ABSOLUTE CONVERGENCE IN ORDERED FIELDS

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Abstract. We explore the distinction between convergence and absolute convergence of series in both Archimedean and non-Archimedean ordered fields and find that the relationship between them is closely connected to sequential (Cauchy) completeness.

1. Introduction

A real series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if the "absolute series" $\sum_{n=1}^{\infty} |a_n|$ converges. This is a strange and rather sneaky terminology; many calculus students have heard "the series is absolutely convergent" as "absolutely, the series is convergent". The language strongly (and somewhat subliminally) hints that an absolutely convergent series should converge. Fortunately this holds. Since $\sum_{n=1}^{\infty} |a_n|$ converges, the partial sums form a Cauchy sequence – for all $\epsilon > 0$, there is $N \in \mathbb{Z}^+$ such that for all $n \geq N$ and $m \geq 0$, we have $\sum_{k=n}^{n+m} |a_k| < \epsilon$. Thus

$$\left| \sum_{k=n}^{m+n} a_k \right| \leq \sum_{k=n}^{m+n} |a_k| < \epsilon,$$

and $\sum_{n=1}^{\infty} a_n$ converges by the Cauchy criterion.

Above we used the convergence of Cauchy sequences – i.e., that $\mathbb{R}$ is **sequentially complete**. In the spirit of "real analysis in reverse" – c.f. [5] and [3] – it is natural to ask about convergence versus absolute convergence in an arbitrary ordered field. In a recent note, Kantrowitz and Schramm ask whether in an ordered field F, every absolutely convergent series is convergent if and only if F is sequentially complete [2, Question 3]. We will answer this question and also determine the ordered fields in which every convergent series is absolutely convergent.

Let us begin with a taxonomic refresher on ordered fields. An ordered field F is **Dedekind complete** if every nonempty subset S of F that is bounded above admits a least upper bound. Dedekind complete ordered fields exist [1, §5], and if F_1 and F_2 are Dedekind complete ordered fields, there is a unique field homomorphism $\iota : F_1 \to F_2$, which is moreover an isomorphism of ordered fields [1, Cor. 3.6, Cor. 3.8]. This essentially unique Dedekind complete ordered field is, of course, denoted by $\mathbb{R}$ and called the field of real numbers.

For $x, y \in F$, we write $x \ll y$ if $n|x| < |y|$ for all $n \in \mathbb{Z}^+$. An ordered field F is **non-Archimedean** if there is $x \in F$ with $1 \ll x$ – equivalently, if $\mathbb{Z}^+$ is bounded above in F – otherwise F is **Archimedean**. Notice that for $x, y \in F$ with $x \neq 0$, $x \ll y$ holds if and only if $1 \ll \frac{y}{x}$. Thus F is non-Archimedean if and only if $x \ll y$ for some $x, y \in F$ with $x \neq 0$.

Example. The formal Laurent series field $\mathbb{R}((t))$ (see e.g. [3, § 3]) has a unique ordering extending the ordering on $\mathbb{R}$ in which $\frac{1}{t}$ is greater than every real number. In particular $1 \ll \frac{1}{t}$, so $\mathbb{R}((t))$ is non-Archimedean.

Explicitly, every nonzero $a \in \mathbb{R}((t))$ has the form

$$a = \sum_{m=M}^{\infty} b_m t^m, \ b_M \neq 0, \tag{1}$$

and then a has the same sign as b_M. Put $v(a) = M$ and also put $v(0) = \infty$. Then a sequence $\{a_n\}$ in $\mathbb{R}((t))$ converges to 0 if and only if $v(a_n)$ converges to ∞.

Let $\{a_n\}$ be a Cauchy sequence in $\mathbb{R}((t))$. Then $\{a_n\}$ is bounded; there is $M \in \mathbb{Z}$ such that for all $n \in \mathbb{Z}^+$ we may write

$$a_n = \sum_{m=M}^{\infty} b_{m,n} t^m.$$

The Cauchy condition also implies that for every $m \geq M$, the real sequence $\{b_{m,n}\}_{n=1}^{\infty}$ is eventually constant, say with value b_m, and thus $\{a_n\}$ converges to $\sum_{m=M}^{\infty} b_m t^m$. Thus $\mathbb{R}((t))$ is sequentially complete. Similarly, a series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $a_n \to 0$, so absolute convergence is equivalent to convergence in $\mathbb{R}((t))$.

An ordered field F is Archimedean if and only if there is a homomorphism $\iota : (F, <) \hookrightarrow (\mathbb{R}, <)$ [1, Thm. 3.5], i.e., if and only if F is isomorphic to a subfield of $\mathbb{R}$ with the induced ordering. An ordered field is Dedekind complete if and only if it is Archimedean and sequentially complete [1, Lemma 3.10, Thm. 3.11]. It follows that the Archimedean ordered fields that are not (sequentially = Dedekind) complete are, up to isomorphism, precisely the proper subfields of $\mathbb{R}$.

Now we can state our main result.

Main Theorem. *Let F be an ordered field.*
a) Suppose F is sequentially complete and Archimedean (so $F \cong \mathbb{R}$). Then:
(i) every absolutely convergent series in F is convergent;
(ii) F admits a convergent series that is not absolutely convergent.
b) Suppose F is sequentially complete and non-Archimedean. Then:
(i) a series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $a_n \to 0$. In particular,
(ii) a series is absolutely convergent if and only if it is convergent.
c) Suppose F is not *sequentially complete. Then:*
(i) F admits an absolutely convergent series that is not convergent;
(ii) F admits a convergent series that is not absolutely convergent.

Part c) gives an affirmative answer to Question 3 of [2].

Part a) of the Main Theorem is familiar from calculus. Part (i) has already been recalled, and for part (ii) we need only exhibit the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. We have included part a) to facilitate comparison with the other cases.

A natural next step is to finish off the case of Archimedean ordered fields by establishing part c) for proper subfields of $\mathbb{R}$. We do so in §2, using arguments that could be presented in an undergraduate honors calculus / real analysis course. (This turns out to be logically superfluous; later we will prove part c) of the Main Theorem for *all* ordered fields. But it still seems like an agreeable way to begin.)

For the remainder of the Main Theorem we need some techniques for constructing sequences in ordered fields. In §3 we examine and classify ordered fields with respect to the existence of sequences of various types. For instance, there are ordered fields in which every convergent sequence is eventually constant. In such a field, a series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $a_n = 0$ for all sufficiently large n if and only if $\sum_{n=1}^{\infty} |a_n|$ is convergent. Thus, in order for the Main Theorem to hold in this case, such a field must be sequentially complete. We will show that every convergent sequence is eventually constant if and only if every Cauchy sequence is eventually constant. In fact we give (Theorem 6) four other conditions equivalent to "there is a convergent sequence that is not eventually constant", including a characterization in terms of the associated topology and a characterization in terms of the cofinality of the underlying ordered set.

We prove parts b) and c) of the Main Theorem in § 4.

2. Subfields of $\mathbb{R}$

Theorem 1. *Let $F \subsetneq \mathbb{R}$ be a proper subfield. There is a series $\sum_{n=1}^{\infty} a_n$ with terms in F that is absolutely convergent but not convergent.*

Proof. Since F is a proper subfield of $\mathbb{R}$, we may choose $x \in [-1,1] \setminus F$. We claim there is a *sign sequence* $\{s_n\}_{n=1}^{\infty}$ – i.e., $s_n \in \{\pm 1\}$ for all n – such that $x = \sum_{n=1}^{\infty} \frac{s_n}{2^n}$. Indeed, for $N \geq 0$, take s_{N+1} to be 1 if $\sum_{n=1}^{N} \frac{s_n}{2^n} < x$ and -1 if $\sum_{n=1}^{N} \frac{s_n}{2^n} > x$. Then we see inductively that $|\sum_{n=1}^{N} \frac{s_n}{2^n} - x| \leq 2^{-N}$, which implies convergence. In fact the sign sequence $\{s_n\}$ is *uniquely determined*; if we take $s_{N+1} = -1$ when $\sum_{n=1}^{N} \frac{s_n}{2^n} < x$ or $S_{N+1} = 1$ when $\sum_{n=1}^{N} \frac{s_n}{2^n} > x$, then we get

$$\left| \sum_{n=1}^{N+1} \frac{s_n}{2^n} - x \right| > 2^{-N-1} = \sum_{n=N+2}^{\infty} 2^{-n} \geq \left| \sum_{n=N+2}^{\infty} \frac{s_n}{2^n} \right|,$$

and we have made an irrevocable mistake! The series $\sum_{n=1}^{\infty} \frac{s_n}{2^n}$ has terms in F and is not convergent in F, but the associated absolute series is $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \in F$. □

The proof of Theorem 1 is reminiscent of that of the **Riemann Rearrangement Theorem**, which states that a real series $\sum_{n=1}^{\infty} a_n$ with $a_n \to 0$ and $\sum_{n=1}^{\infty} |a_n| = \infty$ can be *rearranged* so as to converge to any $L \in \mathbb{R}$. The following result is a variant in which, instead of permuting the terms of a series, we change their signs.

Theorem 2. *Let $\{a_n\}_{n=1}^{\infty}$ be a positive real sequence with $a_n \to 0$ and $\sum_{n=1}^{\infty} a_n = \infty$. For $L \in \mathbb{R}$, there is a sign sequence $s_n \in \{\pm 1\}$ such that $\sum_{n=1}^{\infty} s_n a_n = L$.*

Proof. We may assume $L \in [0, \infty)$. Let N_1 be the least positive integer such that $\sum_{n=1}^{N_1} a_n > L$, and put $s_1 = \ldots = s_{N_1} = 1$. Let N_2 be the least integer greater than N_1 such that $\sum_{n=1}^{N_1} s_n a_n - \sum_{n=N_1+1}^{N_2} a_n < L$, and put $s_{N_1+1} = \ldots = s_{N_2} = -1$. We continue in this manner, taking just enough terms of constant sign to place the partial sum on the opposite side of L as the previous partial sum. The condition $\sum_{n=1}^{\infty} a_n = \infty$ ensures this is well-defined, and the condition $a_n \to 0$ guarantees that the resulting series $\sum_{n=1}^{\infty} s_n a_n$ converges to L. □

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is defined in any subfield $F \subset \mathbb{R}$ and is not absolutely convergent. However, since $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2 \in \mathbb{R} \setminus \mathbb{Q}$, this series need not be convergent in F. Using Theorem 2 we can fix this with a different choice of signs.

Corollary 3. *Let $F \subseteq \mathbb{R}$ be a subfield. Then there is a series in F that is convergent but not absolutely convergent.*

Proof. Apply Theorem 2 with $a_n = \frac{1}{n}$ for all $n \in \mathbb{Z}^+$, and $L = 1$. □

The constructions of this section are intended to complement those of [2]. In particular, the proof of Theorem 1 answers their Question 1 for $b = 2$.

3. Building Sequences and Series in Ordered Fields

We begin by carrying over two results from calculus to the context of ordered fields.

Lemma 4. *Let $(X, <)$ be a totally ordered set, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. Then at least one of the following holds:*
(i) the sequence $\{x_n\}_{n=1}^{\infty}$ admits a constant subsequence;
(ii) The sequence $\{x_n\}_{n=1}^{\infty}$ admits a strictly increasing subsequence;
(iii) The sequence $\{x_n\}_{n=1}^{\infty}$ admits a strictly decreasing subsequence.

Proof. If the image of the sequence is finite, we may extract a constant subsequence. So assume the image is infinite. By passing to a subsequence we may assume $n \mapsto x_n$ is injective. We say $m \in \mathbb{Z}^+$ is a **peak** of the sequence if for all $n > m$, we have $x_n < x_m$. If there are infinitely many peaks, the sequence of peaks forms a strictly decreasing subsequence. So suppose there are only finitely many peaks and thus there is $N \in \mathbb{Z}^+$ such that no $n \geq N$ is a peak. Let $n_1 = N$. Since n_1 is not a peak, there is $n_2 > n_1$ with $x_{n_2} > x_{n_1}$. Since n_2 is not a peak, there is $n_3 > n_2$ with $x_{n_3} > x_{n_2}$. Continuing in this way we build a strictly increasing subsequence. □

Lemma 5. *Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the ordered field F. If $\{a_n\}_{n=1}^{\infty}$ admits a convergent subsequence, then it converges.*

Proof. Let $\{a_{n_k}\}_{k=1}^{\infty}$ be a subsequence converging to $L \in F$. For $\epsilon > 0$, choose $K \in \mathbb{Z}^+$ such that $|a_m - a_n| < \frac{\epsilon}{2}$ for all $m, n \geq K$ and $|a_{n_k} - L| < \frac{\epsilon}{2}$ for all $k \geq K$. Put $n_K \geq K$; then $|a_n - L| \leq |a_n - a_{n_K}| + |a_{n_K} - L| < \epsilon$ for all $n \geq K$. □

Now we introduce an invariant of a linearly ordered set that plays an important role in the theory of ordered fields. A subset S of a linearly ordered set X is **cofinal** if for all $x \in X$ there is $y \in S$ with $x \leq y$. The **cofinality** of X is the least cardinality of a cofinal subset. An ordered field is Archimedean if and only if $\mathbb{Z}^+$ is a cofinal subset, so Archimedean ordered fields have countable cofinality. The subset $\{t^{-n}\}_{n=1}^{\infty}$ of $\mathbb{R}((t))$ is countable and cofinal, so $\mathbb{R}((t))$ is non-Archimedean of countable cofinality. For any regular infinite cardinal κ there is an ordered field of cofinality κ [4, Cor. 2.7].

A **Z-sequence** in F is a sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n > 0$ for all n and $a_n \to 0$. A **ZC-sequence** is a Z-sequence $\{a_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} a_n$ converges.

Theorem 6. *For an ordered field F, the following are equivalent:*
(0) F is first countable (every point admits a countable base of neighborhoods);
(i) F has countable cofinality;
(ii) F admits a convergent sequence that is not eventually constant;

(iii) F admits a ZC-sequence;
(iv) F admits a Z-sequence;
(v) F admits a Cauchy sequence that is not eventually constant.

Proof. (0) $\implies$ (i): Let $\{U_n\}_{n=1}^{\infty}$ be a countable neighborhood base at 0. For $n \in \mathbb{Z}^+$, choose $\epsilon_n > 0$ such that $(-\epsilon_n, \epsilon_n) \subset U_n$. Then $\{\frac{1}{\epsilon_n} \mid n \in \mathbb{Z}^+\}$ is cofinal.
(i) $\implies$ (o): Let $\{s_n\}_{n=1}^{\infty}$ be a cofinal sequence of positive elements. Let $\epsilon_n = \frac{1}{s_n}$ and $U_n = (-\epsilon_n, \epsilon_n)$. Then $\{U_n\}_{n=1}^{\infty}$ is a countable base at zero, and thus for all $x \in F$, the collection $\{U_n + x\}_{n=1}^{\infty}$ is a countable base at x. Thus F is first countable.
(i) $\implies$ (ii): Let $S = \{s_n\}_{n=1}^{\infty}$ be a cofinal sequence. Put $a_1 = \max(1, s_1)$. Having defined a_n, put $a_{n+1} = \max(a_n + 1, s_{n+1})$. Then $\{a_n\}_{n=1}^{\infty}$ is a strictly increasing sequence whose set of terms is a cofinal subset. The sequence $\{a_n^{-1}\}_{n=1}^{\infty}$ converges to 0 and is not eventually constant.
(ii) $\implies$ (iii): Let $\{x_n\}_{n=1}^{\infty}$ be a sequence that is convergent and is not eventually constant; its set of terms must then be infinite. By Lemma 4, $\{x_n\}_{n=1}^{\infty}$ has a subsequence that is either strictly increasing or strictly decreasing; by changing the signs of all terms if necessary and adding a constant we get a strictly increasing convergent sequence $0 < S_1 < S_2 < \ldots$. Put $S_0 = 0$, and for $n \in \mathbb{Z}^+$, put $a_n = S_n - S_{n-1}$. Then $\{a_n\}_{n=1}^{\infty}$ is a ZC-sequence.
(iii) $\implies$ (iv) $\implies$ (v) is immediate.
(v) $\implies$ (i): If $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence and not eventually constant, then

$$\left\{ \frac{1}{|a_m - a_n|} \;\middle|\; m, n \in \mathbb{Z}^+, a_m \neq a_n \right\}$$

is countable and cofinal. Let $\alpha > 0$ in F. There is an $N \in \mathbb{Z}^+$ such that for all $m, n \geq N$, we have $|a_m - a_n| \leq \frac{1}{\alpha}$. Since the sequence is not eventually constant, there are $m, n \geq N$ with $a_m \neq a_n$, and then $\alpha < \frac{1}{|a_m - a_n|}$. □

Theorem 6 is a key step towards establishing the Main Theorem. For starters, it disposes of the case of ordered fields of uncountable cofinality; by (i) $\iff$ (v), such fields must be sequentially complete. Moreover, by (i) $\iff$ (ii) an infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if $a_n = 0$ for all sufficiently large n if and only if $a_n \to 0$. And in the case of countable cofinality it gives us some useful sequences.

Consider the sequence $\{t^n\}_{n=1}^{\infty}$ in $\mathbb{R}((t))$. It converges to 0, and for all $n \in \mathbb{Z}^+$, we have $t^{n+1} \ll t^n$. One can use the sequence $\{t^n\}_{n=1}^{\infty}$ to "test for convergence"; a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}((t))$ converges if and only if for each $N \in \mathbb{Z}^+$, we have $|a_n| \leq t^N$ for all sufficiently large n.

Here is a useful generalization to arbitrary ordered fields. A **test sequence** is a Z-sequence $\{\epsilon_n\}_{n=1}^{\infty}$ such that $\epsilon_{n+1} \ll \epsilon_n$ for all $n \in \mathbb{Z}^+$.

Proposition 7. *For an ordered field F, the following are equivalent:*
(i) F admits a test sequence;
(ii) F is non-Archimedean of countable cofinality.

Proof. (i) $\implies$ (ii): The existence of $0 < \epsilon_2 \ll \epsilon_1$ shows F is non-Archimedean. The implication (iv) $\implies$ (i) of Theorem 6 shows that F has countable cofinality.
(ii) $\implies$ (i): Let $S = \{s_n\}_{n=1}^{\infty} \subset F$ be countable and cofinal. Since F is non-Archimedean, there is $x_1 \in S$ with $1 \ll x_1$ and $s_1 \leq x_1$. Then $x_1 \ll x_1^2$, and by cofinality there is $x_2 \in S$ such that $x_2 \geq \max(x_1^2, s_2)$. Continuing in this manner we get a sequence $\{x_n\}_{n=1}^{\infty}$. Taking $\epsilon_n = x_n^{-1}$ gives a test sequence. □

4. Proof of the Main Theorem

We will now prove part b) of the Main Theorem.

Let F be a sequentially complete non-Archimedean field. First we must show that a series $\sum_{n=1}^{\infty} a_n$ in F is convergent if and only if $a_n \to 0$. That a convergent series has $a_n \to 0$ follows (as usual) from the fact that a convergent sequence is a Cauchy sequence. Suppose $a_n \to 0$. If F has uncountable cofinality, then by the remarks following Theorem 6, a series $\sum_{n=1}^{\infty} a_n$ converges if and only if $a_n = 0$ for all large enough n, hence if and only if $a_n \to 0$. Now suppose F has countable cofinality. By Proposition 7, there is a test sequence $\{\epsilon_n\}_{n=1}^{\infty}$. For $k \in \mathbb{Z}^+$, choose N_k such that for all $n \geq N_k$, we have $|a_n| \leq \epsilon_{k+1}$. Then for all $n \geq N_k$ and $\ell \geq 0$, we have

$$|a_n + a_{n+1} + \ldots + a_{n+\ell}| \leq |a_n| + \ldots + |a_{n+\ell}| \leq (\ell + 1)\epsilon_{k+1} < \epsilon_k.$$

Thus the sequence is a Cauchy sequence, and hence convergent because F is sequentially complete. The fact that a series in F is convergent if and only if it is absolutely convergent follows immediately, since $a_n \to 0$ if and only if $|a_n| \to 0$.

Before proving part c) of the Main Theorem, we want to build one more type of sequence. A **ZD-sequence** is a Z-sequence $\{a_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} a_n$ diverges.

Lemma 8. *For an ordered field F, the following are equivalent:*
(i) F admits a ZD-sequence;
(ii) F is Archimedean or is not *sequentially complete.*

Proof. (i) $\implies$ (ii): Let $\{a_n\}_{n=1}^{\infty}$ be a ZD-sequence in F. By part b) of the Main Theorem, F cannot be non-Archimedean and sequentially complete.
(ii) $\implies$ (i): If F is Archimedean, $\{\frac{1}{n}\}_{n=1}^{\infty}$ is a ZD-sequence. Suppose F is non-Archimedean and not sequentially complete, and let $\{a_n\}_{n=1}^{\infty}$ be a divergent Cauchy sequence. By Lemmas 4 and 5, after passing to a subsequence and possibly changing the sign, we get a strictly increasing, divergent Cauchy sequence $\{S_n\}_{n=1}^{\infty}$. Put $S_0 = 0$, and for $n \in \mathbb{Z}^+$, put $a_n = S_n - S_{n-1}$. Then $\{a_n\}_{n=1}^{\infty}$ is a ZD-sequence. $\square$

Finally we can prove both assertions of part c) of the Main Theorem.

Theorem 9. *For an ordered field F, the following are equivalent:*
(i) F is sequentially complete;
(ii) Every absolutely convergent series in F converges.

Proof. (i) $\implies$ (ii): This was proved at the beginning of §1.
$\neg$ (i) $\implies$ $\neg$ (ii): Let $\{a_n\}_{n=1}^{\infty}$ be a divergent Cauchy sequence in F. By Theorem 6 there is a ZC-sequence $\{c_k\}_{k=1}^{\infty}$. Since $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence, there is a strictly increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that for all $n \geq n_k$, we have $|a_n - a_{n_k}| < c_k$. It follows that $|a_{n_{k+1}} - a_{n_k}| < c_k$ for all k. By Lemma 5, $\{a_{n_k}\}_{k=1}^{\infty}$ is divergent hence so is $\{a_{n_k} - a_{n_1}\}_{k=1}^{\infty}$. For $k \in \mathbb{Z}^+$, put

$$d_{2k-1} = \frac{a_{n_{k+1}} - a_{n_k} + c_k}{2}, \text{ and}$$

$$d_{2k} = \frac{a_{n_{k+1}} - a_{n_k} - c_k}{2}.$$

Then

$$\sum_{i=1}^{k} (d_{2i-1} + d_{2i}) = \sum_{i=1}^{k} \left(a_{n_{i+1}} - a_{n_i}\right) = a_{n_{k+1}} - a_{n_1}.$$

This is a divergent subsequence of the sequence of partial sums associated to $\{d_k\}_{k=1}^{\infty}$, and hence $\sum_{k=1}^{\infty} d_k$ diverges. Since $-c_k < a_{n_{k+1}} - a_{n_k} < c_k$, we have

$$|d_{2k-1}| + |d_{2k}| = d_{2k-1} - d_{2k} = c_k.$$

Hence $\sum_{k=1}^{\infty} |d_k| = \sum_k c_k$ is convergent, i.e., $\sum_{k=1}^{\infty} d_k$ is absolutely convergent. □

Theorem 10. *Let F be an ordered field that is not sequentially complete. Then F admits a convergent series that is not absolutely convergent.*

Proof. By Lemma 8, F admits a ZD-sequence $\{d_n\}_{n=1}^{\infty}$. For $n \in \mathbb{Z}^+$, put

$$a_{2n-1} = \frac{d_n}{2},\ a_{2n} = \frac{-d_n}{2}.$$

Then for all $n \in \mathbb{Z}^+$ we have

$$0 \le \sum_{k=1}^{n} a_k \le \frac{d_{\lceil \frac{n}{2} \rceil}}{2},$$

so $\sum_{n=1}^{\infty} a_n$ converges (to 0). But $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} d_n$ diverges. □

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References

[1] J.F. Hall, Completeness of Ordered Fields (2011), available at http://arxiv.org/pdf/1101.5652v1.pdf.

[2] R. Kantrowitz, M. Schramm, Series that Converge Absolutely but Don't Converge. *College Math. J.* **43** (2012) 331–333.

[3] J. Propp, Real Analysis in Reverse. *Amer. Math. Monthly* **120** (2013) 392–408.

[4] J.H. Schmerl, Models of Peano Arithmetic and a Question of Sikorski on Ordered Fields. Israel J. Math. **50** no. 1-2 (1985) 145–159.

[5] H. Teismann, Toward a More Complete List of Completeness Axioms. *Amer. Math. Monthly* **120** (2013) 99–114.