p$ (here we use that $A[p^\infty]$ has finite exponent) and an injective group homomorphism $\mu : Z_p \hookrightarrow B$, so by Theorem 4.9c), we have
$$\{-\infty, 0, \ldots, p-1\} = \mathcal{D}(Z_p, Z_p) \subset \mathcal{D}(A,B).$$
Since this holds for infinitely many $p \in \mathcal{P}$, we get $\delta^\circ(A,B) = \infty$.

c) There is a surjective group homomorphism $\varepsilon : A \to Z_p$ and for all $\beta \in \mathbb{Z}^+$ an injective group homomorphism $\iota : Z_{p^\beta} \hookrightarrow B$, so as above $\delta^\circ(A,B) \geq \delta^\circ(Z_p, Z_{p^\beta})$. By Theorem 4.9c), we have $\sup_\beta \delta^\circ(Z_p, Z_{p^\beta}) = \infty$, so $\delta^\circ(A,B) = \infty$.

d) It follows from Theorem 2.1 that if $A[p^\infty]$ is infinite and of bounded exponent then for each $d \in \mathbb{Z}^+$ there is a surjective homomorphism $\varepsilon : A \to \bigoplus_{i=1}^d Z_p$ and an injective group homomorphism $\mu : Z_p \hookrightarrow B$. Using Theorem 4.9c) as above, it follows that
$$\delta^\circ(A,B) \geq \sup_d \delta\left(\bigoplus_{i=1}^d Z_p, Z_p\right) = \infty.$$

e) This follows from Theorem 4.9c) and part a). □

**Example 4.12.** *For $p \in \mathcal{P}$, let $C_{p^\infty} = \mathbb{C}^\times[p^\infty]$ be the* ***Prüfer $p$-group****, a $p$-group of infinite exponent. As the identity map is a nonzero homomorphism, we have $\delta^\circ(C_{p^\infty}, C_{p^\infty}) \geq 1$, but we do not know more. This explains the need for the hypothesis that $\exp(A[p^\infty])$ is finite for all $p \in \mathcal{P}$ in Theorem 4.11.*

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