Theorem 1. (Cassels’ Lemma) Let $K$ be a complete local field with finite residue field $\mathbb{F}_q$. Let $\Psi : A \to A'$ be an isogeny of abelian varieties defined over $K$. We assume that $A$ (hence also $A'$) has good reduction and that $\deg \Psi$ is coprime to $q$. Then the image $\mathcal{K}$ of the Kummer map
\[ A'(K)/\Psi(A(K)) \to H^1(K, A[\Psi]) \]
is equal to the subgroup $H^1(K_{unr}/K, A[\Psi])$ of $H^1(K, A[\Psi])$ consisting of classes killed by restriction to the maximal unramified extension $K_{unr}$ of $K$.

First proof: We have a commutative diagram with exact rows
\[
\begin{array}{c}
0 \\
\Rightarrow \\
\Rightarrow \\
A'(K)/\Psi(A(K)) \to H^1(K, A[\Psi]) \xrightarrow{\beta} H^1(K, A[\Psi]) \\
\Rightarrow \\
\Rightarrow \\
A'(K_{unr})/\Psi(A(K_{unr})) \xrightarrow{\beta'} H^1(K_{unr}, A[\Psi]) \xrightarrow{\beta'} H^1(K_{unr}, A[\Psi]) \Rightarrow 0.
\end{array}
\]

Step 1: Let $\xi \in H^1(K, A[\Psi])$ be a class which is in the image of the Kummer map; equivalently, $\beta(\xi) = 0$. We will show that $\text{res} \, \xi = 0$. Because of the commutativity of the diagram, we have $\beta' \circ \text{res} \, \xi = \text{res} \, \beta \circ \xi = 0$. We claim that $\beta'$ is injective, which suffices. Indeed, we will show:

Lemma 2. With hypotheses above, we have
\[ A'(K_{unr})/\Psi(A(K_{unr})) = 0. \]

Proof. Since $A[\Psi] \subset A[\deg \Psi]$, it suffices to show the result for $\Psi = [n]$ on $A$ with $n$ a positive integer coprime to the residue characteristic, $p$. Let $\tilde{A}$ be the (good!) reduction of $A$ modulo the maximal ideal of $\mathcal{O}_K$. We consider the base change to $K_{unr}$: reduction gives a short exact sequence
\[ 0 \to A'(K_{unr}) \to A(K_{unr}) \to \tilde{A}(\mathbb{F}_q) \to 0. \]

Certainly $\tilde{A}(\mathbb{F}_q)$, being the group of points of a connected group scheme over an algebraically closed field, is divisible. On the other hand, the theory of formal groups gives an isomorphism
\[ A^0(K_{unr}) \cong (\mathcal{O}_{K_{unr}}, +) \bigoplus T_p, \]
where $T_p$ is a $p$-primary torsion abelian group. Since $n$ is a unit in the ring $\mathcal{O}_{K_{unr}}$, its additive group is $n$-divisible. Therefore $A(K_{unr})$, being an extension of $n$-divisible abelian groups, is itself $n$-divisible.

Step 2: So far we know that $\mathcal{K}$, the image of the Kummer map, is a subgroup of $H^1(K_{unr}/K, A[\Psi])$. It is easy to see that the latter is a finite abelian group — indeed, the set of continuous cocycles from the topologically cyclic group $\text{Gal}(K_{unr}/K)$ to the finite abelian group $A[\Psi]$ is finite. Hence $\mathcal{K}$ is also finite, and if $H \subset G$ are
finite abelian groups, to show that $H = G$ it suffices to show that $\# H = \# G$.

We recall the following result of basic Galois cohomology of $\hat{\mathbb{Z}} = \text{Gal}(K^{\text{unr}}/K)$. If $M$ is a finite $\hat{\mathbb{Z}}$-module, then

$$H^1(\hat{\mathbb{Z}}, M) \cong M/(F - 1)M,$$

where $F$ is the Frobenius map, the distinguished topological generator of $\hat{\mathbb{Z}}$. Reduction induces an isomorphism on the $n$-torsion, so $A[\psi](K) = A[\psi](K^{\text{unr}})$; we abbreviate this as $A[\psi]$ and write $A[\psi](K)$ for the subgroup of $K$-rational points, which is precisely the kernel of $F - 1$. So we have

$$H^1(K^{\text{unr}}/K, A[\Psi]) \cong A[\psi]/(F - 1)A[\Psi].$$

Next a tiny piece of pure algebra: let $M$ be a finite abelian group and $\varphi$ an endomorphism of $M$. Then contemplation of the short exact sequence

$$0 \to \ker \varphi \to M \xrightarrow{\varphi} M \to \operatorname{coker} \varphi \to 0$$

gives $\# \ker \varphi = \# \operatorname{coker} \varphi$. Applying this with $M = A[\varphi]$ and $\varphi$ equal to $F - 1$, we get

$$\# H^1(K^{\text{unr}}/K, A[\Psi]) = \# A[\Psi]/(F - 1)A[\Psi] = \# \ker \varphi = \# A[\Psi](K) = \# \tilde{A}[\Psi](\mathbb{F}_q).$$

On the other hand, because $\Psi$ is surjective on the kernel of reduction, we have

$$\# \mathcal{K} = \# \tilde{A}'(K)/\Psi(A(K)) = \# \tilde{A}'(\mathbb{F}_q) / \tilde{\Psi} \tilde{A}(\mathbb{F}_q).$$

Note that since $\tilde{A}$ and $\tilde{A}'$ are $\mathbb{F}_q$-rationally isogenous, $\# \tilde{A}(\mathbb{F}_q) = \# \tilde{A}'(\mathbb{F}_q)$. Therefore, applying a similar counting argument to the exact sequence

$$0 \to \tilde{A}(\mathbb{F}_q)[\tilde{\Psi}] \to \tilde{A}(\mathbb{F}_q) \to \tilde{A}'(\mathbb{F}_q) \to \# \tilde{A}'(\mathbb{F}_q) / \tilde{\Psi} \tilde{A}(\mathbb{F}_q) = 0,$$

we conclude

$$\# H^1(K^{\text{unr}}/K, A[\Psi]) = \# \tilde{A}(\mathbb{F}_q)[\tilde{\Psi}] = \# \tilde{A}'(\mathbb{F}_q) / \tilde{\Psi} \tilde{A}(\mathbb{F}_q) = \# \mathcal{K},$$

qed.