A NOTE ON GOLOMB TOPOLOGIES

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Abstract. In 1959 Golomb defined a connected topology on $\mathbb{Z}$. An analogous *Golomb topology* on an arbitrary integral domain was defined first by Knopfmacher-Porubský [KP97] and then again in a recent work of Clark [Cl17]. Here we study properties of Golomb topologies.

1. Introduction

In 1955, H. Furstenberg published his now-famous proof of the infinitude of the prime numbers by means of a topology on $\mathbb{Z}$ [Fu55]. Clark generalized Furstenberg's proof to show that if R is a semiprimitive domain that is not a field in which every nonzero nonunit element has at least one irreducible divisor, then R has infinitely many nonassociate irreducibles [Cl17, Thm. 3.1] by means of an *adic topology* on R. However adic topologies — while arising naturally in commutative algebra — are not so interesting *as topologies*: cf. §3.3. In [Go59] and [Go62], S.W. Golomb defined a new topology on the positive integers $\mathbb{Z}^+$. It retains enough features of Furstenberg's topology to yield a proof of the infinitude of primes — if there were only finitely many primes, then $\{1\}$ would be open, whereas nonempty open sets are infinite. But Golomb's topology makes $\mathbb{Z}^+$ into a connected Hausdorff space.

A *Golomb topology* on any domain R was defined by Knopfmacher-Porubský [KP97] and is also considered by Clark in [Cl17, §3.7]. (See §3.4 for more information on [KP97] and subsequent work of Marko-Porubský.) Clark mentions that the Golomb topolgy on R is never Hausdorff and suggests studying the induced topology on $R^\bullet \coloneqq R \setminus \{0\}$ instead.

In this note we study these Golomb topologies in detail. We find that the Golomb topology on $R^\bullet$ is Hausdorff iff R is semiprimitive. Since a PID is semiprimitive iff it has infinitely many nonassociate prime elements, we again see an interplay between arithmetic and topology. To any countably infinite semiprimitive domain that is not a field, we have associated a connected, countably infinite Hausdorff space, and we ask how many homeomorphism types of spaces arise in this way. We are not able to resolve this problem completely, but we give a partial result that is inspired by some striking recent work of Banakh-Mioduszewski-Turek [BMT17].

2. The Golomb Topology on a Domain

A *domain* is a nonzero commutative ring in which no nonzero element is a zero-divisor. For a subset S of a ring R, we put $S^\bullet = S \setminus \{0\}$.

For a domain R, let $\mathcal{B}_R$ be the family of **coprime cosets** $\{x + I\}$ where $x \in R$, I is a nonzero ideal of R, and $\langle x, I\rangle = R$. Suppose $x_1 + I_1$, $x_2 + I_2 \in \mathcal{B}_R$, and let $z \in (x_1 + I_1) \cap (x_2 + I_2)$. For $i = 1, 2$, since $z \in x_i + I_i$ we have $z - x_i \in I_i$ and thus $\langle z, I_i \rangle = R$. It follows [Cl-CA, Lemma 3.17c)] that $\langle z, I_1 I_2 \rangle = R$. Since R is a domain and I_1 and I_2 are nonzero ideals, we have $\{0\} \subsetneq I_1 I_2 \subset I_1 \cap I_2$. Thus

$$z + I_1 I_2 \in \mathcal{B}_R, \quad z + I_1 I_2 \subset (x_1 + I_1) \cap (x_2 + I_2).$$

It follows that $\mathcal{B}_R$ is the base for a topology on R, which we denote by $\widetilde{G(R)}$. The *Golomb topology* $G(R)$ is the subspace topology on $R^\bullet$.

Recall that if R is a domain that is not a field, then R is infinite and every nonzero ideal I of R is infinite (and indeed $\#I = \#R$): if $x \in I^\bullet$, then $R \to I$ by $r \mapsto rx$ is an injection. Thus, as for the topologies originally defined by Furstenberg and Golomb, nonempty open subsets in $\widetilde{G(R)}$ and in $G(R)$ are infinite.

Lemma 1. *For a domain R, the following are equivalent:*

(i) *The space $\widetilde{G(R)}$ is indiscrete (there are no proper, nonempty open subsets).*
(ii) *The space $G(R)$ is indiscrete.*
(iii) *The ring R is a field.*

Proof. (i) $\implies$ (ii): Every subspace of an indiscrete space is indiscrete.
(ii) $\implies$ (iii): We prove the contrapositive. If R is not a field, then it has a nonzero proper ideal I, and $1 + I$ is a nonempty, proper open subset of $R^\bullet$.
(iii) $\implies$ (i): For a field R, the only nonzero ideal is R itself, so for all $x \in R$ the only neighborhood of x is $x + R = R$. $\square$

For the remainder of the section, R denotes a domain that is not a field.

A point x of a topological space X is *indiscrete* if the only open neighborhood of x is X itself. (Thus X is indiscrete iff every point of X is indiscrete.)

Let M be an R-module and N an R-submodule of M. The *Jacobson radical of* $N \subset M$, denoted $\mathcal{J}_M(N)$, is the intersection of the maximal submodules of M that contain N. The *Jacobson radical of* R is $\mathcal{J}(R) \coloneqq \mathcal{J}_R((0))$, i.e., the intersection of the maximal ideals of R. We say R is *semiprimitive* if $\mathcal{J}(R) = \{0\}$.

Proposition 2. *The indiscrete points of $\widetilde{G(R)}$ [resp. of $G(R)$] are precisely the elements of $\mathcal{J}(R)$ [resp. of $\mathcal{J}(R)^\bullet$].*

Proof. First we work in $\widetilde{G(R)}$. Let $x \in \mathcal{J}(R)$. If I is a proper ideal of R, there is a maximal ideal $\mathfrak{m}$ containing I. (Since R is not a field, $\mathfrak{m} \neq \{0\}$.) Then

$$\langle x, I \rangle \subset \langle x, \mathfrak{m} \rangle = \mathfrak{m} \subsetneq R.$$

So the only open neighborhood of x in $\widetilde{G(R)}$ is $x + R = R$. If $x \in R \setminus \mathcal{J}(R)$, then there is a maximal ideal $\mathfrak{m}$ that does not contain x. Then $\langle x, \mathfrak{m} \rangle = R$, so $x + \mathfrak{m}$ is a proper open neighborhood of x in R.

Now we work in $G(R)$. If $x \in \mathcal{J}(R)^\bullet$, then x is indiscrete in $\widetilde{G(R)}$ hence also in $G(R)$. If $x \in R^\bullet \setminus \mathcal{J}(R)^\bullet$, there is a maximal ideal $\mathfrak{m}$ not containing x. Then $x + \mathfrak{m}$ is a neighborhood of x in $G(R)$; since $\mathfrak{m}^\bullet \subset R^\bullet \setminus (x + \mathfrak{m})$, we have $x + \mathfrak{m} \subsetneq R^\bullet$. $\square$

A topological space X is *Kolmogorov* if for all $x \neq y$ in X there is an open set containing exactly one of x and y. A topological space X is *separated* if for all $x \neq y$ in X there is an open set containing x and not y and also an open set containing y and not x; equivalently, for all $x \in X$, $\{x\}$ is closed.

Corollary 3. *The space $\widetilde{G(R)}$ is not separated.*

Proof. If X is a topological space with $\# X \geq 2$ and an indiscrete point 0, then X is not separated: for all $x \neq 0 \in X$, we have $0 \in \overline{\{x\}}$. $\square$

Theorem 4. *Let I be an ideal of R, and let $x \in R$.*

a) *In $\widetilde{G(R)}$, we have $\overline{x+I} \supset \mathcal{J}_R(I)$.*
b) *In $\widetilde{G(R)}$, we have $\overline{I} = \mathcal{J}_R(I)$.*
c) *Suppose $x \notin I$. Then in $G(R)$ we have $\overline{(x+I)^\bullet} \supset \mathcal{J}_R(I)^\bullet$.*

Proof. a) Let $y \in \mathcal{J}_R(I)$, and let U be a neighborhood of y. Then U contains a coprime coset $y + J$. We claim that I and J are comaximal. For if not, there is a maximal ideal $\mathfrak{m}$ of R such that $\langle I, J \rangle \subset \mathfrak{m}$. But by definition of the Jacobson radical, we must have $y \in \mathfrak{m}$ and thus $\langle y, J \rangle \subset \mathfrak{m}$, contradicting the fact that $y+J$ is a coprime coset. Since I and J are comaximal, by the Chinese Remainder Theorem [Cl-CA, Thm. 4.19] the cosets $x+I$ and $y+J$ intersect. Thus $y \in \overline{x+I}$.
b) By part a) we have $\overline{I} \supset \mathcal{J}_R(I)$. Now suppose $y \in R \setminus \mathcal{J}_R(I)$. Then there is a maximal ideal $\mathfrak{m}$ of R containing I but not y, so the coprime coset $y + \mathfrak{m}$ is disjoint from $\mathfrak{m}$ hence also from I. Thus $y \notin \overline{I}$.
c) Since $x \notin I$ we have $x + I \subset R^\bullet$. The result now follows from part a). $\square$

Theorem 5. *The following are equivalent:*

(i) *R is semiprimitive: $\mathcal{J}(R) = \{0\}$.*
(ii) *$G(R)$ is Hausdorff.*
(iii) *$G(R)$ is separated.*
(iv) *$G(R)$ is Kolmogorov.*
(v) *In $\widetilde{G(R)}$, we have $\overline{\{0\}} = \{0\}$.*
(vi) *The space $\widetilde{G(R)}$ is Kolmogorov.*

Proof. We first argue that all of (i)–(v) are equivalent.
(i) $\implies$ (ii): Let $x \neq y \in R^\bullet$. Since R is a domain, $xy(x-y) \in R^\bullet$, and since R is semiprimitive, there is a maximal ideal $\mathfrak{m}$ of R such that $xy(x-y) \notin \mathfrak{m}$. Then $x + \mathfrak{m}$ and $y + \mathfrak{m}$ are disjoint open neighborhoods of x and y.
(ii) $\implies$ (iii) $\implies$ (iv) holds for all topological spaces.
(iv) $\implies$ (i): We argue by contraposition. Suppose that $\mathcal{J}(R) \neq \{0\}$. Then $\mathcal{J}(R)$ is infinite, so Proposition 2 implies that $G(R)$ has infinitely many indiscrete points. A topological space with at least two indiscrete points is not Kolmogorov.
(i) $\iff$ (v): By Theorem 4 we have $\overline{\{0\}} = \mathcal{J}_R((0)) = \mathcal{J}(R)$.
It remains to show the equivalence of (i)–(v) with (vi).
(vi) $\implies$ (iv): Subspaces of Kolmogorov spaces are Kolmogorov.
(iv, v) $\implies$ (vi): Take any pair of points $x \neq y \in R$. If $x, y \neq 0$, (iv) shows there is an open subset U of $G(R)$, hence also of $\widetilde{G(R)}$, which contains exactly one of x and y. To handle the remaining cases, observe that by (v), $G(R)$ is open in $\widetilde{G(R)}$, and for every $x \in R^\bullet$, $G(R)$ contains x and not 0. $\square$

Corollary 6.

a) *If $G(R)$ is Hausdorff (or separated, or Kolmogorov), then R has infinitely many maximal ideals.*
b) *If R is a Dedekind domain, then $G(R)$ is Hausdorff (or...) iff R has infinitely many maximal ideals.*

c) *If R is a PID, then $G(R)$ is Hausdorff (or...) iff R has infinitely many nonassociate prime elements.*

Proof. a) If $G(R)$ is Kolmogorov, then $\mathcal{J}(R) = \{0\}$ and R has infinitely many maximal ideals [Cl17, Cor 4.3].
b) A Dedekind domain has infinitely many maximal ideals iff it is semiprimitive [Cl17, Thm. 4.8].
c) A PID has infinitely many maximal ideals iff it has infinitely many nonassociate prime elements. $\square$

A topological space is *quasi-regular* if for for every point x and every neighborhood V of x there is a closed neighborhood N of x such that $N \subset V$; it is *regular* if it is quasi-regular Hausdorff.

A *Brown space* is a topological space X such that for all nonempty open subsets $U, V \subset X$, we have $\overline{U} \cap \overline{V} \neq \varnothing$.

Proposition 7.

a) *Brown spaces are connected.*
b) *Every space with an indiscrete point is a Brown space.*
c) *A Brown space is quasi-regular iff it is indiscrete.*

Proof. a) If X is disconnected, there are disjoint nonempty open subsets $U, V \subset X$ with $X = U \cup V$. Then $\overline{U} \cap \overline{V} = U \cap V = \varnothing$, and X is not a Brown space.
b) An indiscrete point lies in the closure of every nonempty open set.
c) Indiscrete spaces are clearly quasi-regular. Now suppose that X is quasi-regular and not indiscrete. Let U be a nonempty proper open subset, let $x \in U$, and let N_x be a closed neighborhood of x that is contained in U. Let $y \in X \setminus N_x$, and let N_y be a closed neighborhood of y that is contained in $X \setminus N_x$. Let N_x° [resp. N_y°] be the interior of N_x (resp. of N_y). Then N_x°, N_y° are nonempty open sets and

$$\overline{N_x^\circ} \cap \overline{N_y^\circ} \subset N_x \cap N_y = \varnothing;$$

hence, X is not a Brown space. $\square$

Theorem 8.

a) *The spaces $\widetilde{G(R)}$ and $G(R)$ are Brown spaces.*
b) *The spaces $\widetilde{G(R)}$ and $G(R)$ are connected.*
c) *The spaces $\widetilde{G(R)}$ and $G(R)$ are not quasi-regular.*

Proof. a) Since $0 \in \widetilde{G(R)}$ is indiscrete, Proposition 7b) implies that $\widetilde{G(R)}$ is a Brown space. Now consider $G(R)$. The condition for a Brown space can be checked on the elements of a base for the topology, so let $x_1 + I_1$ and $x_2 + I_2$ be two coprime cosets of nonzero ideals. By Theorem 4 we have

$$\overline{(x_1 + I_1)^\bullet} \cap \overline{(x_2 + I_2)^\bullet} \supset I_1^\bullet \cap I_2^\bullet \supset (I_1 I_2)^\bullet \supsetneq \varnothing.$$

b) This is immediate from part a) and Proposition 7a).
c) This is immediate from part a), Proposition 7b) and Lemma 1. $\square$

A topological space is *quasi-compact* if every open cover admits a finite subcover; it is *compact* if it is quasi-compact Hausdorff.

Theorem 9.

a) *If R is not semiprimitive, then $G(R)$ is quasi-compact.*
b) *If R is semiprimitive, then $G(R)$ is not quasi-compact; in fact, every quasi-compact subset of $G(R)$ has empty interior.*

Proof. a) If R is not semiprimitive, then $G(R)$ has an indiscrete point, so every open covering of $G(R)$ has a singleton subcovering!
b) Let K be a quasi-compact subset of $G(R)$. Since R is semiprimitive, $G(R)$ is Hausdorff (Theorem 5). Thus, K is compact, and K is closed as a subset of $G(R)$. Moreover, $\overline{K^\circ}$ is compact, being a closed subset of the compact set K.

For each $x \in \overline{K^\circ}$, choose a maximal ideal $\mathfrak{m}_x$ not containing x. Clearly, the coprime cosets $x+\mathfrak{m}_x$ constitute an open cover of $\overline{K^\circ}$. We now argue that if K° is nonempty, then this cover has no finite subcover. In this case, Theorem 4 implies that $\overline{K^\circ} \supset I^\bullet$ for some nonzero ideal I of R. So if $\overline{K^\circ}$ is covered by $x_1 + \mathfrak{m}_{x_1}, \ldots, x_r + \mathfrak{m}_{x_r}$, then

$$I^\bullet \subset \bigcup_{i=1}^{r} (x_i + \mathfrak{m}_{x_i}).$$

But none of the (infinitely many) elements of $I \cap \mathfrak{m}_{x_1} \cap \cdots \cap \mathfrak{m}_{x_r}$ are contained in the right-hand union. □

Let X be a topological space, and let x be a point of X. *X is locally connected at x* if x admits a neighborhood base of connected *open* sets: that is, for every open subset V containing x, there is a connected open set U with

$$x \in U \subset V.$$

X is totally disconnected at x if there is a neighborhood V of x such that no connected subspace of V has more than one point.

Proposition 10.

a) *For all $x \in \mathcal{J}(R)^\bullet$, the space $G(R)$ is locally connected at x.*
b) *Let $x \in R^\bullet \setminus \mathcal{J}(R)$, and suppose there is a nonzero ideal I such that $\langle x, I \rangle = R$ and $\bigcap_{n=1}^\infty I^n = \{0\}$. Then the space $G(R)$ is totally disconnected at x.*

Proof. a) If $x \in \mathcal{J}(R)^\bullet$, then the only neighborhood of x is X itself (Proposition 2), which is connected (Theorem 8b)). So X is locally connected at x.
b) The coprime coset $V = x + I$ is an open neighborhood of x. Let C be a subset of V containing distinct points y, z. Since $y, z \in V$, we have $y + I = z + I = x + I$. By hypothesis, there is an $n \in \mathbb{Z}^+$ such that $y + I^n \neq z + I^n$. Let

$$\mathcal{U}_1 := y + I^n, \quad \mathcal{U}_2 := (x + I) \setminus (y + I^n).$$

By [Cl-CA, Lemma 3.17c)], if $w \in R$ is such that $\langle w, I \rangle = R$, then $w + I^n$ is a coprime coset. Thus $\mathcal{U}_1$ is an open neighborhood of y. Moreover, $\mathcal{U}_2$ is a union of cosets $w + I^n$ for elements $w \in R$ such that $\langle w, I \rangle = R$, so $\mathcal{U}_2$ is an open neighborhood of z. Thus $(\mathcal{U}_1 \cap C, \mathcal{U}_2 \cap C)$ is a separation of C. □

Corollary 11. *If R is Noetherian and semiprimitive, then $G(R)$ is totally disconnected at each of its points.*

Proof. Let $x \in R^\bullet$. Since R is semiprimitive, there is a maximal ideal $\mathfrak{m}$ with $\langle x, \mathfrak{m} \rangle = R$. Since R is Noetherian, we have $\bigcap_{n=1}^\infty \mathfrak{m}^n = \{0\}$ [Cl-CA, Cor. 8.44]. Proposition 10b) applies to show that $G(R)$ is totally disconnected at x. □

Thus if R is Noetherian and semiprimitive, the space $G(R)$ is connected but totally disconnected at each of its points. This sounds rather pathological, but in fact such spaces are not so exotic: there is a nonempty, connected subset of the Euclidean plane for which each bounded subset is totally disconnected [Ma21].

Recall that if C is an open cover of a topological space X, a refinement of C is an open cover C' such that every element of C' is contained in an element of C. We say that X has (Lebesgue covering) *dimension* $d \in \mathbb{N}$ if every open cover C of X has a refinement with every point of X belonging to at most $d+1$ elements of C', and d is minimal with this property. If no such d exists, we say that that X is infinite dimensional.

Theorem 12. *If R is not semiprimitive, then $G(R)$ has dimension 0. Otherwise, $G(R)$ is infinite dimensional.*

Proof. As seen above, when R is not semiprimitive, every open cover of $G(R)$ has a singleton subcover. It follows immediately that $G(R)$ has dimension zero.

Now suppose that X is semiprimitive. Cover $G(R)$ by open sets $x + \mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal chosen with $x \notin \mathfrak{m}$. Calling this cover C, we will show that for any refinement C' of C, and any positive integer r, there is an element of $R^\bullet$ belonging to more than r elements of C'.

Pick an arbitrary $x_0 \in R^\bullet$. There is a nonzero ideal I_0 with $\langle x_0, I_0 \rangle = R$ and an element $A_0 \in C'$ with $x_0 + I_0 \subset A_0$. Moreover, $A_0 \subset y_0 + \mathfrak{m}_0$, where $y_0 + \mathfrak{m}_0$ is an element of the original cover C. So we have

$$x_0 + I_0 \subset A_0 \subset y_0 + \mathfrak{m}_0.$$

Suppose we have defined x_i, I_i, A_i, y_i, and $\mathfrak{m}_i$ for $i = 0, \ldots, j$. Choose a nonzero $x_{j+1} \in \prod_{i=0}^{j} I_i \mathfrak{m}_i$. Then for some nonzero ideal I_{j+1} with $\langle x_{j+1}, I_{j+1} \rangle = R$, some element $A_{j+1} \in C'$, and some element $y_{j+1} + \mathfrak{m}_{j+1}$ of our original cover, we have

$$x_{j+1} + I_{j+1} \subset A_{j+1} \subset y_{j+1} + \mathfrak{m}_{j+1}.$$

We continue until x_i, I_i, A_i, y_i, and $\mathfrak{m}_i$ have been defined for all of $i = 0, \ldots, r$.

Suppose that $0 \le i < j \le r$. Then $x_j \in I_i \mathfrak{m}_i \subset I_i$; hence, $R = x_j + I_j \subset I_i + I_j$. Thus, I_i and I_j are comaximal. We can also see easily that the sets A_i and A_j are distinct. Indeed, every element of A_i belongs to an invertible residue class modulo $\mathfrak{m}_i$ (the class of y_i), whereas $x_j \in A_j$ and $x_j \equiv 0 \pmod{\mathfrak{m}_i}$.

The Chinese Remainder Theorem now yields the existence of an $x \in R^\bullet$ with

$$x \equiv x_i \bmod I_i \quad \text{for all } i = 0, 1, \ldots, r.$$

Any such x belongs to the $r+1$ distinct sets $A_0, \ldots, A_r \in C'$. □

3. Nonhomeomorphic Golomb topologies

3.1. The homeomorphism problem.

Much of the interest in Golomb's topology $G(\mathbb{Z}^+)$ stems from the fact that it is a countably infinite, connected Hausdorff space. That such spaces exist is not obvious; the first example was constructed by Urysohn in 1925 [Ur25].

There is a large literature on countably infinite, connected Hausdorff spaces and also an enormous number of such spaces: there are $2^{\mathfrak{c}} = 2^{2^{\aleph_0}}$ homeomorphism types [KR72]. Our work gives a machine for producing such spaces: start with a countably infinite, semiprimitive domain R that is not a field and take $G(R)$.

If S is a subset of the prime numbers with infinite complement, then

$$\mathbb{Z}_S := \mathbb{Z}[\frac{1}{p} \mid p \in S]$$

is a semiprimitive PID, and $\mathbb{Z}_{S_1} \cong \mathbb{Z}_{S_2} \iff S_1 = S_2$. On the other hand, the set of all pairs of binary operations on a countably infinite set has cardinality $\mathfrak{c}$. So the number of isomorphism types of countably infinite semiprimitive PIDs, of countably infinite semiprimitive domains and of countably infinite rings are all $\mathfrak{c}$.

So apparently we are producing a large class of countably infinite, connected Hausdorff spaces via Golomb topologies on semiprimitive countably infinite domains. However there is a catch: the spaces look different, but they may nevertheless be homeomorphic. This raises a natural problem.

Problem 1 (Homeomorphism Problem)**.**
Let R and S be countably infinite semiprimitive domains. Decide whether the Golomb topologies $G(R)$ and $G(S)$ are homeomorphic.

Certainly if R and S are isomorphic, then $G(R)$ and $G(S)$ are homeomorphic. So far as we know, the converse might also be true. We can however make one contribution to the Homeomorphism Problem, via the following result.

Theorem 13. *Let R and S be infinite Dedekind domains, and let $h\colon G(R) \to G(S)$ be a homeomorphism of Golomb topologies. Then h restricts to a bijection from the unit group $R^\times$ of R to the unit group $S^\times$ of S.*

The proof of Theorem 13 will be given in §3.5. Three preliminary lemmas are given in §3.2, §3.3 and §3.4. In its structure our argument closely follows recent work of Banakh, Mioduszewski and Turek [BMT17] on the "classical Golomb space" $G(\mathbb{Z}^+)$ (cf. §4.1), showing that every homeomorphism of $G(\mathbb{Z}^+)$ fixes 1.

From Theorem 13 we deduce:

Corollary 14. *As q ranges over all prime powers, the Golomb topologies $G(\mathbb{F}_q[t])$ are pairwise nonhomeomorphic.*

Proof. If q_1 and q_2 are prime powers and $G(\mathbb{F}_{q_1}[t])$ and $G(\mathbb{F}_{q_2}[t])$ are homeomophic, then by Theorem 13 we have

$$q_1 - 1 = \#\mathbb{F}_{q_1}^\times = \#\mathbb{F}_{q_1}[t]^\times = \#\mathbb{F}_{q_2}[t]^\times = q_2 - 1,$$

so $q_1 = q_2$. $\square$

Corollary 14 and the previous discussion show that the number of homeomorphism types of Golomb topologies associated to countably infinite semiprimitive domains is between $\aleph_0$ and $\mathfrak{c}$. In the absence of any known homeomorphisms between Golomb topologies of nonisomorphic rings, we are inclined to believe that the truth is $\mathfrak{c}$.

3.2. The closure of a coprime coset.

The following result is modelled on [BMT17, Lemma 1].

Lemma 15. *Let R be a Dedekind domain, let I be a nonzero ideal of R, and let $\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}$ be its prime power factorization. Let $x \in R$ be such that $x+I$ is a coprime coset. Then we have*

$$\overline{(x+I)^\bullet} = (\bigcap_{i=1}^{g} (\mathfrak{p}_i \cup (x + \mathfrak{p}_i^{e_i})))^\bullet.$$

Proof. Since $x+I$ is a coprime coset, it is equivalent to show that

$$\overline{x+I} = \bigcap_{i=1}^{g} (\mathfrak{p}_i \cup (x + \mathfrak{p}_i^{e_i})).$$

Let $y \in R$. Suppose first that for some $1 \leq i \leq g$ we have $y \notin \mathfrak{p}_i \cup (x + \mathfrak{p}_i^{e_i})$. Then $y + \mathfrak{p}_i^{e_i}$ is a coprime coset disjoint from $x + \mathfrak{p}_i^{e_i}$, hence also from $x+I$. This shows

$$\overline{x+I} \subset \bigcap_{i=1}^{g} (\mathfrak{p}_i \cup (x + \mathfrak{p}_i^{e_i})).$$

Suppose that $y \in \mathfrak{p}_i \cup (x + \mathfrak{p}_i^{e_i})$ for all $1 \leq i \leq g$, and let U_y be a neighborhood of y in $\widetilde{G(R)}$, so U_y contains a coprime coset $y+J$. Let $A \coloneqq \{1 \leq i \leq g \mid y \in \mathfrak{p}_i\}$. For $i \in A$, since $y+J$ is a coprime coset we have $\langle \mathfrak{p}_i, J \rangle = R$, and thus

$$\langle I, J \rangle \supset \prod_{i \in \{1,\ldots,g\} \setminus A} \mathfrak{p}_i^{e_i}.$$

As $x \equiv y \pmod{\mathfrak{p}_i^{e_i}}$ for all $i \in \{1,\ldots,g\} \setminus A$, we get $x \equiv y \pmod{\langle I, J \rangle}$ and thus [Cl-CA, Lemma 21.5] we have $(x+I) \cap (y+J) \neq \varnothing$. □

3.3. **The Brown filter.**

Above we defined a topological space to be a Brown space if the closures of any two nonempty open sets intersect. A *strongly Brown space* is a space X such that for any finite collection $U_1, \ldots, U_n$ of nonempty open sets, we have that $\bigcap_{i=1}^n \overline{U_i} \neq \varnothing$. Being strongly Brown is precisely the condition for the sets $\{\overline{U} \mid U$ is nonempty and open$\}$ to be the subbase for a filter on X; explicitly, the *Brown filter* is

$$\mathcal{B}(X) \coloneqq \{A \subset X \mid \exists \text{ nonempty open subsets } U_1, \ldots, U_n \text{ such that } A \supset \bigcap_{i=1}^{n} \overline{U_i}\}.$$

The crux of our proof of Theorem 13 lies in the following observation: if $h : X \to Y$ is a homeomorphism of strongly Brown spaces, then the pushforward $h_*(\mathcal{B}(X)) = \{h(A) \mid A \in \mathcal{B}(X)\}$ of the Brown filter on X is the Brown filter $\mathcal{B}(Y)$ on Y. This is immediate from the topological nature of the definition.

Now let R be a domain that is not a field. The proof of Theorem 8a) works verbatim to show that the spaces $\widetilde{G(R)}$ and $G(R)$ are strongly Brown.

For any family $\{I_i\}$ of nonzero ideals, the family $\{I_i^\bullet\}$ satisfies the finite intersection proprety and is thus the subbase for a filter on $R^\bullet$. The *radical filter* $\mathcal{R}(R)$ is the filter generated by the family of all $\{I^\bullet\}$ such that I is a nonzero radical ideal.

The following result is modelled on [BMT17, Lemma 2].

Lemma 16. *Let R be a Dedekind domain. Then the Brown filter $\mathcal{B}(G(R))$ coincides with the radical filter $\mathcal{R}(R)$.*

Proof. In a Dedekind domain, the Jacobson radical of every nonzero ideal coincides with its radical, namely the product of the distinct maximal ideals it contains.

Let U be a nonempty open subset of $G(R)$. Then U contains a coprime coset $x+I$ and thus, by Theorem 4c), also $U \supset (\operatorname{rad} I)^\bullet$, so $U \in \mathcal{R}(R)$. Thus $\mathcal{R}(R)$ contains every element of a subbase for $\mathcal{B}(R)$ so $\mathcal{R}(R) \supset \mathcal{B}(R)$.

Since R is Dedekind, the nonzero prime ideals $\mathfrak{p}$ of R form a subbase for $\mathcal{R}(R)$, so it is enough to show that $\mathfrak{p}^\bullet \in \mathcal{B}(R)$. Choose $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$. By Lemma 15 we have

$$\overline{(1+\mathfrak{p}^2)^\bullet} = ((1+\mathfrak{p}^2) \cup \mathfrak{p})^\bullet,$$

$$\overline{(1+\pi+\mathfrak{p}^2)^\bullet} = ((1+\pi+\mathfrak{p}^2) \cup \mathfrak{p})^\bullet,$$

so

$$\mathfrak{p}^\bullet = \overline{(1+\mathfrak{p}^2)^\bullet} \cap \overline{(1+\pi+\mathfrak{p}^2)^\bullet} \in \mathcal{B}(R). \qquad \square$$

3.4. **A criterion for coprime cosets.**

The following result is modelled on [BMT17, Lemmas 3-4].

Lemma 17. *Let R be a Dedekind domain, let x, y be distinct elements of $R^\bullet$, and I be a nonzero radical ideal of R. The following are equivalent:*

(i) *There are neighborhoods U_x of x and U_y of y in $G(R)$ with $\overline{U_x} \cap \overline{U_y} \subset I^\bullet$.*
(ii) *We have $\langle x, I\rangle = \langle y, I\rangle = R$.*

Proof. (i) $\implies$ (ii): It is enough to assume that neighborhoods U_x and U_y as in (i) exist but that $\langle x, I\rangle \subsetneq R$ and deduce a contradiction. Let $\mathfrak{p}$ be a maximal ideal of R such that $\langle x, I\rangle \subset \mathfrak{p}$. Shrinking U_x and U_y if necessary, we may assume that

$$U_x = (x+I_1)^\bullet,\ U_y = (y+I_2)^\bullet, \quad \text{where} \quad \langle x, I_1\rangle = \langle y, I_2\rangle = R.$$

Then Theorem 4c) gives

$$I_1 I_2 \subset \operatorname{rad}(I_1) \cap \operatorname{rad}(I_2) \subset (\overline{U_x} \cap \overline{U_y}) \cup \{0\} \subset I \subset \mathfrak{p};$$

since $\langle \mathfrak{p}, I_1\rangle = R$, we get $\mathfrak{p} \supset I_2$. Choose $e \in \mathbb{Z}^+$ such that $I_2 \subset \mathfrak{p}^e$ and $I_2 \not\subset \mathfrak{p}^{e+1}$ and write $I_2 = \mathfrak{p}^e J$. By the Chinese Remainder Theorem there is $t \in R$ such that $t \in (y+\mathfrak{p}^e) \cap I_1 J$. Since $I_2 \subset \mathfrak{p}^e \subset \mathfrak{p}$ and $\langle y, I_2\rangle = R$, we have $y \notin \mathfrak{p}$ and thus $t \notin \mathfrak{p}$. Because $t \in I_1$, we have $t \in \overline{U_x}$; because $t \in (y+\mathfrak{p}^e) \cap J$, we have $t \in \overline{U_y}$. However, since $t \notin \mathfrak{p}$, we have $t \notin I$, contradicting $\overline{U_x} \cap \overline{U_y} \subset I^\bullet$.
(ii) $\implies$ (i): For each maximal ideal $\mathfrak{p} \supset I$, choose $e_\mathfrak{p} \in \mathbb{Z}^+$ large enough so that $x - y \notin \mathfrak{p}^{e_\mathfrak{p}}$, and put $J := \prod_\mathfrak{p} \mathfrak{p}^{e_\mathfrak{p}}$. Since $\operatorname{rad} J = \operatorname{rad} I$ and $\langle x, I\rangle = 1$, we have $\langle x, J\rangle = 1$ [Cl-CA, Prop. 4.17]. Put $U_x := (x+J)^\bullet$ and $U_y := (y+J)^\bullet$. If $\alpha \in \overline{U_x}$, then by Lemma 15, for each maximal ideal $\mathfrak{p} \supset I$, either $\alpha \in \mathfrak{p}$ or $\alpha \equiv x \pmod{\mathfrak{p}^{e_\mathfrak{p}}}$. Similarly for $\alpha \in \overline{U_y}$. So if $\alpha \in \overline{U_x} \cap \overline{U_y}$ and $\alpha \notin \mathfrak{p}$ for some maximal ideal $\mathfrak{p} \supset I$, then $x \equiv \alpha \equiv y \pmod{\mathfrak{p}^{e_\mathfrak{p}}}$, contradiction. Thus

$$\overline{U_x} \cap \overline{U_y} \subset \left(\bigcap_{\mathfrak{p} \supset I} \mathfrak{p}\right)^\bullet = I^\bullet. \qquad \square$$

3.5. **The proof of Theorem 13.**

The following proof is modelled on [BMT17, Lemma 5].

Let R and S be Dedekind domains, and let $h\colon G(R) \to G(S)$ be a homeomorphism. It is enough to show that $h(R^\times) \subset S^\times$: for if so, then also $h^{-1}(S^\times) \subset R^\times$, so $h|_{R^\times}$ induces a bijection from $R^\times$ to $S^\times$.

Seeking a contradiction, suppose there is $u \in R^\times$ such that $h(u) \in S^\bullet \setminus S^\times$, and let $\mathfrak{p}$ be a maximal ideal of S containing $h(u)$. By the work of §3.2 we have

$$\mathfrak{p}^\bullet \in \mathcal{R}(S) = \mathcal{B}(S) = h_*(\mathcal{B}(R)) = h_*(\mathcal{R}(R)),$$

so there is a nonzero radical ideal I of R such that $h(I^\bullet) \subset \mathfrak{p}^\bullet$.

Choose $y \in 1 + I$ with $y \neq u$. Then u and y are distinct elements with $\langle u, I\rangle = \langle y, I\rangle = R$, so by Lemma 17 there are neighborhoods U_u of u and U_y of y in $G(R)$ such that $\overline{U_u} \cap \overline{U_y} \subset I^\bullet$. It follows that

$$\overline{h(U_u)} \cap \overline{h(U_y)} \subset h(I^\bullet) \subset \mathfrak{p}^\bullet.$$

Now $h(U_u)$ is a neighborhood of $h(u)$ in $G(S)$ and $h(U_y)$ is a neighborhood of $h(y)$ in $G(S)$, so applying Lemma 17 again we get $\langle h(u), \mathfrak{p}\rangle = S$, contradicting our choice of $\mathfrak{p}$. This completes the proof of Theorem 13.

4. Remarks

4.1. **Work of Brown, Golomb and Kirch.**

Golomb defined [Go59] a topology $G(\mathbb{Z}^+)$ on $\mathbb{Z}^+$ by taking as a base the sets

$$\mathfrak{b}_{a,b} = \{an + b \mid n \in \mathbb{N}\}$$

as a, b range over coprime positive integers. He shows that his topology is Hausdorff (using the infinitude of prime numbers) and connected but not regular or compact. It is not hard to see that $G(\mathbb{Z}^+)$ is precisely the subspace topology on $\mathbb{Z}^+ \subset G(\mathbb{Z})$.

It seems that Golomb's topology had been defined several years ealier by Morton Brown. Brown did not publish his work, but he spoke about it at the April, 1953 AMS meeting in New York. Here is his abstract:

> *A countable connected Hausdorff space.* The points are the positive integers. Neighborhoods are sets of integers $\{a+bx\}$, where a and b are relatively prime to each other ($x = 1, 2, 3, \ldots$). Let $\{a+bx\}$ and $\{c+dx\}$ be two neighborhoods. It is shown that bd is a limit point of both neighborhoods. Thus, the closures of any two neighborhoods have a nonvoid intersection. This is a sufficient condition that a space be connected.

This should serve to explain our use of the term "Brown space."

No countably infinite connected space can be regular [Ur25]. However, the proof of nonregularity given by Golomb adapts to show that $G(R)$ is not quasi-regular unless R is a field. This was our initial approach; however, our current approach using Brown spaces seems more precise and perhaps more interesting.

In [Ki69], Kirch shows that in $G(\mathbb{Z}^+)$, the point 1 admits no connected neighborhood. Proposition 10b) comes from analyzing his proof. Kirch then coarsens Golomb's topology by taking as a base the coprime cosets $b + \langle a\rangle$ for squarefree

a and shows that his topology is Hausdorff and locally connected. One can carry over Kirch's topology and its properties to any semiprimitive Dedekind domain.

4.2. **Adic topologies revisited.**

For a domain R that is not a field, the adic topology on R has as its base all cosets $x+I$ of nonzero ideals I of R. The adic topology makes $(R,+)$ into a Hausdorff topological group, hence a uniformizable space, hence a regular space.

Suppose now that R is countably infinite. Since for all $a,b \in R^{\bullet}$ we have $(ab) \subset (a) \cap (b)$, the cosets of nonzero *principal* ideals form a countable base for the adic topology on R, so the adic topology is metrizable by Urysohn's Theorem. Since nonempty open subsets are infinite, there are no isolated points. A theorem of Sierpiński [Si20] implies that R is homeomorphic to $\mathbb{Q}$ with the Euclidean topology. (This was proven by Broughan when $R = \mathbb{Z}$ [Br03, Thms. 2.3 and 2.4], and the above argument largely follows his.) These adic topologies thus have dimension zero, unlike the Golomb topology on a semiprimitive domain that is not a field.

4.3. **Work of Knopfmacher–Porubský and Marko–Porubský.**

In several papers [KP97], [Po01], [MP12], [MP15] Knopfmacher–Porubský, Porubský and Marko–Porubský define and study a class of coset topologies on a domain R that includes the Golomb topology. The first author was unaware of this work when he considered the Golomb topology in [Cl17, §3.7], and much of the present work was completed before we became aware of it.

Thus the present work concerns objects defined and studied by Knopfmacher, Marko and Porubský. Nevertheless the only overlap in the results obtained is that our Theorem 8b) is also [KP97, Thm. 12]: both are direct generalizations of Golomb's argument. All the other facts on the Golomb topology obtained here are new, although some of them generalize results of [KP97], [MP12], [MP15]. Notably, [KP97, Thm. 21] and [MP15, Thms. 4.8 and 4.9] give rather specific sufficient conditions on R for $G(R)$ to be Hausdorff, whereas our Theorem 5 shows that this holds precisely when R is semiprimitive. In turn the works [KP97] and especially [MP15] explore rather general classes of coset topologies, not just the Golomb topology.

All in all, we believe that readers conversant with [KP97], [Po01], [MP12] and [MP15] will be in a position to better appreciate the present work. Conversely, readers who have made it this far will probably be interested in these other works.

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