# TORSION POINTS AND GALOIS REPRESENTATIONS ON CM ELLIPTIC CURVES

ABBEY BOURDON AND PETE L. CLARK

Abstract. We prove several results on torsion points and Galois representations for complex multiplication (CM) elliptic curves over a number field containing the CM field. The first, a close relative of a result of Stevenhagen [St01], computes the Weber function field of the $N$-torsion subgroup of a CM elliptic curve over an arbitrary order, generalizing the classical Main Theorem of Complex Multiplication. The second determines the degrees in which such an elliptic curve has a rational point of order $N$. The third bounds the size of the torsion subgroup of an elliptic curve with CM by a nonmaximal order in terms of the torsion subgroup of an elliptic curve with CM by the maximal order. We give several applications.

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## 1. Introduction

1.1. **Overview.** Let $F$ be a field of characteristic 0, and let $E_{/F}$ be an elliptic curve. We say $E$ has **complex multiplication (CM)** if the endomorphism algebra

$$\operatorname{End}^0 E = \operatorname{End}(E_{/\overline{F}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is strictly larger than $\mathbb{Q}$, in which case it is necessarily an imaginary quadratic field $K$ and $\mathcal{O} := \operatorname{End}(E_{/\overline{F}})$ is a $\mathbb{Z}$-order in $K$. This paper continues a program of study of torsion points and Galois representations on CM elliptic curves defined over number fields. Contributions have been made by Olson [Ol74], Silverberg [Si88], [Si92], Parish [Pa89], Aoki [Ao95], [Ao06], Ross [Ro94], Kwon [Kw99], Prasad-Yogananda [PY01], Stevenhagen [St01], Breuer [Br10], Lombardo [Lo15], Lozano-Robledo [LR], Gaudron-Rémond [GR18] and the present authors and our collaborators [CCS13], [CCRS14], [BCS17], [CP15], [BCP17], [BP16]. Á. Lozano-Robledo has informed us that he has also done work on the image of the adelic Galois representation in the CM case.

Two long-term goals of this program are on the one hand to completely understand the adelic Galois representation on any CM elliptic curve defined over a number field and on the other hand to determine all degrees of CM points on modular curves associated to congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. These two problems are closely related. An archetypical example is the following case of the **First Main Theorem of Complex Multiplication** (the full statement is reproduced as Theorem 2.8): if $K$ is an imaginary quadratic field and $E_{/K(j(E))}$ is an $\mathcal{O}_K$-CM elliptic curve, then for all $N \in \mathbb{Z}^+$ the field obtained by adjoining to $K(j(E))$ the Weber function of the $N$-torsion subgroup is $K^{(N)}$, the $N$-ray class field of $K$. For all $N \geq 3$, we have (see Lemma 2.10)

$$[K^{(N)} : K(j(E))] = \frac{\#(\mathcal{O}_K/N\mathcal{O}_K)^\times}{\#\mathcal{O}_K^\times}.$$

This implies that the mod $N$ Galois representation on an $\mathcal{O}_K$-CM elliptic curve $E_{/K(j(E))}$ is as large as possible *up to twisting*, and we will show there is an $\mathcal{O}_K$-CM elliptic curve $E_{/K(j(E))}$ such that the mod $N$ Galois representation surjects onto the mod $N$ Cartan subgroup $(\mathcal{O}/N\mathcal{O})^\times$ (see Corollary 1.5). This is a sharp version of Serre's Open Image Theorem in the $\mathcal{O}_K$-CM case. The corresponding result on the modular curve side is: the field of moduli of an $\mathcal{O}_K$-CM point on $X(N)_{/K(\zeta_N)}$ is $K^{(N)}$.

The above results restrict to the case of the maximal order $\mathcal{O}_K$, as does most of the classical theory. Here we work in the context of an arbitrary order $\mathcal{O}$, of conductor $\mathfrak{f}$, in an imaginary quadratic field $K$. (This notation is fixed throughout the remainder of the introduction.) Let $F \supset K$ be a number field, and let $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve. For any positive integer $N$, we define the **reduced mod N Cartan subgroup** to be the quotient of $C_N(\mathcal{O}) = (\mathcal{O}/N\mathcal{O})^\times$ by the image of $\mathcal{O}^\times$ under the natural map $q_N : \mathcal{O} \to \mathcal{O}/N\mathcal{O}$. That is,

$$\overline{C_N(\mathcal{O})} = C_N(\mathcal{O})/q_N(\mathcal{O}^\times).$$

(The map $q_N^\times : \mathcal{O}^\times \to (\mathcal{O}/N\mathcal{O})^\times$ is injective when $N \geq 3$; when $N = 2$ its kernel is $\{\pm 1\}$.) We define the **reduced Galois representation** to be the following composite homomorphism:

$$\overline{\rho_N} : \mathfrak{g}_F \xrightarrow{\rho_N} C_N(\mathcal{O}) \to \overline{C_N(\mathcal{O})}.$$

1.2. **Extending the First Main Theorem of Complex Multiplication.** The reduced Galois representation depends only on $j(E)$; it is independent of the $F$-rational model. Its fixed field is the field obtained by adjoining to $K(j(E))$ the values of a Weber function on $E$ evaluated at the points of order $N$. In particular, it is a subextension of the $(N\mathfrak{f})$-ray class field of $K$. The following result determines it explicitly and thus gives a generalization of the First Main Theorem of Complex Multiplication.

**Theorem 1.1.** *Let $E_{/\mathbb{C}}$ be an $\mathcal{O}$-CM elliptic curve, and let $N \in \mathbb{Z}^+$. Then the Weber function field $K(j(E))(\mathfrak{h}(E[N]))$ is the compositum of the $N$-ray class field $K^{(N)}$ of $K$ with the $N\mathfrak{f}$-ring class field $K(N\mathfrak{f})$ of $K$. Moreover, we have $[K(j(E))(\mathfrak{h}(E[N])) : K(j(E))] = \#\overline{C_N(\mathcal{O})}$.*

**Corollary 1.2.** *For any $\mathcal{O}$-CM elliptic curve $E_{/K(j(E))}$, the reduced mod $N$ Galois representation $\overline{\rho_N} : \mathfrak{g}_{K(j(E))} \to \overline{C_N(\mathcal{O})}$ is surjective.*

**Corollary 1.3.** *(Uniform Open Image Theorem) For all number fields $F \supset K$ and all $\mathcal{O}$-CM elliptic curves $E_{/F}$ we have*

$$[C_N(\mathcal{O}) : \rho_N(\mathfrak{g}_F)] \mid \#\mathcal{O}^\times[F : K(j(E))] \leq 6[F : K(j(E))].$$

**Corollary 1.4.** *Let $N \in \mathbb{Z}^+$. There is a number field $F \supset K$ and an $\mathcal{O}$-CM elliptic curve $E_{/F}$ such that $E[N] = E[N](F)$ and $[F : K(j(E))] = \#\overline{C_N(\mathcal{O})}$.*

**Corollary 1.5.** *For all $N \in \mathbb{Z}^+$, there is an $\mathcal{O}$-CM elliptic curve $E_{/K(\mathfrak{f})}$ such that $\rho_{E,N}(\mathfrak{g}_{K(\mathfrak{f})}) = C_N(\mathcal{O})$.*

Theorem 1.1 is closely related to work of P. Stevenhagen [St01], as we now explain. Let $\widehat{\mathcal{O}} = \mathcal{O} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ be the profinite completion of $\mathcal{O}$. Class field theory gives a canonical isomorphism

$$\Psi : \operatorname{Aut}(K^{\mathrm{ab}}/K(\mathfrak{f})) \xrightarrow{\sim} \widehat{\mathcal{O}}^\times/\mathcal{O}^\times \tag{1}$$

[St01, (3.2)]. For $N \in \mathbb{Z}^+$, we have a quotient map from $\widehat{\mathcal{O}}^\times/\mathcal{O}^\times$ to $\overline{C_N(\mathcal{O})} = (\mathcal{O}/N\mathcal{O})^\times/q_N(\mathcal{O}^\times)$, and thus via $\Psi$ there is a finite Galois extension $H_{N,\mathcal{O}}/K(\mathfrak{f})$, the **$N$-ray class field of $\mathcal{O}$**, such that

$$\operatorname{Aut}(H_{N,\mathcal{O}}/K(\mathfrak{f})) = \overline{C_N(\mathcal{O})}.$$

Stevenhagen uses Shimura's reciprocity law to show that for any $\mathcal{O}$-CM elliptic curve $E$, we have

$$K(\mathfrak{f})(\mathfrak{h}(E[N])) = H_{N,\mathcal{O}}.$$

From this, Corollary 1.2 follows. (Indeed it follows that $\Psi$ may be identified with the adelic reduced Galois representation on any $\mathcal{O}$-CM elliptic curve $E_{/K(\mathfrak{f})}$.) To deduce Theorem 1.1 one must show

$$H_{N,\mathcal{O}} = K^{(N)}K(N\mathfrak{f}). \tag{2}$$

Thus Theorem 1.1 is really a variant of work of Stevenhagen, though (rather curiously) our particular "neoclassical" formulation seems to be new.

Stevenhagen's approach is clean and elegant. On the other hand, Galois representations do not explicitly appear in his work. Here we will prove Theorem 1.1 by a different approach, proceeding via the observation that the two fields $K^{(N)}$ and $K(N\mathfrak{f})$ can be identified as subfields of the Weber function field $K(\mathfrak{f})(\mathfrak{h}(E[N]))$ – the former by a connection to the $\mathcal{O}_K$-CM case and the latter by its relation to the projective torsion point field, following work of Parish [Pa89]. Thus we get

$$K(\mathfrak{f})(\mathfrak{h}(E[N])) \supset K^{(N)}K(N\mathfrak{f}). \tag{3}$$

The formalism of reduced Galois representations gives

$$[K(j(E))(\mathfrak{h}(E[N])) : K(j(E))] \leq \#\overline{C_N(\mathcal{O})}.$$

Moreover, combining (2) and (3) we get

$$[K^{(N)}K(N\mathfrak{f}) : K(j(E))] = \#\overline{C_N(\mathcal{O})},$$

which gives Theorem 1.1. This proof is mostly independent of Stevenhagen's; the only common ingredient is the purely class field theoretic (1).

### 1.3. **Applications of Theorem 1.1.**

It follows from Corollary 1.3 that if $E_{/F}$ is an $\mathcal{O}$-CM elliptic curve and $F \supset K$ is a number field, the index of the image of the adelic Galois representation on $E$ in the Cartan subgroup $\widehat{C} = \widehat{O}^\times$ divides $\#\mathcal{O}^\times[F : K(j(E))]$. If $F$ does not contain $K$, then the image of the adelic Galois representation has index dividing $[F : \mathbb{Q}(j(E))]\#\mathcal{O}^\times$ in a subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ that contains the adelic Cartan $\widehat{C}$ with index 2. This is close to being a complete description of the adelic Galois representation on any CM elliptic curve defined over a number field. It falls short in two aspects: first, for a fixed $N \geq 3$, to get a mod $N$ Galois representation with index $\#\mathcal{O}^\times$ in the mod $N$ Cartan, our construction takes $F$ to be a proper extension of the minimal possible ground field $K(j(E))$. Second, it shows that at any finite level $N$ the index of the mod $N$ representation in the

Cartan can be any divisor of $\#\mathcal{O}^\times$ but does not address whether this can happen for the adelic Galois representation. These do not impact the second aspect of our program, which studies degrees of level $N$ structures of CM elliptic curves and fields of moduli of CM points on modular curves and studies *all* pairs $(E, L_N)_{/F}$ for a level $N$ structure $L_N$ over a number field $F \supset K(j(E))$, not just pairs in which the underlying elliptic curve $E$ arises from base extension of an elliptic curve $E_{/K(j(E))}$.

We propose to use the above results to determine all degrees of CM points on modular curves. To do so requires further work of a more algebraic nature: an analysis of orbits of the mod $N$ Cartan subgroup $C_N(\mathcal{O})$ on level $N$ structures. To understand the relevance of this, let $E_{/K(j(E))}$ be an $\mathcal{O}$-CM elliptic curve. If $P \in E[\text{tors}]$ is a point of order $N$, the field of moduli of the point $(E, P)$ on $X_1(N)$ depends only on the $\mathcal{O}^\times$ orbit $\overline{P}$ of $P$. Since the reduced Galois representation is surjective, the degree of this field over $K(j(E))$ may be computed by determining the size of the orbit of $\overline{C_N(\mathcal{O})}$ on $\overline{P}$.

We give an analysis of Cartan orbits on $\mathcal{O}/N\mathcal{O}$ in §6 and §7. The algebra is much simpler when $\mathcal{O}$ is maximal, and in this case our analysis is complete. When $\mathcal{O}$ is nonmaximal we give substantial, but not full, information on the structure of the Cartan orbits, enough to yield the following result.

**Theorem 1.6.** *Let $N \in \mathbb{Z}^{\geq 2}$. There is an integer $T(\mathcal{O}, N)$, explicitly computed in §7, such that:*
*(i) if $F \supset K$ is a number field and $E_{/F}$ is an $\mathcal{O}$-CM elliptic curve with an $F$-rational point of order $N$, then $T(\mathcal{O}, N) \mid [F : K(j(E))]$, and*
*(ii) there is a number field $F \supset K$ and an $\mathcal{O}$-CM elliptic curve $E_{/F}$ such that $[F : K(j(E))] = T(\mathcal{O}, N)$ and $E(F)$ contains a point of order $N$.*

Theorem 1.6 should be compared to Theorem 6.2, a refinement of bounds of Silverberg [Si88], [Si92], Prasad-Yogananda [PY01] and Gaudron-Rémond [GR18]. Theorem 6.2 also gives a divisibility on $[F : K(j(E))]$ imposed by the existence of an $F$-rational point of order $N$: in the current notation, Theorem 6.2 asserts

$$\varphi(N) \mid \#\mathcal{O}^\times \cdot T(\mathcal{O}, N).$$

This bound is "homogeneous" in the sense that it is a single bound that holds in all cases. Theorem 1.6 gives the optimal divisibility in *all* cases.

We give two other applications of our Cartan orbit analysis: the determination of all possible torsion subgroups of a $K$-CM elliptic curve $E_{/K(j(E))}$ (§6.6) and the set of $N \in \mathbb{Z}^+$ for which there is a $K(j(E))$-rational cyclic $N$-isogeny (Theorem 6.18). S. Kwon gave a classification of degrees of cyclic isogenies rational over $\mathbb{Q}(j(E))$ [Kw99]. Our Theorem 6.18 is the analogue over $K(j(E))$.

1.4. **The Isogeny Torsion Theorem.** Although we seek *results* which treat elliptic curves with CM by a nonmaximal order on an equal footing with the $\mathcal{O}_K$-CM case, in most cases the *proofs* use "change of order" functorialities. Let $E$ be an $\mathcal{O}$-CM elliptic curve defined over a number field $F$, and let $\mathfrak{f}'$ be a positive integer that divides $\mathfrak{f}$. Then by [BP16, Prop. 2.2] there is an elliptic curve $(E_{\mathfrak{f}'})_{/F}$ such that $\operatorname{End} E_{\mathfrak{f}'}$ is the order of conductor $\mathfrak{f}'$ in $K$ and an $F$-rational isogeny $\iota_{\mathfrak{f}'} : E \to E_{\mathfrak{f}'}$ that is cyclic of degree $\frac{\mathfrak{f}}{\mathfrak{f}'}$. So the induced $\mathfrak{g}_F$-module map $E[N] \to E_{\mathfrak{f}'}[N]$ is an isomorphism iff $\gcd(\frac{\mathfrak{f}}{\mathfrak{f}'}, N) = 1$; otherwise there is a nontrivial kernel. But nevertheless there are relations between the mod $N$ Galois representations on $E$ and $E_{\mathfrak{f}'}$. Here is the last main result of this paper:

**Theorem 1.7.** *(Isogeny Torsion Theorem) Let $\mathcal{O}$ be an order in an imaginary quadratic field $K$, of conductor $\mathfrak{f}$, and let $\mathfrak{f}'$ be a positive integer dividing $\mathfrak{f}$. Let $F \supset K$ be a number field, and let $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve. Let $\iota_{\mathfrak{f}'} : E \to E_{\mathfrak{f}'}$ be the $F$-rational isogeny to an elliptic curve $E_{\mathfrak{f}'}$ with CM by the order in $K$ of conductor $\mathfrak{f}'$, as described above. Then we have*

$$\#E(F)[\text{tors}] \mid \#E_{\mathfrak{f}'}(F)[\text{tors}].$$

In particular, taking $\mathfrak{f}' = 1$, we see that $\#E(F)[\text{tors}]$ is bounded by $\#E_1(F)[\text{tors}]$, where $(E_1)_{/F}$ is an elliptic curve with CM by the maximal order in $K$. We give examples where the exponent of $E_{\mathfrak{f}'}(F)[\text{tors}]$ is strictly smaller than that of $E(F)[\text{tors}]$, showing in general we cannot view $E(F)[\text{tors}]$ as a subgroup of $E_{\mathfrak{f}'}(F)[\text{tors}]$, and we prove that $\frac{\#E_{\mathfrak{f}'}(F)[\text{tors}]}{\#E(F)[\text{tors}]}$ can be arbitrarily large (see Propositions 6.8 and 6.9). Moreover, the statement is false if we do not require $F \supset K$. Despite the fact that this relationship is not as strong as one might hope, Theorem 1.7 has applications to determining fields of moduli of partial level $N$ structures (§6.2, §6.3).

A paper of R. Ross [Ro94] contains a result related to Theorem 1.7: in the notation of Theorem 1.7, Ross's assertion implies that the groups $E(F)[\mathrm{tors}]$ and $E_{\mathfrak{f}'}(F)[\mathrm{tors}]$ have the same exponent. This is false: Proposition 6.8 gives counterexamples. Still, Ross's work led us to Theorem 1.7.

1.5. **Acknowledgments.** We are deeply indebted to R. Broker for making us aware of Stevenhagen's important work [St01]. Several anonymous referees provided crucial critical feedback; in one case, this led us to the understanding that our Theorem 1.1 is essentially a variant of Stevenhagen's work.

## 2. Preliminaries

2.1. **Foundations.** We begin by setting some terminology for orders in imaginary quadratic fields. Let $K$ be an imaginary quadratic field and $\mathcal{O}$ a $\mathbb{Z}$-order in $K$. We put

$$\mathfrak{f} = [\mathcal{O}_K : \mathcal{O}],$$

the **conductor** of $\mathcal{O}$. Then

$$\mathcal{O} = \mathbb{Z} + \mathfrak{f}\mathcal{O}_K,\ \Delta(\mathcal{O}) = \mathfrak{f}^2\Delta_K.$$

Conversely, for fixed $K$ and $\mathfrak{f} \in \mathbb{Z}^+$ there is a unique order $\mathcal{O}(\mathfrak{f})$ in $K$ of conductor $\mathfrak{f}$. Thus an imaginary quadratic order is determined by its discriminant $\Delta$, a negative integer which is 0 or 1 modulo 4. Conversely, for any negative integer $\Delta$ which is 0 or 1 modulo 4, we put

$$\tau_\Delta = \frac{\Delta + \sqrt{\Delta}}{2},$$

and then $\mathbb{Z}[\tau_\Delta]$ is an order in $K$ of discriminant $\Delta$.

Throughout this paper we will use the following terminological convention: by "an order $\mathcal{O}$" we always mean a $\mathbb{Z}$-order $\mathcal{O}$ in an imaginary quadratic field, which is determined as the fraction field of $\mathcal{O}$ and denoted by $K$. We may specify an order $\mathcal{O}$ by giving its discriminant, which also determines $K$. If $K$ is already given, then we specify an order $\mathcal{O}$ in $K$ by giving the conductor $\mathfrak{f}$.

For any $\mathcal{O}$-CM elliptic curve $E$ we have $K(j(E)) = K(\mathfrak{f})$, the ring class field of $K$ of conductor $\mathfrak{f}$ ([Co89, Thm. 11.1]). We may thus determine $[K(j(E)) : K]$ via the following formula:

**Theorem 2.1.** *For $N \in \mathbb{Z}^+$, let $K(N)$ denote the $N$-**ring class field** of $K$. Then $K(1) = K^{(1)}$ is the Hilbert class field of $K$, and for all $N \geq 2$ we have*

$$[K(N) : K^{(1)}] = \frac{2}{w_K} N \prod_{p \mid N} \left(1 - \left(\frac{\Delta_K}{p}\right)\frac{1}{p}\right).$$

*Proof.* See e.g. [Co89, Cor. 7.24]. □

For number field $F$, a positive integer $N$, and $E_{/F}$ an elliptic curve, we denote by $\rho_N$ the homomorphism

$$\mathfrak{g}_F \to \operatorname{Aut} E[N] \cong \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}),$$

the **modulo N Galois representation**. If $E_{/F}$ has CM by the order $\mathcal{O}$ in $K$, then $E[N] \cong_{\mathcal{O}} \mathcal{O}/N\mathcal{O}$ (see [Pa89, Lemma 1], generalized in Lemma 2.4 below), and provided $F \supset K$ we have

$$\rho_N : \mathfrak{g}_F \hookrightarrow \operatorname{Aut}_{\mathcal{O}} E[N] \cong \mathrm{GL}_1(\mathcal{O}/N\mathcal{O}) = (\mathcal{O}/N\mathcal{O})^\times.$$

In other words, the image of the mod $N$ Galois representation lands in the **mod N Cartan subgroup**

$$C_N(\mathcal{O}) = (\mathcal{O}/N\mathcal{O})^\times.$$

**Lemma 2.2.** *Let $\mathcal{O}$ be an order of discriminant $\Delta$, and let $N = p_1^{a_r} \cdots p_r^{a_r} \in \mathbb{Z}^+$.*
*a) We have $C_N(\mathcal{O}) = \prod_{i=1}^r C_{p_i^{a_i}}(\mathcal{O})$ (canonical isomorphism).*
*b) We have $\# C_N(\mathcal{O}) = N^2 \prod_{p \mid N} \left(1 - \left(\frac{\Delta}{p}\right)\frac{1}{p}\right)\left(1 - \frac{1}{p}\right)$.*

*Proof.* a) It suffices to tensor the Chinese Remainder Theorem isomorphism $\mathbb{Z}/N\mathbb{Z} = \prod_{i=1}^{r} \mathbb{Z}/p_i^{a_i}\mathbb{Z}$ with the $\mathbb{Z}$-module $\mathcal{O}$ and pass to the unit groups.
b) By [CCS13], for any prime number $p$ we have

$$\# C_p(\mathcal{O}) = p^2 \left(1 - \left(\frac{\Delta}{p}\right)\frac{1}{p}\right)\left(1 - \frac{1}{p}\right).$$

The natural map $C_{p^a}(\mathcal{O}) \to C_p(\mathcal{O})$ is surjective with kernel of size $p^{2a-2}$ [CP15, p. 3]. Together with part a) this shows that if $N = p_1^{a_1}\cdots p_r^{a_r}$ then

$$\# C_N(\mathcal{O}) = \prod_{i=1}^{r} p_i^{2a_i-2}(p_i - 1)\left(p_i - \left(\frac{\Delta}{p_i}\right)\right) = N^2 \prod_{p \mid N}\left(1 - \left(\frac{\Delta}{p}\right)\frac{1}{p}\right)\left(1 - \frac{1}{p}\right). \qquad \square$$

2.2. **Torsion Kernels.** Let $E_{/\mathbb{C}}$ be an $\mathcal{O}$-CM elliptic curve. For a nonzero ideal $I$ of $\mathcal{O}$, we define the **I-torsion kernel**

$$E[I] = \{P \in E \mid \forall \alpha \in I,\ \alpha P = 0\}.$$

There is an invertible ideal $\Lambda \subset \mathcal{O}$ such that

$$E \cong \mathbb{C}/\Lambda.$$

If we put

$$(\Lambda : I) = \{x \in \mathbb{C} \mid xI \subset \Lambda\} = \{x \in K \mid xI \subset \Lambda\}$$

then we have (immediately) that

$$E[I] = \{x \in \mathbb{C}/\Lambda \mid xI \subset \Lambda\} = (\Lambda : I)/\Lambda.$$

Let $|I| = \#\mathcal{O}/I$.

**Lemma 2.3.** *Let $I, J \subset \mathcal{O}$ be nonzero ideals and $E_{/\mathbb{C}}$ be an $\mathcal{O}$-CM elliptic curve.*
*a) If $I \subset J$, then $E[J] \subset E[I]$.*
*b) We have $E[I] \subset E[|I|]$. In particular*

$$\# E[I] \leq |I|^2.$$

*Proof.* a) This is immediate from the definition. b) By Lagrange's Theorem, every element of $\mathcal{O}/I$ is killed by $|I|$, so $|I| \subset |I|\mathcal{O} \subset I$. Apply part a). $\square$

**Lemma 2.4.** *If $I$ is an invertible $\mathcal{O}$-ideal, then*

$$E[I] = I^{-1}\Lambda/\Lambda \cong_{\mathcal{O}} \mathcal{O}/I.$$

*In particular $\# E[I] = |I| = \#\mathcal{O}/I$.*

*Proof.* An ideal $I$ is invertible iff there is an $\mathcal{O}$-submodule $I^{-1}$ of $K$ such that $II^{-1} = \mathcal{O}$. If so, then for $x \in K$ we have

$$xI \subset \Lambda \iff xII^{-1} = x\mathcal{O} \subset I^{-1}\Lambda \iff x \in I^{-1}\Lambda,$$

giving $E[I] = I^{-1}\Lambda/\Lambda$. Because $\Lambda$ is a locally free $\mathcal{O}$-module, for all $\mathfrak{p} \in \operatorname{Spec}\mathcal{O}$ we have $\Lambda_{\mathfrak{p}} \cong \mathcal{O}_{\mathfrak{p}}$ and thus $(I^{-1}\Lambda/\Lambda)_{\mathfrak{p}} \cong (I^{-1}/\mathcal{O})_{\mathfrak{p}} \cong (\mathcal{O}/I)_{\mathfrak{p}}$. Thus $I^{-1}\Lambda/\Lambda$ is locally free of rank 1 as an $\mathcal{O}/I$-module. But the ring $\mathcal{O}/I$ is semilocal, hence has trivial Picard group: any locally free rank 1 $\mathcal{O}/I$-module is isomorphic to $\mathcal{O}/I$ [CA, Cor. 13.38]. $\square$

**Lemma 2.5.** *Let $R$ be a Dedekind domain, and let $M$ be a cyclic torsion $R$-module, and let $N \subset M$ be an $R$-submodule. Then:*
*a) $N$ is also a cyclic $R$-module.*
*b) We have $N \cong R/\operatorname{ann} N$.*

*Proof.* Let $I = \operatorname{ann} M$. Since $M$ is a finitely generated torsion module over a domain, we have $I \neq 0$ and $M \cong R/I$. Thus $N \cong I'/I$ for some ideal $I' \supset I$. The ring $R/I$ is principal Artinian [CA, Thm. 20.11], so the ideal $I'/I$ of $R/I$ is principal. Thus $N$ is a cyclic, torsion $R$-module, so $N \cong R/\operatorname{ann} N$. $\square$

**Theorem 2.6.** *Let $E_{/\mathbb{C}}$ be an $\mathcal{O}_K$-CM elliptic curve, and let $M \subset E(\mathbb{C})$ be a finite $\mathcal{O}_K$-submodule. Then $M = E[\operatorname{ann} M] \cong_{\mathcal{O}} \mathcal{O}/\operatorname{ann} M$ and thus $\# M = |\operatorname{ann} M|$.*

*Proof.* That $M \subset E[\operatorname{ann} M]$ is a tautology. Because $\mathcal{O} = \mathcal{O}_K$ every nonzero $\mathcal{O}$-ideal is invertible, so by Lemma 2.4 we have $\# E[\operatorname{ann} M] = |\operatorname{ann} M|$. On the other hand, let $\mathfrak{t} = \# M$. Then $M \subset E[\mathfrak{t}] \cong_{\mathcal{O}_K} \mathcal{O}_K/\mathfrak{t}\mathcal{O}_K$, a finite cyclic $\mathcal{O}_K$-module. By Lemma 2.5 we have $M \cong \mathcal{O}_K/\operatorname{ann} M$ so $\# M = |\operatorname{ann} M|$. Thus $M = E[\operatorname{ann} M]$, hence Lemma 2.4 gives $M \cong \mathcal{O}/\operatorname{ann} M$ and $\# M = |\operatorname{ann} M|$. □

**Remark 2.7.** *Let $\mathcal{O}$ be a nonmaximal order in $K$. There is nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$ such that the local ring $\mathcal{O}_{\mathfrak{p}}$ is not a DVR. If $\mathfrak{p} \cap \mathbb{Z} = (\ell)$, then $\mathcal{O}/\mathfrak{p} \cong \mathbb{Z}/\ell\mathbb{Z}$. Since every ideal of $\mathcal{O}$ can be generated by two elements, we have $\dim_{\mathcal{O}/\mathfrak{p}} \mathfrak{p}/\mathfrak{p}^2 = 2$. Thus $\#\mathcal{O}/\mathfrak{p}^2 = \ell^3$ and $(\ell^3) \subset \mathfrak{p}^2$. It follows that in the quotient ring $\mathcal{O}/\ell^3\mathcal{O}$ the maximal ideal $\mathfrak{p} + \ell^3\mathcal{O}$ is not principal. Let $E_{/\mathbb{C}}$ be an $\mathcal{O}$-CM elliptic curve, so $E[\ell^3] \cong_{\mathcal{O}} \mathcal{O}/\ell^3\mathcal{O}$. So the $\mathcal{O}$-submodule $M = \mathfrak{p}E[\ell^3]$ of $E[\ell^3]$ is not cyclic and thus not isomorphic to $\mathcal{O}/\operatorname{ann} M$.*

Now we recall an important classical result.

**Theorem 2.8.** *(First Main Theorem of Complex Multiplication) Let $E_{/\mathbb{C}}$ be an $\mathcal{O}_K$-CM elliptic curve, and let $I$ be a nonzero ideal of $\mathcal{O}_K$. Let $\mathfrak{h} : E \to \mathbb{P}^1$ be a Weber function. Then:*

$$K^{(1)}(\mathfrak{h}(E[I])) = K^I.$$

*Proof.* See e.g. [Si94, Thm. II.5.6]. □

Combining Theorems 2.6 and 2.8, we get the class-field theoretic containment corresponding to any finite $\mathcal{O}_K$-submodule of $E(\overline{F})$, for any $\mathcal{O}_K$-CM elliptic curve $E$ defined over a number field $F \supset K$. Theorem 2.8 implies that whenever $E$ is an $\mathcal{O}_K$-CM elliptic curve, $K^{(1)}(\mathfrak{h}(E[N])) = K^{(N)}$. In the case of CM by an arbitrary order in $K$, a containment has previously been established.

**Theorem 2.9.** [BCS17, Thm. 3.16] *Let $E$ be a $K$-CM elliptic curve defined over a number field $F \supset K$. Then we have*

$$F(\mathfrak{h}(E[N])) \supset K^{(N)}. \tag{4}$$

For convenience, we record here the formulas for $[K^I : K^{(1)}]$. Here, $\varphi$ denotes Euler's totient function and $\varphi_K(I)$ the natural generalization for a nonzero ideal $I$ of $\mathcal{O}_K$. That is,

$$\varphi_K(I) = \#(\mathcal{O}_K/I)^\times = |I| \prod_{\mathfrak{p}|I} \left(1 - \frac{1}{|\mathfrak{p}|}\right),$$

where $|I| = \#\mathcal{O}_K/I$.

**Lemma 2.10.** *Let $I$ be a nonzero ideal of $K$, and let $K^I$ be the $I$-**ray class field**. We put $U(K) = \mathcal{O}_K^\times$ and $U_I(K) = \{x \in U(K) \mid x - 1 \in I\}$.*
*a) We have*

$$[K^I : K^{(1)}] = \frac{\varphi_K(I)}{[U(K) : U_I(K)]}.$$

*b) If $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$, then*

$$[K^I : K^{(1)}] = \begin{cases} \varphi_K(I) & I \mid (2) \\ \frac{\varphi_K(I)}{2} & I \nmid (2) \end{cases}.$$

*c) If $K = \mathbb{Q}(\sqrt{-1})$, then*

$$[K^I : K^{(1)}] = \begin{cases} \varphi_K(I) & I \mid (1+i) \\ \frac{\varphi_K(I)}{2} & I \nmid (1+i) \text{ and } I \mid (2) \\ \frac{\varphi_K(I)}{4} & I \nmid (2) \end{cases}.$$

*d) If $K=\mathbb{Q}(\sqrt{-3})$, then*

$$[K^I:K^{(1)}]=\begin{cases}1 & I=(1)\\ \frac{\varphi_K(I)}{2} & I\neq(1) \text{ and } I\mid(\zeta_3-1)\\ \frac{\varphi_K(I)}{3} & I=(2)\\ \frac{\varphi_K(I)}{6} & \text{otherwise}\end{cases}.$$

*Proof.* Parts b)-d) can be deduced from a), which appears as [Co00, Cor. 3.2.4]. □

### 2.3. On Weber Functions.

**Theorem 2.11.** *(Weber Function Principle) Let $N\in\mathbb{Z}^{\geq 2}$, let $\mathcal{O}$ be the order of conductor $\mathfrak{f}$ in $K$, and let $F=K(\mathfrak{f})$. For an $\mathcal{O}$-CM elliptic curve $E_{/F}$, fix an embedding $F\hookrightarrow\mathbb{C}$ such that $j(E)=j(\mathbb{C}/\mathcal{O})$. Define*

$$W(N,\mathcal{O})=K(\mathfrak{f})(\mathfrak{h}(E[N])).$$

*a) $W(N,\mathcal{O})$ is a subfield of $F(E[N])$ and $[F(E[N]):W(N,\mathcal{O})]\mid\begin{cases}\#\mathcal{O}^\times & N\geq 3\\ \frac{\#\mathcal{O}^\times}{2} & N=2\end{cases}$.*

*b) There is an elliptic curve $E_{/F}$ such that*

$$[F(E[N]):W(N,\mathcal{O})]=\begin{cases}\#\mathcal{O}^\times & N\geq 3\\ \frac{\#\mathcal{O}^\times}{2} & N=2\end{cases}.$$

*c) As we range over all elliptic curves $E_{/F}$ with $j(E)=j(\mathbb{C}/\mathcal{O})$, we have*

$$\bigcap_E F(E[N])=W(N,\mathcal{O}).$$

*Proof.* a) Let $w=\begin{cases}\#\mathcal{O}^\times & N\geq 3\\ \frac{\#\mathcal{O}^\times}{2} & N=2\end{cases}$. Let $\mu_w$ be the image of $\mathcal{O}^\times\to C_N(\mathcal{O})$, a cyclic group of order $w$. The field $F(E[N])/F$ is Galois with Galois group $\rho_N(\mathfrak{g}_F)\subset C_N(\mathcal{O})$. Since $\mathfrak{h}(P)=\mathfrak{h}(Q)$ for points $P,Q$ on $E$ if and only if there is $\xi\in\mathcal{O}^\times$ such that $\xi(P)=Q$ (e.g. [La87, Thm. I.7]), it follows that

$$W(N,\mathcal{O})=F(E[N])^{\rho_N(\mathfrak{g}_F)\cap\mu_w}.$$

Thus

$$[F(E[N]):W(N,\mathcal{O})]\mid w.$$

b), c) If $E_{/F}$, $E'_{/F}$ with $j(E)=j(E')$, then $K(\mathfrak{f})(\mathfrak{h}(E[N]))=K(\mathfrak{f})(\mathfrak{h}(E'[N]))$ by the model independence of the Weber function. So $W(N,\mathcal{O})\subset\bigcap_E F(E[N])$. To see that equality holds, let $E_{/F}$ have $j(E)=j(\mathbb{C}/\mathcal{O})$. Let $\mathfrak{p}$ be a prime of $\mathcal{O}_F$ that is unramified in $F'=F(E[N])$. By weak approximation, there is $\pi\in\mathfrak{p}\setminus\mathfrak{p}^2$. Put $L=F(\pi^{\frac{1}{w}})$, and let $\chi:\mathfrak{g}_F\to\mu_w$ be a character with splitting field $\overline{F}^{\ker\chi}=L$. Then $L/F$ is totally ramified over $\mathfrak{p}$, so $F'$ and $L$ are linearly disjoint over $F$. It follows that

$$\rho_{N,E^\chi}(\mathfrak{g}_{F'})=(\rho_{N,E_{/F'}}\otimes\chi)(\mathfrak{g}_{F'})=\chi(\mathfrak{g}_{F'})=\mu_w.$$

Thus

$$w=[F(E^\chi[N]):F(E[N])\cap F(E^\chi[N])]\mid[F(E^\chi[N]):W(N,\mathcal{O})]\mid w,$$

so $F(E^\chi[N])$ has degree $w$ over $W(N,\mathcal{O})=F(E[N])\cap F(E^\chi[N])$. □

## 3. PROOF OF THE ISOGENY TORSION THEOREM

For a positive integer $d$, we will write $\mathcal{O}(d)$ for the order in $K$ of conductor $d$. There is a field embedding $F\hookrightarrow\mathbb{C}$ such that $E_{/\mathbb{C}}\cong\mathbb{C}/\mathcal{O}(\mathfrak{f})$, $(E_{\mathfrak{f}'})_{/\mathbb{C}}\cong\mathbb{C}/\mathcal{O}(\mathfrak{f}')$ and $(\iota_{\mathfrak{f}'})_{/\mathbb{C}}$ is the quotient map $\mathbb{C}/\mathcal{O}(\mathfrak{f})\to\mathbb{C}/\mathcal{O}(\mathfrak{f}')$. Put $\tau_K=\frac{\Delta_K+\sqrt{\Delta_K}}{2}$, so $\mathcal{O}(\mathfrak{f})=\mathbb{Z}[\mathfrak{f}\tau_K]$ and $\mathcal{O}(\mathfrak{f}')=\mathbb{Z}[\mathfrak{f}'\tau_K]$. For $N\in\mathbb{Z}^+$, let

$$e_{1,\mathfrak{f}}(N):=\frac{1}{N}+\mathcal{O}(\mathfrak{f}),\ e_{2,\mathfrak{f}}(N):=\frac{\mathfrak{f}\tau_K}{N}+\mathcal{O}(\mathfrak{f}),$$

$$e_{1,\mathfrak{f}'}(N) := \frac{1}{N} + \mathcal{O}(\mathfrak{f}'),\ e_{2,\mathfrak{f}'}(N) := \frac{\mathfrak{f}'\tau_K}{N} + \mathcal{O}(\mathfrak{f}').$$

Then $\ker(E[N] \xrightarrow{\iota_{\mathfrak{f}'}} E_{\mathfrak{f}'}[N])$ is cyclic of order $\gcd(N, \frac{\mathfrak{f}}{\mathfrak{f}'})$, generated by $\frac{N}{\gcd(N,\frac{\mathfrak{f}}{\mathfrak{f}'})} e_{2,\mathfrak{f}}(N)$, and $\iota_{\mathfrak{f}'}(e_{1,\mathfrak{f}}(N)) = e_{1,\mathfrak{f}'}(N)$.

For finite commutative groups $T_1$ and $T_2$, we have $\#T_1 \mid \#T_2$ if and only if $\#T_1[\ell^\infty] \mid \#T_2[\ell^\infty]$ for all prime numbers $\ell$. So it suffices to show: for all prime numbers $\ell$, we have $\#E(F)[\ell^\infty] \mid \#E_{\mathfrak{f}'}(F)[\ell^\infty]$. Write $\mathfrak{f} = \ell^{c_1}\bar{\mathfrak{f}}$ with $\gcd(\bar{\mathfrak{f}}, \ell) = 1$ and $\mathfrak{f}' = \ell^{c_2}\overline{\mathfrak{f}'}$ with $\gcd(\overline{\mathfrak{f}'}, \ell) = 1$. Then we have

$$\#E(F)[\ell^\infty] = \#E_{\ell^{c_1}}(F)[\ell^\infty],\ \#E_{\mathfrak{f}'}(F)[\ell^\infty] = \#E_{\ell^{c_2}}(F)[\ell^\infty],$$

so we may assume that $\mathfrak{f} = \ell^{c_1}$ and $\mathfrak{f}' = \ell^{c_2}$. Indeed, it is enough to treat the case $c_2 = c_1 - 1$, since repeated application of this case yields the general case. So suppose $\mathfrak{f} = \ell^c$ for some $c \in \mathbb{Z}^+$ and $\mathfrak{f}' = \ell^{c-1}$. By (e.g.) the Mordell-Weil Theorem, there are integers $0 \leq a \leq b$ such that

$$E(F)[\ell^\infty] \cong \mathbb{Z}/\ell^a\mathbb{Z} \oplus \mathbb{Z}/\ell^b\mathbb{Z}.$$

In particular, $Q := \frac{1}{\ell^a} + \mathcal{O}(\mathfrak{f}) \in E(F)$ and $Q' := \iota_{\mathfrak{f}'}(Q) = \frac{1}{\ell^a} + \mathcal{O}(\mathfrak{f}')$ generates $E_{\mathfrak{f}'}[\ell^a]$ as an $\mathcal{O}(\mathfrak{f}')$-module, so $E_{\mathfrak{f}'}[\ell^a] = E_{\mathfrak{f}'}(F)[\ell^a]$. If $a = b$, it follows that $\#E(F)[\ell^\infty] \mid \#E_{\mathfrak{f}'}(F)[\ell^\infty]$, so we may assume $b > a$. Since $\ker(E[\ell^\infty] \xrightarrow{\iota_{\mathfrak{f}'}} E_{\mathfrak{f}'}[\ell^\infty])$ has order $\ell$, we have $\mathbb{Z}/\ell^a\mathbb{Z} \oplus \mathbb{Z}/\ell^{b-1}\mathbb{Z} \hookrightarrow E_{\mathfrak{f}'}(F)[\ell^\infty]$. Thus it suffices to show that $E_{\mathfrak{f}'}(F)$ has either a point of order $\ell^b$ or has full $\ell^{a+1}$-torsion.

Let $P = E(F)$ be a point of order $\ell^b$, and write $P = \alpha e_{1,\mathfrak{f}}(\ell^b) + \beta e_{2,\mathfrak{f}}(\ell^b)$ with $\alpha, \beta \in \mathbb{Z}/\ell^b\mathbb{Z}$. If $\ell \nmid \alpha$ then $\iota_{\mathfrak{f}'}(P) = \alpha e_{1,\mathfrak{f}'}(\ell^b) + \ell\beta e_{2,\mathfrak{f}'}(\ell^b)$ has order $\ell^b$ and we are done, so we may assume that $\ell \mid \alpha$, in which case $\ell \nmid \beta$. With respect to the basis $e_{1,\mathfrak{f}}(\ell^b), e_{2,\mathfrak{f}}(\ell^b)$ of $E[\ell^b]$, the image of the mod $\ell^b$ Galois representation on $E$ consists of matrices of the form

$$\begin{bmatrix} a & b\ell^{2c}\frac{\Delta_K - \Delta_K^2}{4} \\ b & a + b\ell^c\Delta_K \end{bmatrix} \text{ with } a, b \in \mathbb{Z}/\ell^b\mathbb{Z}. \tag{5}$$

Since $E(F)$ has full $\ell^a$-torsion, we have $a \equiv 1 \pmod{\ell^a}$ and $b \equiv 0 \pmod{\ell^a}$. Thus

$$\rho_{\ell^{a+1}}(\mathfrak{g}_F) \subset \left\{ \begin{bmatrix} 1 + \ell^a A & 0 \\ \ell^a B & 1 + \ell^a A \end{bmatrix} \middle| A, B \in \mathbb{Z}/\ell^{a+1}\mathbb{Z} \right\}.$$

Since $\ell^{b-a-1}P = \alpha e_{1,\mathfrak{f}}(\ell^{a+1}) + \beta e_{2,\mathfrak{f}}(\ell^{a+1})$ is $F$-rational, all such matrices in the image of Galois satisfy

$$\begin{bmatrix} 1 + \ell^a A & 0 \\ \ell^a B & 1 + \ell^a A \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

and thus $\ell^a\alpha B + \beta + \ell^a A\beta \equiv \beta \pmod{\ell^{a+1}}$. Since $\ell \mid \alpha$, we get

$$\ell^a A\beta \equiv -\ell^a\alpha B \equiv 0 \pmod{\ell^{a+1}},$$

and thus $\ell \mid A$ and $\rho_{\ell^{a+1}}(\mathfrak{g}_F)$ consists of matrices of the form $\begin{bmatrix} 1 & 0 \\ \ell^a B & 1 \end{bmatrix}$ for $B \in \mathbb{Z}/\ell^{a+1}\mathbb{Z}$. It follows that for all $\sigma \in \mathfrak{g}_F$, there is $B \in \mathbb{Z}/\ell^{a+1}\mathbb{Z}$ such that

$$\begin{aligned} \sigma(\iota_{\mathfrak{f}'}(e_{1,\mathfrak{f}}(\ell^{a+1}))) &= \iota_{\mathfrak{f}'}(e_{1,\mathfrak{f}}(\ell^{a+1}) + B\ell^a\iota_{\mathfrak{f}'}(e_{2,\mathfrak{f}}(\ell^{a+1})) \\ &= \iota_{\mathfrak{f}'}(e_{1,\mathfrak{f}}(\ell^{a+1})) + B\ell^a(\ell e_{2,\mathfrak{f}'}(\ell^{a+1})) = \iota_{\mathfrak{f}'}(e_{1,\mathfrak{f}}(\ell^{a+1})). \end{aligned}$$

Thus $e_{1,\mathfrak{f}'}(\ell^{a+1}) = \iota_{\mathfrak{f}'}(e_{1,\mathfrak{f}}(\ell^{a+1})) \in E_{\mathfrak{f}'}(F)$. Since the $\mathcal{O}(\mathfrak{f}')$-submodule generated by $e_{1,\mathfrak{f}'}(\ell^{a+1})$ is $E_{\mathfrak{f}'}[\ell^{a+1}]$, we get $\mathbb{Z}/\ell^{a+1}\mathbb{Z} \oplus \mathbb{Z}/\ell^{a+1}\mathbb{Z} \hookrightarrow E_{\mathfrak{f}'}(F)$, completing the proof of Theorem 1.7.

## 4. The Projective Torsion Point Field

Let $F$ be a field. For a positive integer $N$ not divisible by the characteristic of $F$ and $E_{/F}$ an elliptic curve, we define the **projective modulo N Galois representation** as the composite map

$$\mathbb{P}\rho_N : \mathfrak{g}_F \xrightarrow{\rho_N} \operatorname{Aut} E[N] \cong \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to \mathrm{PGL}_2(\mathbb{Z}/N\mathbb{Z}) := \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/(\mathbb{Z}/N\mathbb{Z})^\times.$$

The **projective torsion field** is

$$F(\mathbb{P}E[N]) = \overline{F}^{\ker \mathbb{P}\rho_N}.$$

Thus $F(\mathbb{P}E[N])$ is the unique minimal field extension of $F$ on which the image of $\rho_N$ consists of scalar matrices. It follows that $F(E[N])/F(\mathbb{P}E[N])$ is a Galois extension with automorphism group a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$.

Observe that the projective Galois representation and thus the projective torsion field are unchanged by *quadratic* twists. If $E_{/F}$ has CM by an order of discriminant $\Delta = \mathfrak{f}^2\Delta_K \neq -3, -4$ and $F \supset K$, then the projective $N$-torsion field is a well-defined abelian extension of $K(\mathfrak{f})$. A result of Parish identifies this projective torsion field with a suitable ring class field. When $\Delta = -4$ (resp. $\Delta = -3$) we have quartic twists (resp. sextic twists) which can change the projective Galois representation and the projective torsion field.

**Theorem 4.1.** *Let $\mathcal{O}$ be an order of discriminant $\Delta = \mathfrak{f}^2\Delta_K$. Let $E$ be an $\mathcal{O}$-CM elliptic curve defined over $F = K(\mathfrak{f})$. Let $N \geq 2$.*
*a) We have $F(\mathbb{P}E[N]) \supset K(N\mathfrak{f})$. Thus we may put*

$$d(E,N) = [F(\mathbb{P}E[N]) : K(N\mathfrak{f})].$$

*b) If $\Delta \notin \{-3,-4\}$, then $d(E,N) = 1$, i.e., $F(\mathbb{P}E[N]) = K(N\mathfrak{f})$.*
*c) If $\Delta = -4$, then $d(E,N) \mid 2$.*
*d) If $\Delta = -3$, then $d(E,N) \mid 3$.*

*Proof.* For $N \in \mathbb{Z}^+$, let $\mathcal{O}(N)$ be the order of conductor $N$ in $K$. Thus $\mathcal{O} = \mathcal{O}(\mathfrak{f})$.
Step 1: We show that $F(\mathbb{P}E[N]) \supset K(N\mathfrak{f})$ in all cases.
There is a field embedding $F \hookrightarrow \mathbb{C}$ such that $E_{/\mathbb{C}} \cong \mathbb{C}/\mathcal{O}$. The $\mathbb{C}$-linear map $z \mapsto Nz$ carries $\mathcal{O}(\mathfrak{f})$ into $\mathcal{O}(N\mathfrak{f})$ and induces a cyclic $N$-isogeny $\mathbb{C}/\mathcal{O}(\mathfrak{f}) \to \mathbb{C}/\mathcal{O}(N\mathfrak{f})$. Let $C$ be the kernel of this isogeny, viewed as a finite étale subgroup scheme of $E_{/\mathbb{C}}$. Then $C$ has a (unique) minimal field of definition $F(C) \subset F(E[N])$, hence of finite degree over $F$. The field $F(\mathbb{P}E[N])$ is precisely the compositum of the minimal fields of definition of all order $N$ cyclic subgroup schemes $C \subset E_{/\mathbb{C}}$, so $F(C) \subset F(\mathbb{P}E[N])$. Since $C$ is $F(\mathbb{P}E[N])$-rational, the elliptic curve $E/C$ has a model over this field, and thus

$$F(\mathbb{P}E[N]) \supset K(j(E/C)) = K(N\mathfrak{f}).$$

Step 2: In view of Step 1, we have $F(\mathbb{P}E[N]) \supset K(N\mathfrak{f}) \supset K(\mathfrak{f}) = K(j(E))$, so we have $F(\mathbb{P}E[N]) = K(N\mathfrak{f})$ iff $[F(\mathbb{P}E[N]) : K(\mathfrak{f})] \leq [K(N\mathfrak{f}) : K(\mathfrak{f})]$. We have

$$[F(\mathbb{P}E[N]) : K(\mathfrak{f})] = \#\mathbb{P}\rho_N(\mathfrak{g}_F) \leq \#(\mathcal{O}/N\mathcal{O})^\times/(\mathbb{Z}/N\mathbb{Z})^\times = N\prod_{p\mid N}\left(1 - \left(\frac{\Delta}{p}\right)\frac{1}{p}\right).$$

- Suppose $\mathfrak{f} > 1$. Using Theorem 2.1 to compute $[K(N\mathfrak{f}) : K^{(1)}]$ and $[K(\mathfrak{f}) : K^{(1)}]$ gives

$$[K(N\mathfrak{f}) : K(\mathfrak{f})] = \frac{[K(N\mathfrak{f}) : K^{(1)}]}{[K(\mathfrak{f}) : K^{(1)}]} = N\prod_{p\mid N,\ p\nmid\mathfrak{f}}\left(1 - \left(\frac{\Delta}{p}\right)\frac{1}{p}\right) = N\prod_{p\mid N}\left(1 - \left(\frac{\Delta}{p}\right)\frac{1}{p}\right),$$

because $1 - \left(\frac{\Delta}{p}\right)\frac{1}{p} = 1$ for all $p \mid \mathfrak{f}$. Thus $d(E,N) = 1$ in this case.
- Suppose $\mathfrak{f} = 1$, so $\Delta = \Delta_K$. Then

$$[K(N\mathfrak{f}) : K(\mathfrak{f})] = [K(N) : K^{(1)}] = \frac{2}{w_K}N\prod_{p\mid N}\left(1 - \left(\frac{\Delta}{p}\right)\frac{1}{p}\right).$$

If $\Delta \notin \{-3,-4\}$ then $\frac{2}{w_K} = 1$, and again we get $d(E,N) = 1$. If $\Delta = -4$ then $\frac{2}{w_K} = \frac{1}{2}$, so the calculation shows $d(E,N) \in \{1,2\}$, and if $\Delta = -3$ then $\frac{2}{w_K} = \frac{1}{3}$, so the calculation shows $d(E,N) \in \{1,3\}$. $\square$

The following result is an an analogue of [BCS17, Thm. 5.6] for higher twists.

**Proposition 4.2.** *(Higher Twisting at the Bottom)*
*For $M \in \mathbb{Z}^+$, we denote the mod $M$ cyclotomic character by $\chi_M$.*

*a) Let* $K = \mathbb{Q}(\sqrt{-1})$ *and let* $\ell \equiv 5 \pmod 8$ *be a prime number. There is a character* $\Psi : \mathfrak{g}_K \to (\mathbb{Z}/\ell\mathbb{Z})^\times$ *of order* $\frac{\ell-1}{4}$ *and an* $\mathcal{O}_K$*-CM elliptic curve* $E_{/K}$ *such that the mod* $\ell$ *Galois representation is*

$$\sigma \mapsto \rho_\ell(\sigma) = \begin{bmatrix} \Psi(\sigma) & 0 \\ 0 & \Psi^{-1}(\sigma)\chi_\ell(\sigma) \end{bmatrix}.$$

*b) Let* $K = \mathbb{Q}(\sqrt{-3})$ *and let* $\ell \equiv 7, 31 \pmod{36}$ *be a prime number. There is a character* $\Psi : \mathfrak{g}_K \to (\mathbb{Z}/\ell\mathbb{Z})^\times$ *of order* $\frac{\ell-1}{6}$ *and an* $\mathcal{O}_K$*-CM elliptic curve* $E_{/K}$ *such that the mod* $\ell$ *Galois representation is*

$$\sigma \mapsto \rho_\ell(\sigma) = \begin{bmatrix} \Psi(\sigma) & 0 \\ 0 & \Psi^{-1}(\sigma)\chi_\ell(\sigma) \end{bmatrix}.$$

*Proof.* a) Because $\ell \equiv 1 \pmod 4$, the Cartan subgroup $C_\ell(\mathcal{O})$ is split, and for an $\mathcal{O}_K$-CM elliptic curve $(E_1)_{/K}$, the mod $\ell$ Galois representation has the form

$$\sigma \mapsto \rho_\ell(\sigma) = \begin{bmatrix} \Psi_1(\sigma) & 0 \\ 0 & \Psi_1^{-1}(\sigma)\chi_\ell(\sigma) \end{bmatrix}$$

for a character $\Psi_1 : \mathfrak{g}_K \to (\mathbb{Z}/\ell\mathbb{Z})^\times$. Under this isomorphism, the matrix representation of $i \in \mathcal{O}_K$ is a diagonal matrix $\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$, where $z$ is a primitive 4th root of unity in $\mathbb{Z}/\ell\mathbb{Z}$. A general $\mathcal{O}_K$-CM elliptic curve over $K$ is of the form $E_1^\psi$ for a character $\psi : \mathfrak{g}_K \to \mu_4 \subset (\mathbb{Z}/\ell\mathbb{Z})^\times$. Let $Q_4(\ell) = (\mathbb{Z}/\ell\mathbb{Z})^\times/(\mathbb{Z}/\ell\mathbb{Z})^{\times 4}$. Then the image of $z$ in $Q_4(\ell)$ has order 4: if not, there is $w \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ such that $z = w^2$, and then $w$ has order 8 in $(\mathbb{Z}/\ell\mathbb{Z})^\times$, contradicting the assumption that $\ell \equiv 5 \pmod 8$. Thus the natural map $\mu_4 \to Q_4(\ell)$ given by $i \mapsto z \pmod{(\mathbb{Z}/\ell\mathbb{Z})^{\times 4}}$ is an isomorphism; we denote the inverse isomorphism $Q_4(\ell) \to \mu_4$ by $\iota$. Now take

$$\psi : \mathfrak{g}_K \overset{\Psi_1^{-1}}{\to} (\mathbb{Z}/\ell\mathbb{Z})^\times \overset{q}{\to} Q_4(\ell) \overset{\iota}{\to} \mu_4.$$

Let $\Psi_2 = \psi\Psi_1$. Then the twist $E_1^\psi$ has mod $\ell$ Galois representation

$$\sigma \mapsto \rho_\ell(\sigma) = \begin{bmatrix} \Psi_2(\sigma) & 0 \\ 0 & \Psi_2^{-1}(\sigma)\chi_\ell(\sigma) \end{bmatrix}.$$

The composite $\Psi_2 : \mathfrak{g}_K \to (\mathbb{Z}/\ell\mathbb{Z})^\times \to Q_4(\ell)$ is trivial, so $\Psi_2(\mathfrak{g}_K)$ has order $c \mid \frac{\ell-1}{4}$. Thus

$$\#\rho_{\ell,E_1^\psi}(\mathfrak{g}_K) \mid c(\ell-1) \mid \frac{(\ell-1)^2}{4} = [K^{(\ell)} : K^{(1)}] = [K^{(\ell)} : K].$$

Because $K(E_1^\psi[\ell]) \supset K^{(\ell)}$, we have $\#\rho_{\ell,E_1^\psi}(\mathfrak{g}_K) = \frac{(\ell-1)^2}{4}$ and $c = \frac{\ell-1}{4}$.
b) Since $\ell \equiv 1 \pmod 3$, we have a primitive 6th root of unity $z$ in $\mathbb{Z}/\ell\mathbb{Z}$. Since $\ell \equiv 7, 31 \pmod{36}$, we have $4, 9 \nmid \ell - 1$, so $z$ has order 6 in $Q_6(\ell) = (\mathbb{Z}/\ell\mathbb{Z})^\times/(\mathbb{Z}/\ell\mathbb{Z})^{\times 6}$. Also $\frac{(\ell-1)^2}{6} = [K^{(\ell)} : K^{(1)}]$. The argument of part a) carries over. □

**Example 4.3.** *a) Let* $K = \mathbb{Q}(\sqrt{-1})$*, and let* $\ell \equiv 5 \pmod 8$*. Let* $E_{/K}$ *be an* $\mathcal{O}_K$*-CM elliptic curve with mod* $\ell$ *Galois representation as in Proposition 4.2a). Then for a number field* $L \supset K$*,* $\rho_\ell|_{\mathfrak{g}_L}$ *has scalar image iff* $\chi_\ell\Psi^{-2}|_{\mathfrak{g}_L}$ *is trivial. Since* $\chi_\ell : \mathfrak{g}_K \to (\mathbb{Z}/\ell\mathbb{Z})^\times$ *has order* $\ell - 1$ *– that is, for all* $1 \le k < \ell - 1$*,* $\chi_\ell^k \neq 1$ *– and* $\Psi^{-2}$ *has order dividing* $\frac{\ell-1}{4}$*, the character* $\chi_\ell\Psi^{-2}$ *has order* $\ell - 1$*. Thus* $[K(\mathbb{P}E[\ell]) : K] = \ell - 1$*,whereas* $[K(\ell) : K] = \frac{\ell-1}{2}$*. So* $d(E, \ell) = 2$*.*
*b) Let* $K = \mathbb{Q}(\sqrt{-3})$*, and let* $\ell \equiv 7, 31 \pmod{36}$*. Let* $E_{/K}$ *be an* $\mathcal{O}$*-CM elliptic curve with mod* $\ell$ *Galois representation as in Proposition 4.2b). As in part a), we have* $[K(\mathbb{P}E[\ell]) : K] = \ell - 1$ *and* $[K(\ell) : K] = \frac{\ell-1}{3}$*. So* $d(E, \ell) = 3$*.*

**Remark 4.4.** *a) Parts a) and b) of Theorem 4.1 are due to Parish* [Pa89, Prop. 3]*. However, Parish alludes to a calculation of the above sort rather than explicitly carrying it out. Since Theorem 4.1 will play an important role in the proof of Theorem 1.1, we have given a complete proof.*
*b) In* [Pa89, Prop. 3]*, Parish assumes* $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$*. Later on* [Pa89, p. 263]*, he claims:*
- *If* $\Delta = -4$ *then* $F(\mathbb{P}E[N]) = K(N)$ *for all* $N \geq 3$*, and*

- *If* $\Delta = -3$ *then* $F(\mathbb{P}E[N]) = K(N)$ *for all* $N \geq 4$.

*As Example 4.3 shows, both claims are false.*

**Proposition 4.5.** *Let* $\mathcal{O}$ *be an order of discriminant* $\Delta = \mathfrak{f}^2\Delta_K$*, and let* $N \in \mathbb{Z}^+$*. Then there is an* $\mathcal{O}$*-CM elliptic curve* $E_{/K(N\mathfrak{f})}$ *such that the mod* $N$ *Galois representation consists of scalar matrices.*

*Proof.* When $\Delta \notin \{-3,-4\}$, this is immediate from Theorem 4.1b): in that case, the elliptic curve has a model defined over $K(\mathfrak{f})$. Thus we may assume that $\Delta \in \{-3,-4\}$, so $\mathfrak{f} = 1$. Let $\zeta \in \mathcal{O}_K^\times$ be a primitive $w_K$th root of unity. Let $\mathcal{O}$ be the order in $K$ of conductor $N$, let $\tilde{E}_{/K(N)}$ be an $\mathcal{O}$-CM elliptic curve, and let $\iota : \tilde{E} \to E$ be the canonical $K(N)$-rational isogeny to an $\mathcal{O}_K$-CM elliptic curve $E$, let $\iota^\vee : E \to \tilde{E}$ be the dual isogeny, and let $C$ be the kernel of $\iota^\vee$. Identifying $E[N]$ with $N^{-1}\mathcal{O}_K/\mathcal{O}_K \subset \mathbb{C}/\mathcal{O}_K$, $\iota^\vee : \mathbb{C}/\mathcal{O}_K \to \mathbb{C}/\mathcal{O}$ is the map $z + \mathcal{O}_K \mapsto Nz + \mathcal{O}$, so $C$ is the $\mathbb{Z}$-submodule of $\mathbb{C}/\mathcal{O}_K$ generated by $P_1 = \frac{1}{N} + \mathcal{O}_K$. Because $C$ is stable under the action of $\mathfrak{g}_{K(N)}$, this action is given by an isogeny character, say

$$\sigma(P_1) = \Psi(\sigma)P_1.$$

Let $P_2 = \zeta P_1$. Then $\{P_1, P_2\}$ is a $\mathbb{Z}/N\mathbb{Z}$-basis for $E[N]$. Moreover, for $\sigma \in \mathfrak{g}_{K(N)}$,

$$\sigma P_2 = \sigma\zeta P_1 = \zeta\sigma P_1 = \zeta\Psi(\sigma)P_1 = \Psi(\sigma)\zeta P_1 = \Psi(\sigma)P_2.$$

It follows that $\sigma \in \mathfrak{g}_{K(N)}$ acts on $E[N]$ via the scalar matrix $\Psi(\sigma)$. □

## 5. Proof of Theorem 1.1 and Its Corollaries

5.1. **An Equality of Class Fields.** Let $\mathcal{O}$ and $\mathcal{O}'$ be orders in an imaginary quadratic field $K$ of conductors $\mathfrak{f}$ and $N\mathfrak{f}$, respectively. Here we prove (2): $H_{N,\mathcal{O}} = K^{(N)}K(N\mathfrak{f})$. We may assume that $N \geq 2$. Via class field theory, it suffices to prove an equality of open subgroups of $\widehat{\mathcal{O}_K}^\times/\mathcal{O}_K^\times$. We abbreviate

$$\mathcal{O}_p := \mathcal{O} \otimes \mathbb{Z}_p.$$

Put

$$A := \{x \in \widehat{\mathcal{O}}^\times \mid x \equiv 1 \pmod{N}\} = \prod_{p\nmid N} \mathcal{O}_p^\times \times \prod_{p \mid N}(1 + N\mathcal{O}_p),\ \tilde{A} := A\mathcal{O}_K^\times,$$

$$B := \widehat{\mathcal{O}'}^\times = \prod_p (\mathcal{O}')_p^\times,\ \tilde{B} := B\mathcal{O}_K^\times,$$

$$C := \{x \in \widehat{\mathcal{O}_K}^\times \mid x \equiv 1 \pmod{N}\} = \prod_{p\nmid N}(\mathcal{O}_K)_p^\times \times \prod_{p\mid N}(1 + N(\mathcal{O}_K)_p),\ \tilde{C} := C\mathcal{O}_K^\times.$$

Under class field theory, the field $H_{N,\mathcal{O}}$ corresponds to $\tilde{A}$ (cf. [St01, p. 9]), the field $K(N\mathfrak{f})$ corresponds to $\tilde{B}$ and the field $K^{(N)}$ corresponds to $\tilde{C}$, so $H_{N,\mathcal{O}} = K^{(N)}K(N\mathfrak{f})$ is equivalent to

$$\tilde{A} = \tilde{B} \cap \tilde{C}.$$

Step 1: We show that $A = B \cap C$. Writing $A_p$, $B_p$ and $C_p$ for the components of $p$ of each of these groups, it is enough to show that

$$A_p = B_p \cap C_p \text{ for all primes } p.$$

Case 1: Suppose $p \nmid N$. Then

$$A_p = \mathcal{O}_p^\times,$$
$$B_p = (\mathcal{O}')_p^\times = A_p,$$
$$C_p = (\mathcal{O}_K)_p^\times,$$

so $C_p \supset A_p = B_p$ and thus $B_p \cap C_p = A_p$.

Case 2: Suppose $p \mid N$. Write $\mathcal{O}_K = \mathbb{Z}1 + \mathbb{Z}\tau_K$, so $\mathcal{O} = \mathbb{Z}1 + \mathbb{Z}\mathfrak{f}\tau_K$. We have

$$A_p = 1 + N\mathcal{O}_p = 1 + N\mathbb{Z}_p1 + N\mathfrak{f}\mathbb{Z}_p\tau_K,$$

$$B_p = (1 + \mathbb{Z}_p1 + N\mathfrak{f}\mathbb{Z}_p\tau_K)^\times,$$

$$C_p = 1 + N(\mathcal{O}_K)_p = 1 + N\mathbb{Z}_p 1 + N\mathbb{Z}_p \tau_K,$$

so indeed we have $B_p \cap C_p = A_p$.

It follows that $\tilde{B} \cap \tilde{C} = B\mathcal{O}_K^\times \cap C\mathcal{O}_K^\times \supset A\mathcal{O}_K^\times = \tilde{A}$, so it remains to show that $\tilde{B} \cap \tilde{C} \subset \tilde{A}$.

Step 2: Suppose $\Delta_K < -4$, so $\mathcal{O}_K^\times = \{\pm 1\}$. Then $\tilde{B} = B$, so if $z \in \tilde{B} \cap \tilde{C}$, then there is $\epsilon \in \{\pm 1\}$ such that $z \in B$, $-z \in B$ and $\epsilon z \in C$, so $\epsilon z \in B \cap C = A$ and thus $z \in \tilde{A}$.

Step 3: Suppose $K = \mathbb{Q}(\sqrt{-1})$ and let $\zeta$ be a primitive 4th root of unity, so $\mathcal{O}_K = \mathbb{Z}1 + \mathbb{Z}\zeta$ and $\mathcal{O}_K^\times = \{1, \zeta, \zeta^2, \zeta^3\}$ . Suppose $z \in \tilde{B} \cap \tilde{C}$. Then there are $i, j \in \{0, 1, 2, 3\}$, $b \in B$ and $c \in C$ such that

$$z = \zeta^i b = \zeta^j c.$$

We have $z \in \tilde{A}$ iff $\zeta^{-j} z \in \tilde{A}$, so we may assume that $j = 0$. If $i$ is even we may argue as in Step 2, so assume that $i \in \{1, 3\}$, and thus we have either $\zeta b = c$ or $\zeta c = b$. But we claim that there are no such elements $b$ and $c$, which will complete the argument in this case. Indeed, choose a prime $p$ dividing $N$, and let $b_p$ and $c_p$ be the components at $p$. There is a reduction map

$$(\mathcal{O}_K)_p \to \mathcal{O}_K \otimes \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/p\mathbb{Z}1 + \mathbb{Z}/p\mathbb{Z}\zeta.$$

Under this map, every element of $B_p \cup C_p$ lands in $\mathbb{Z}/p\mathbb{Z}1$, so $b_p, c_p \in \mathbb{Z}/p\mathbb{Z}1$ while $\zeta b_p, \zeta c_p \in \mathbb{Z}/p\mathbb{Z}\zeta$. Thus we cannot have $\zeta b_p = c_p$ or $\zeta c_p = b_p$.

If $K = \mathbb{Q}(\sqrt{-3})$, then we let $\zeta$ be a primitive 6th root of unity, so $\mathcal{O}_K = \mathbb{Z}1 + \mathbb{Z}\zeta$ and $\mathcal{O}_K^\times = \{1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5\}$, and the argument is very similar: we cannot have $\pm\zeta b_p = c_p$ or $\pm b_p = \zeta c_p$.

5.2. **Proof of Theorem 1.1.** By Theorems 2.9 and 4.1a) and (2), we have

$$K(\mathfrak{f})(\mathfrak{h}(E[N])) \supset K^{(N)}K(N\mathfrak{f}) = H_{N,\mathcal{O}}.$$

For any $\mathcal{O}$-CM elliptic curve $E_{/K(\mathfrak{f})}$, the splitting field $\overline{K(\mathfrak{f})}^{\ker \overline{\rho_N}}$ of the reduced mod $N$ Galois representation on $E$ is $K(\mathfrak{f})(\mathfrak{h}(E[N]))$, so

$$[K(\mathfrak{f})(\mathfrak{h}(E[N])) : K(\mathfrak{f})] \leq \#\overline{C_N(\mathcal{O})}.$$

As described in the introduction, it is immediate from (1) and the definition of $H_{N,\mathcal{O}}$ that

$$\mathrm{Aut}(H_{N,\mathcal{O}}/K(\mathfrak{f})) = \overline{C_N(\mathcal{O})},$$

and thus it follows that

$$K(\mathfrak{f})(\mathfrak{h}(E[N])) = K^{(N)}K(N\mathfrak{f}).$$

### 5.3. Proof of Corollaries 1.2, 1.3, 1.4 and 1.5.

5.3.1. *Proof of Corollary 1.2.* Corollary 1.2 is an immediate consequence of Theorem 1.1.

5.3.2. *Proof of Corollary 1.3.* Since the reduced modulo $N$ Galois representation is independent of the rational model, Corollary 1.2 implies that for any number field $F \supset K(j(E))$ and $N \in \mathbb{Z}^+$, we have

$$[\overline{C_N(\mathcal{O})} : \overline{\rho_N}(\mathfrak{g}_F)] \mid [F : K(j(E))]$$

and thus

$$[C_N(\mathcal{O}) : \rho_N(\mathfrak{g}_F)] \mid \#\mathcal{O}^\times [F : K(j(E))] \leq 6[F : K(j(E))].$$

5.3.3. *Proof of Corollary 1.4.* We may of course assume that $N \geq 2$. Let $w = \#q_N(\mathcal{O}^\times)$, so

$$w = \begin{cases} \#\mathcal{O}^\times & N \geq 3 \\ \frac{\#\mathcal{O}^\times}{2} & N = 2 \end{cases}.$$

Then we have an injection $\mu_w \hookrightarrow C_N(\mathcal{O})$. Let $E_{/K(\mathfrak{f})}$ be any $\mathcal{O}$-CM elliptic curve. We may view $G = \mathrm{Aut}(K(\mathfrak{f})(E[N])/K(\mathfrak{f}))$ as a subgroup of $C_N(\mathcal{O})$. Let $H = G \cap \mu_w$ and $L = (K(\mathfrak{f})(E[N]))^H$, so a suitable twist $(E')_{/L}$ of $E_{/L}$ has trivial mod $N$ Galois representation. As shown in the proof of Theorem 2.11, we have $L = K(\mathfrak{f})(\mathfrak{h}(E[N]))$, so by Theorem 1.1 we have $[L : K(\mathfrak{f})] = \#\overline{C_N(\mathcal{O})}$.

5.3.4. *Proof of Corollary 1.5.* We may assume that $N \geq 2$. Let $q_N : \mathcal{O}^\times \to C_N(\mathcal{O})$ be the natural homomorphism. By Theorem 2.11b), there is an elliptic curve $E_{/K(\mathfrak{f})}$ such that $[K(\mathfrak{f})(E[N]) : K(\mathfrak{f})(\mathfrak{h}(E[N]))] = \# q_N(\mathcal{O}^\times) = \begin{cases} \#\mathcal{O}^\times & N \geq 3 \\ \frac{\#\mathcal{O}^\times}{2} & N = 2 \end{cases}$. By Theorem 1.1 we have $[K(\mathfrak{f})(\mathfrak{h}(E[N])) : K(\mathfrak{f})] = \#\overline{C_N(\mathcal{O})}$. Thus $\rho_{E,N}(\mathfrak{g}_{K(\mathfrak{f})}) = C_N(\mathcal{O})$.

## 6. Applications

### 6.1. SPY Divisibilities.

**Lemma 6.1.** *Let $H, K$ be subgroups of a group $G$. If $H$ is normal and $H \cap K = \{1\}$, then $\#K \mid [G : H]$.*

*Proof.* The composite homomorphism $K \hookrightarrow G \to G/H$ is an injection. □

The following result extends [BCP17, Cor. 2.5] from maximal orders to all imaginary quadratic orders, thereby confirming the expectation expressed in [BCP17, Remarks 2.2].

**Theorem 6.2.** *Let $\mathcal{O}$ be an order in an imaginary quadratic field $K$, and let $E$ be an $\mathcal{O}$-CM elliptic curve defined over a number field $F \supset K$. If $E(F)$ has a point of order $N \in \mathbb{Z}^+$, then*

$$\varphi(N) \mid \frac{\#\mathcal{O}^\times}{2} \frac{[F : \mathbb{Q}]}{\# \operatorname{Pic} \mathcal{O}}.$$

*Proof.* Let $\mathcal{I}_N = [C_N(\mathcal{O}) : \rho_N(\mathfrak{g}_F)]$ be the index of the mod $N$ Galois representation in the Cartan subgroup. By Corollary 1.3 we have

$$\mathcal{I}_N \mid \#\mathcal{O}^\times [F : K(j(E))] = \frac{\#\mathcal{O}^\times}{2} \frac{[F : \mathbb{Q}]}{\# \operatorname{Pic} \mathcal{O}}.$$

Since there is a rational point of order $N$, $\rho_N(\mathfrak{g}_F)$ contains no scalar matrices other than the identity, so by Lemma 6.1 we have $\varphi(N) \mid \mathcal{I}_N$, and we're done. □

6.2. **A Theorem of Franz.** Let $\mathcal{O}$ be an order in $K$, of conductor $\mathfrak{f}$, and let $E_{/K(\mathfrak{f})}$ be an $\mathcal{O}$-CM elliptic curve. Choose a field embedding $K(\mathfrak{f}) \hookrightarrow \mathbb{C}$ such that $j(E) = j(\mathbb{C}/\mathcal{O})$ and an isomorphism $E_{/\mathbb{C}} \xrightarrow{\sim} \mathbb{C}/\mathcal{O}$. This induces an isomorphism $E(\overline{K(\mathfrak{f})})[\text{tors}] \xrightarrow{\sim} \mathbb{C}/\mathcal{O}[\text{tors}]$, which we use to view (the image in $\mathbb{C}/\mathcal{O}$ of) $\tau_K = \frac{\Delta_K + \sqrt{\Delta_K}}{2}$ as a point of $E(\overline{K(\mathfrak{f})})[\text{tors}]$ of order $\mathfrak{f}$.

**Theorem 6.3.** *(Franz* [Fr35]*) With notation as above, we have*

$$K(\mathfrak{f})(\mathfrak{h}(\tau_K)) = K^{(\mathfrak{f})}.$$

*Proof.* As in the proof of Theorem 1.7, over $\mathbb{C}$ we may view the canonical isogeny as $\iota : \mathbb{C}/\mathcal{O} \to \mathbb{C}/\mathcal{O}_K$. We take $e_1 = \frac{1}{\mathfrak{f}} + \mathcal{O}$ and $e_2 = \tau_K + \mathcal{O}$ as a basis for $E[\mathfrak{f}]$. Then $e_2$ generates $\ker(\iota)$, a $K(\mathfrak{f})$-rational cyclic subgroup of order $\mathfrak{f}$, and there is a character $\Psi : \mathfrak{g}_F \to (\mathbb{Z}/\mathfrak{f}\mathbb{Z})^\times$ such that

$$\rho_{E,\mathfrak{f}}(\sigma) = \begin{bmatrix} \Psi(\sigma) & 0 \\ * & \Psi(\sigma) \end{bmatrix}.$$

If $\mathfrak{f} \leq 2$, then $K(\mathfrak{f})(\mathfrak{h}(\tau_K)) = K(\mathfrak{f}) = K^{(\mathfrak{f})}$ and the result holds. Thus we may assume $\mathfrak{f} \geq 3$. Let $L := K(\mathfrak{f})(\mathfrak{h}(e_2))$. Since $j(E) \neq 0, 1728$, the restriction $\Psi|_{\mathfrak{g}_L} : \mathfrak{g}_L \to \{\pm 1\}$ defines a quadratic character $\chi$, and on the twist $E^\chi$ of $E_{/L}$ the point $e_2$ becomes $L$-rational. As in the proof of Theorem 5.5 of [BCS17], let $\Psi^\pm : \mathfrak{g}_{K(\mathfrak{f})} \to (\mathbb{Z}/\mathfrak{f}\mathbb{Z}^\times)/\{\pm 1\}$ denote the composition of $\Psi$ with the natural map $(\mathbb{Z}/\mathfrak{f}\mathbb{Z})^\times \to (\mathbb{Z}/\mathfrak{f}\mathbb{Z})^\times/\pm 1$. Then $L \subset (\overline{K(\mathfrak{f})})^{\ker \Psi^\pm}$, so $[L : K(\mathfrak{f})] \mid \frac{\varphi(\mathfrak{f})}{2}$. If $\iota : E^\chi \to E'$ is the canonical isogeny, then the proof of Theorem 1.7 shows that $\iota(e_1)$ is an element of $E'(L)$ which generates $E'[\mathfrak{f}]$ as an $\mathcal{O}_K$-module. Thus $E'$ has full $\mathfrak{f}$-torsion over $L$, so by Theorem 2.8, $K^{(\mathfrak{f})} \subset L$. So

$$[L : K(\mathfrak{f})] \geq [K^{(\mathfrak{f})} : K(\mathfrak{f})] = \frac{\varphi(\mathfrak{f})}{2} \geq [L : K(\mathfrak{f})],$$

and thus $K(\mathfrak{f})(\mathfrak{h}(e_2)) = L = K^{(\mathfrak{f})}$. □

6.3. **The Field of Moduli of a Point of Prime Order.** In the introduction, we discussed a program to determine fields of moduli of all CM points on modular curves. Theorem 1.1 carries out this program for the curves $X(N)$. In this section we will obtain a result on the curves $X_1(N)$.

Let $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ be an imaginary quadratic field, and let $\mathcal{O} \subset K$ be the order of conductor $\mathfrak{f}$. Here we use Theorem 1.7 to determine the smallest field $F \supset K$ for which there exists an $\mathcal{O}$-CM elliptic curve $E_{/F}$ with an $F$-rational point of order $\ell > 2$.

**Lemma 6.4.** *Let $K$ be an imaginary quadratic field, let $\mathfrak{f} \in \mathbb{Z}^+$, and let $\ell > 2$ be prime. Then $K^{(\ell)} \cap K(\ell\mathfrak{f}) = K(\ell)$.*

*Proof.* Let $\Delta = \mathfrak{f}^2\Delta_K$. The statement is immediate if $\mathfrak{f} = 1$, so suppose $\mathfrak{f} > 1$. By Theorem 2.1,

$$[K(\ell\mathfrak{f}) : K(\mathfrak{f})] = \ell - \left(\frac{\Delta}{\ell}\right).$$

Since $[K^{(\ell)}K(\ell\mathfrak{f}) : K(\mathfrak{f})] = \#C_\ell(\mathcal{O})/2$ by Theorem 1.1, we have in both cases that

$$[K^{(\ell)}K(\ell\mathfrak{f}) : K(\ell\mathfrak{f})] = \frac{\#C_\ell(\mathcal{O})}{2[K(\ell\mathfrak{f}) : K(\mathfrak{f})]} = \frac{1}{2}(\ell - 1).$$

Thus $[K^{(\ell)} : K^{(\ell)} \cap K(\ell\mathfrak{f})] = [K^{(\ell)}K(\ell\mathfrak{f}) : K(\ell\mathfrak{f})] = \frac{1}{2}(\ell - 1)$. As we have $K(\ell) \subset K^{(\ell)} \cap K(\ell\mathfrak{f})$ and $[K^{(\ell)} : K(\ell)] = \frac{1}{2}(\ell - 1)$, the result follows. □

**Theorem 6.5.** *Let $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ be an imaginary quadratic field, and let $\mathcal{O}$ be the order of conductor $\mathfrak{f}$ in $K$. Let $F \supset K$.*
*a) Let $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve such that $E(F)$ contains a point of prime order $\ell > 2$. Then there is a prime $\mathfrak{p}$ of $\mathcal{O}_K$ lying over $\ell$ such that $K(\mathfrak{f})K^{\mathfrak{p}} \subset F$.*
*b) If $\left(\frac{\Delta}{\ell}\right) \neq -1$, then there is a prime $\mathfrak{p}$ is a prime of $\mathcal{O}_K$ lying over $\ell$ and an $\mathcal{O}$-CM elliptic curve $E_{/K(\mathfrak{f})K^{\mathfrak{p}}}$ such that $E(K(\mathfrak{f})K^{\mathfrak{p}})$ has a point of order $\ell$.*

If $\left(\frac{\Delta}{\ell}\right) = -1$, then an $\mathcal{O}$-CM elliptic curve $E_{/F}$ with an $F$-rational point of order $\ell$ must have full $\ell$-torsion (see [BCS17, Thm. 4.8] or Lemma 6.12). In this case, $K(\ell\mathfrak{f})K^{(\ell)} \subset F$ by Theorem 1.1. The existence of an elliptic curve $E_{/K(\ell\mathfrak{f})K^{(\ell)}}$ with full $\ell$-torsion is guaranteed by Corollary 1.4.

*Proof.* a) Let $F \supset K$ and $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve with an $F$-rational point of order $\ell$. By Theorem 1.7, there is an $\mathcal{O}_K$-CM elliptic curve $E'_{/F}$ with an $F$-rational point $P$ of order $\ell$. If $M$ is the $\mathcal{O}_K$-submodule of $E'(F)$ generated by $P$, then $M = E'[\operatorname{ann} M]$ and $\#M = |\operatorname{ann} M|$ by Theorem 2.6. Since $\ell \mid \#M$, we must have $\mathfrak{p} \mid \operatorname{ann} M$ for some prime $\mathfrak{p}$ of $\mathcal{O}_K$ above $\ell$. By Theorem 2.8 we have

$$K(\mathfrak{f})K^{\mathfrak{p}} \subset K(\mathfrak{f})K^{\operatorname{ann} M} = K(j(E))K^{(1)}(\mathfrak{h}(E'[\operatorname{ann} M])) \subset F.$$

b) If $\left(\frac{\Delta}{\ell}\right) \neq -1$, then an $\mathcal{O}$-CM elliptic curve $E_{/K(\mathfrak{f})}$ possesses a $K(\mathfrak{f})$-rational cyclic subgroup of order $\ell$. (See e.g. [CCS13, p.13]. This is also a special case of Theorem 6.18.) By [BCS17, Thm. 5.5], there is an extension $L/K(\mathfrak{f})$ of degree $(\ell - 1)/2$ and a quadratic twist $(E_1)_{/L}$ such that $E_1(L)$ has a point of order $\ell$. By part a), there is a prime $\mathfrak{p}$ of $\mathcal{O}_K$ lying over $\ell$ such that $K(\mathfrak{f})K^{\mathfrak{p}} \subset L$, so it will suffice to show that $[K(\mathfrak{f})K^{\mathfrak{p}} : K(\mathfrak{f})] \geq \frac{\ell-1}{2}$.

If $\ell \nmid \mathfrak{f}$, then $\ell$ is unramified in $K(\mathfrak{f})$. Thus $K(\mathfrak{f})$, $K^{\mathfrak{p}}$ are linearly disjoint over $K^{(1)}$, and we have $[K(\mathfrak{f})K^{\mathfrak{p}} : K(\mathfrak{f})] = [K^{\mathfrak{p}} : K^{(1)}] = \frac{1}{2}(\ell - 1)$ since $\left(\frac{\Delta_K}{\ell}\right) = \left(\frac{\Delta}{\ell}\right) \neq -1$. If $\ell \mid \mathfrak{f}$, then by Lemma 6.4 we have

$$K^{\mathfrak{p}} \cap K(\mathfrak{f}) \subset K^{(\ell)} \cap K(\mathfrak{f}) = K(\ell).$$

Thus $K^{\mathfrak{p}} \cap K(\mathfrak{f}) = K^{\mathfrak{p}} \cap K(\ell)$, so

$$[K(\mathfrak{f})K^{\mathfrak{p}} : K(\mathfrak{f})] = [K^{\mathfrak{p}} : K^{\mathfrak{p}} \cap K(\mathfrak{f})] = [K^{\mathfrak{p}} : K^{\mathfrak{p}} \cap K(\ell)] = [K(\ell)K^{\mathfrak{p}} : K(\ell)]$$

and it is enough to show that $[K(\ell)K^{\mathfrak{p}} : K(\ell)] \geq \frac{\ell-1}{2}$.

- $\left(\frac{\Delta_K}{\ell}\right) = 1$: We will prove that $K^{\mathfrak{p}} \cap K(\ell) = K^{(1)}$ using CM elliptic curves. Let $(E_0)_{/K^{(1)}}$ be an $\mathcal{O}_K$-CM elliptic curve. Then $E_0[\mathfrak{p}]$ is stable under the action of $\mathfrak{g}_{K^{(1)}}$ and generated by a point $P$ of order $\ell$. By [BCS17, Thm. 5.5], there is an extension $L/K^{(1)}$ of degree $(\ell-1)/2$ and a quadratic twist $(E_1)_{/L}$ such that $P$ becomes $L$-rational. By Theorem 2.8 we have $K^{\mathfrak{p}} \subset L$, and $K^{\mathfrak{p}} = L$ since $[K^{\mathfrak{p}} : K^{(1)}] = \frac{1}{2}(\ell-1)$. Over $K(\ell)K^{\mathfrak{p}}$, the curve $E_1$ has a rational point of order $\ell$, and the mod $\ell$ Galois representation is scalar by Theorem 4.1. Thus $E_1$ has full $\ell$-torsion over $K(\ell)K^{\mathfrak{p}}$, and $K^{(\ell)} \subset K(\ell)K^{\mathfrak{p}}$. This implies $\frac{1}{2}(\ell-1) \mid [K(\ell)K^{\mathfrak{p}} : K(\ell)] = [K^{\mathfrak{p}} : K^{\mathfrak{p}} \cap K(\ell)]$. Since $[K^{\mathfrak{p}} : K^{(1)}] = \frac{1}{2}(\ell-1)$, we have $K^{\mathfrak{p}} \cap K(\ell) = K^{(1)}$, and $[K(\mathfrak{f})K^{\mathfrak{p}} : K(\mathfrak{f})] = [K^{\mathfrak{p}} : K^{(1)}] = \frac{1}{2}(\ell-1)$.
- $\left(\frac{\Delta_K}{\ell}\right) = -1$: In this case, $K^{\mathfrak{p}} = K^{(\ell)}$, so $K^{\mathfrak{p}} \cap K(\ell) = K(\ell)$. This implies $[K(\mathfrak{f})K^{\mathfrak{p}} : K(\mathfrak{f})] = [K^{\mathfrak{p}} : K(\ell)] = \frac{1}{2}(\ell-1)$.
- $\left(\frac{\Delta_K}{\ell}\right) = 0$: Since $[K(\ell) : K^{(1)}] = \ell$ and $[K^{\mathfrak{p}} : K^{(1)}] = \frac{1}{2}(\ell-1)$, we have $K^{\mathfrak{p}} \cap K(\ell) = K^{(1)}$. Thus $[K(\mathfrak{f})K^{\mathfrak{p}} : K(\mathfrak{f})] = [K^{\mathfrak{p}} : K^{(1)}] = \frac{1}{2}(\ell-1)$. □

**Remark 6.6.** *Assume the setup of Theorem 6.5 but take $K = \mathbb{Q}(\sqrt{-1})$ or $K = \mathbb{Q}(\sqrt{-3})$. Then the assertion of Theorem 6.5b) is false. Indeed, if $\ell \geq 5$ and $\left(\frac{\Delta}{\ell}\right) \neq -1$, we have $[K(\mathfrak{f})K^{\mathfrak{p}} : K(\mathfrak{f})] \mid \frac{1}{w_K}(\ell-1)$. (See Lemma 2.10.) Suppose $F \supset K$, and let $E_{/F}$ be an elliptic curve with CM by the order in $K$ of conductor $\mathfrak{f}$. If $E(F)$ contains a rational point of order $\ell$, then Theorem 6.2 implies $\frac{1}{2}(\ell-1) \mid [F : K(\mathfrak{f})]$. Thus $F$ must properly contain $K(\mathfrak{f})K^{\mathfrak{p}}$.*

### 6.4. **Sharpness in the Isogeny Torsion Theorem.**

The following result was established during the proof of Theorem 1.7.

**Lemma 6.7.** *Let $E$ be an $\mathcal{O}$-CM elliptic curve defined over a number field $F$ containing the CM field $K$, and for a positive integer $\mathfrak{f}'$ dividing the conductor $\mathfrak{f}$ of $\mathcal{O}$, let $\iota : E \to E'$ be the canonical $F$-rational isogeny to an elliptic curve $E'$ with CM by the order in $K$ of conductor $\mathfrak{f}'$. Write*

$$E(F)[\mathrm{tors}] = \mathbb{Z}/s\mathbb{Z} \times \mathbb{Z}/e\mathbb{Z},\ E'(F)[\mathrm{tors}] = \mathbb{Z}/s'\mathbb{Z} \times \mathbb{Z}/e'\mathbb{Z},$$

*where $s \mid e$ and $s' \mid e'$. Then $s \mid s'$.*

In [Ro94, §4], Ross claims that if $E$ is a CM elliptic curve defined over a number field $F$ containing the CM field, then the exponent of the finite group $E(F)[\mathrm{tors}]$ is an invariant of the $F$-rational isogeny class. In the setting of Lemma 6.7, this would give $e = e'$, and combining this with the conclusion of Lemma 6.7 we would get an injective group homomorphism $E(F)[\mathrm{tors}] \hookrightarrow E(F')[\mathrm{tors}]$. This conclusion is stronger than that of Theorem 1.7.

Unfortunately Ross's claim is false: in the setup of Lemma 6.7 one can have $e' < e$ (in which case there is no injective group homomorphism $E(F)[\mathrm{tors}] \hookrightarrow E'(F)[\mathrm{tors}]$), as the following result shows.

**Proposition 6.8.** *Let $\ell > 3$ be a prime number, let $K = \mathbb{Q}(\sqrt{-\ell})$, let $n \in \mathbb{Z}^{\geq 3}$, let $\mathcal{O}$ be the order in $K$ of conductor $\mathfrak{f} = \ell^{\lfloor \frac{n}{2} \rfloor}$, and let $F = K(\mathfrak{f})$. For any $\mathcal{O}$-CM elliptic curve $E_{/F}$, there is an extension $L/F$ of degree $\varphi(\ell^n)$ such that $E(L)$ has a point of order $\ell^n$, and no $\mathcal{O}_K$-CM elliptic curve has an $L$-rational point of order $\ell^k$ for $k > \frac{1}{2}\left(n+1+\lfloor \frac{n}{2} \rfloor\right)$ (hence no $L$-rational point of order $\ell^n$).*

*Proof.* Let $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve. As in (5) we may choose a basis $\{e_1, e_2\}$ for $E[\ell^n]$ so that the image of the mod $\ell^n$ Galois representation consists of matrices

$$\left[\begin{array}{cc} a & b\mathfrak{f}^2\frac{\Delta_K-\Delta_K^2}{4} \\ b & a+b\mathfrak{f}\Delta_K \end{array}\right] \mid a, b \in \mathbb{Z}/\ell^n\mathbb{Z}.$$

Since $\ell$ ramifies in $K$ and $\mathfrak{f} = \ell^{\lfloor \frac{n}{2} \rfloor}$, we have $\mathrm{ord}_\ell(b\mathfrak{f}^2\frac{\Delta_K-\Delta_K^2}{4}) = 1 + 2\lfloor \frac{n}{2} \rfloor \geq n$, so the matrices have the form

$$\left[\begin{array}{cc} a & 0 \\ b & a+b\mathfrak{f}\Delta_K \end{array}\right] \mid a, b \in \mathbb{Z}/\ell^n\mathbb{Z}.$$

The action of $\mathfrak{g}_F$ on $\langle e_2\rangle$ gives a character $\Phi : \mathfrak{g}_F \to (\mathbb{Z}/\ell^n\mathbb{Z})^\times$. Take $M = (\overline{F})^{\ker\Phi}$. Then $[M : F] \mid \varphi(\ell^n)$ and $\Phi|_{\mathfrak{g}_M}$ is trivial. Thus there exists an extension $L/F$ with $[L : F] = \varphi(\ell^n)$ such that $E(L)$ contains $e_2$.

Let $E'_{/L}$ be an $\mathcal{O}_K$-CM elliptic curve, and suppose $E'(L)$ contains a point $P$ of order $\ell^k$. Let $\mathfrak{p}$ be the prime ideal of $\mathcal{O}_K$ such that $\ell\mathcal{O}_K = \mathfrak{p}^2$. We claim that the $\mathcal{O}_K$-submodule $M = \langle P\rangle_{\mathcal{O}_K}$ of $E'(L)$ generated by $P$ contains $E[\mathfrak{p}^{2k-1}]$ and thus, by Theorem 2.8, that $K^{\mathfrak{p}^{2k-1}} \subset L$. Indeed, by Theorem 2.6, we have $M = E[I]$ for some ideal $I$ of $\mathcal{O}_K$ such that $(\mathcal{O}_K/I, +)$ has $\ell$-power order and exponent $\ell^k$. Since $\ell$ ramifies in $\mathcal{O}_K$, this forces $I$ to be of the form $\mathfrak{p}^a$ for some $a \in \mathbb{Z}^+$, and the smallest $a$ such that $(\mathcal{O}_K/\mathfrak{p}^a, +)$ has exponent $\ell^k$ is $a = 2k - 1$, establishing the claim. Thus

$$\operatorname{ord}_\ell([K^{\mathfrak{p}^{2k-1}} : K^{(1)}]) = 2k - 2 \le \operatorname{ord}_\ell([L : K^{(1)}]) = \left\lfloor \frac{n}{2} \right\rfloor + n - 1,$$

so $k \le \frac{1}{2}(n + 1 + \lfloor \frac{n}{2} \rfloor)$. □

In the setting of Theorem 1.7, one wonders whether $\#E(F)[\mathrm{tors}] = \#E'(F)[\mathrm{tors}]$. In fact $\frac{\#E'(F)[\mathrm{tors}]}{\#E(F)[\mathrm{tors}]}$ can be arbitrarily large:

**Proposition 6.9.** *Let $\ell$ be an odd prime, let $K \ne \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ be an imaginary quadratic field, let $\mathcal{O}$ be the order in $K$ of conductor $\ell$, and let $F = K(\ell)$. For any $\mathcal{O}$-CM elliptic curve $E_{/F}$ there is an extension $L/F$ such that if $\iota : E \to E'$ is the canonical isogeny to an $\mathcal{O}_K$-CM elliptic curve $E$, then*

$$\ell \mid \frac{\#E'(L)[\mathrm{tors}]}{\#E(L)[\mathrm{tors}]}.$$

*Proof.* Let $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve. As above, there is a basis $\{e_1, e_2\}$ for $E[\ell]$ such that

$$\rho_\ell(\mathfrak{g}_F) \subset \left\{ \begin{bmatrix} a & 0 \\ b & a \end{bmatrix} \mid a, b \in \mathbb{Z}/\ell\mathbb{Z} \right\}$$

and there is an extension $L/F$ with $[L : F] = \ell - 1$ such that $E(L)$ contains $e_2$. In fact, $E(L)[\ell^\infty] \cong \mathbb{Z}/\ell\mathbb{Z}$. Indeed, $E$ does not have full $\ell$-torsion over $L$ since Theorem 1.1 would imply $K^{(\ell)}K(\ell^2) \subset L$ and $\frac{1}{2}\ell(\ell - 1) = [K^{(\ell)}K(\ell^2) : K(\ell)]$. In addition, $E$ has no point of order $\ell^2$ by Theorem 6.2.

Let $\iota : E \to E'$ be the canonical $L$-rational isogeny from $E_{/L}$ to $E'_{/L}$, where $E'$ has $\mathcal{O}_K$-CM. Since $e_2 \in E(L)$, the proof of Theorem 1.7 shows $\iota(e_1) \in E'(L)$, and $\iota(e_1)$ generates $E'[\ell]$ as an $\mathcal{O}_K$-module. In other words, $\mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z} \hookrightarrow E'(L)[\mathrm{tors}]$. It follows that $\ell \mid \frac{\#E'(L)[\mathrm{tors}]}{\#E(L)[\mathrm{tors}]}$. □

Finally, Theorem 1.7 requires $K \subset F$. This hypothesis cannot be omitted:

**Proposition 6.10.** *Let $\ell > 3$ be a prime with $\ell \equiv 3 \pmod 4$ and let $n \in \mathbb{Z}^{\ge 3}$. Let $K = \mathbb{Q}(\sqrt{-\ell})$, and let $\mathcal{O}$ be the order in $K$ of conductor $\mathfrak{f} = \ell^{\lfloor \frac{n}{2} \rfloor}$. Let $F = \mathbb{Q}(j(\mathbb{C}/\mathcal{O}))$. There is an elliptic curve $E_{/F}$ and an extension $L/F$ of degree $\frac{\varphi(\ell^n)}{2}$ such that:*
*(i) $L \not\supset K$,*
*(ii) $E(L)$ has a point of order $\ell^n$, and*
*(iii) for every $\mathcal{O}_K$-CM elliptic curve $E'_{/L}$ we have $\ell^n \nmid \#E'(L)[\mathrm{tors}]$.*

*Proof.* Let $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve. By [Kw99, Corollary 4.2], $E$ has an $F$-rational subgroup which is cyclic of order $\ell^n$. It follows from [BCS17, Theorem 5.6] that there is a twist $E_1$ of $E_{/F}$ and an extension $L/F$ of degree $\varphi(\ell^n)/2$ such that $E_1(L)$ has a point of order $\ell^n$. Note $[L : \mathbb{Q}] = h_K \ell^{\lfloor \frac{n}{2} \rfloor} \frac{\varphi(\ell^n)}{2}$ is odd (see [Co89, Proposition 3.11]) , so $K \not\subset L$.

Let $E'_{/L}$ be an $\mathcal{O}_K$-CM elliptic curve. Since $[L : \mathbb{Q}]$ is odd, $E'(L)[\ell^\infty]$ must be cyclic, as full $\ell^k$-torsion would imply $\mathbb{Q}(\zeta_{\ell^k}) \subset L$ by the Weil pairing. As in the proof of Proposition 6.8, $E'(LK)$ contains no point of order $\ell^n$. Hence $E'(L)$ contains no point of order $\ell^n$, and $\ell^n \nmid \#E'(L)[\mathrm{tors}]$. □

6.5. **Minimal and Maximal Cartan Orbits.** Let $\mathcal{O}$ be an order, let $N \in \mathbb{Z}^+$, and let $P \in \mathcal{O}/N\mathcal{O}$ be a point of order $N$. Since $C_N(\mathcal{O})$ contains all scalar matrices, if $P \in \mathcal{O}/N\mathcal{O}$ has order $N$, then the orbit of $C_N(\mathcal{O})$ on $P$ has size at least $\varphi(N)$. On the other hand, the orbit of $C_N(\mathcal{O})$ on $P$ is certainly no larger than the number of order $N$ points of $\mathcal{O}/N\mathcal{O}$.

In this section we will find all pairs $(\mathcal{O}, N)$ for which there exists a Cartan orbit of this smallest possible size and also all pairs for which there exists a Cartan orbit of this largest possible size.

We introduce the shorthand $H(\mathcal{O}, N)$ to mean: *there is a point $P$ of order $N$ in $\mathcal{O}/N\mathcal{O}$ such that the $C_N(\mathcal{O})$-orbit of $P$ has size $\varphi(N)$.*

**Lemma 6.11.** *Let $\mathcal{O}$ be an order, and let $N = \ell_1^{a_1} \cdots \ell_r^{a_r} \in \mathbb{Z}^+$. Then $H(\mathcal{O}, N)$ holds iff $H(\mathcal{O}, \ell_i^{a_i})$ holds for all $1 \leq i \leq r$.*

*Proof.* This is an easy consequence of the Chinese Remainder Theorem. □

**Lemma 6.12.** *Let $\mathcal{O}$ be the order of discriminant $\Delta$, $\ell$ a prime number and $a \in \mathbb{Z}^+$.*
*a) If $\left(\frac{\Delta}{\ell}\right) = 1$, there is an $\mathcal{O}$-submodule of $\mathcal{O}/\ell^a\mathcal{O}$ with underyling $\mathbb{Z}$-module $\mathbb{Z}/\ell^a\mathbb{Z}$.*
*b) If $\left(\frac{\Delta}{\ell}\right) = -1$, then $C_{\ell^a}(\mathcal{O})$ acts simply transitively on the order $\ell^a$ elements of $\mathcal{O}/\ell^a\mathcal{O}$.*

*Proof.* a) if $\left(\frac{\Delta}{\ell}\right) = 1$, then $\mathcal{O}/\ell\mathcal{O} = \mathcal{O}_K/\ell\mathcal{O}_K \cong \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$, so $\mathcal{O} \otimes \mathbb{Z}_\ell$ is isomorphic as a ring to $\mathbb{Z}_\ell \times \mathbb{Z}_\ell$ (see e.g. [Ei, Cor. 7.5]) and thus $\mathcal{O}/\ell^a\mathcal{O}$ is isomorphic as a ring to $\mathbb{Z}/\ell^a\mathbb{Z} \times \mathbb{Z}/\ell^a\mathbb{Z}$.
b) If $\left(\frac{\Delta}{\ell}\right) = -1$, then $\mathcal{O} \otimes \mathbb{Z}_\ell = \mathcal{O}_K \otimes \mathbb{Z}_\ell$ is a complete DVR with uniformizer $\ell$, so the ring $\mathcal{O}/\ell^a\mathcal{O}$ is finite, local and principal with maximal ideal $\langle \ell \rangle$. An element of $\mathcal{O}/\ell^a\mathcal{O}$ has order $\ell^a$ iff it lies in the unit group $C_{\ell^a}(\mathcal{O})$. □

**Lemma 6.13.** *Let $\mathcal{O}$ be the order of discriminant $\Delta$, and let $N \in \mathbb{Z}^+$. The following are equivalent:*
*(i) If $2 \mid N$, then $\left(\frac{\Delta}{2}\right) \neq 1$.*
*(ii) The $\mathbb{Z}/N\mathbb{Z}$-subalgebra of $\mathcal{O}/N\mathcal{O}$ generated by $C_N(\mathcal{O})$ is $\mathcal{O}/N\mathcal{O}$.*

*Proof.* Using the Chinese Remainder Theorem we reduce to the case of $N = \ell^a$ a power of a prime number $\ell$. Let $B$ be the $\mathbb{Z}/\ell^a\mathbb{Z}$-subalgebra generated by $C_{\ell^a}(\mathcal{O})$, so $\#B = \ell^b$ for some $b \leq 2a$.
(i) $\implies$ (ii): Since $0 \in B \setminus C_{\ell^a}(\mathcal{O})$, we have

$$\#B \geq \#C_{\ell^a}(\mathcal{O}) + 1$$

$$= \ell^{2a}\left(1 - \frac{1}{\ell}\right)\left(1 - \left(\frac{\Delta}{\ell}\right)\frac{1}{\ell}\right) + 1 \geq \begin{cases} \frac{4}{9}\ell^{2a} + 1 > \ell^{2a-1}, & \text{if } \ell \geq 3 \\ \frac{1}{2}\ell^{2a} + 1 > \ell^{2a-1}, & \text{if } \ell = 2 \text{ and } \left(\frac{\Delta}{2}\right) \neq 1 \end{cases}.$$

Thus $b = 2a$ and $B = \mathcal{O}/\ell^a\mathcal{O}$.
$\neg$ (i) $\implies$ $\neg$ (ii): If $\ell = 2$ and $\left(\frac{\Delta}{2}\right) = 1$, then

$$\mathcal{O}/2^a\mathcal{O} \cong \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \mid \alpha, \beta \in \mathbb{Z}/2^a\mathbb{Z} \right\}$$

and $C_{2^a}(\mathcal{O})$ consists of the set of such matrices with $\alpha, \beta \in (\mathbb{Z}/2^a\mathbb{Z})^\times$. Thus $C_{2^a}(\mathcal{O})$ is contained in the subalgebra

$$\mathcal{B} = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \mid \alpha, \beta \in \mathbb{Z}/2^a\mathbb{Z} \text{ and } \alpha \equiv \beta \pmod 2 \right\}$$

of order $2^{2a-1}$, so $B \subset \mathcal{B} \subsetneq \mathcal{O}/2^a\mathcal{O}$.[1] □

**Lemma 6.14.** *For an order $\mathcal{O}$ and $N \in \mathbb{Z}^+$, the following are equivalent:*
*(i) There is an ideal $I$ of $\mathcal{O}$ with $\mathcal{O}/I \cong \mathbb{Z}/N\mathbb{Z}$.*
*(ii) There is an $\mathcal{O}$-submodule of $\mathcal{O}/N\mathcal{O}$ with underlying commutative group $\mathbb{Z}/N\mathbb{Z}$.*
*(iii) $H(\mathcal{O}, N)$ holds.*

---

[1] Since $\#B \geq \#C_{2^a}(\mathcal{O}) + 1 = 2^{2a-2} + 1 > 2^{2a-2}$, in fact we have $B = \mathcal{B}$.

*Proof.* (i) $\iff$ (ii):
Step 1: Let $\Lambda$ be a free, rank 2 $\mathbb{Z}$-module, and let $\Lambda'$ be a $\mathbb{Z}$-submodule of $\Lambda$ containing $N\Lambda$. By the structure theory of modules over a PID, there is a $\mathbb{Z}$-basis $e_1, e_2$ for $\Lambda$ and positive integers $a \mid b$ such that $ae_1, be_2$ is a $\mathbb{Z}$-basis for $\Lambda'$. Thus

$$\Lambda/\Lambda' \cong \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z},\ \Lambda'/N\Lambda \cong \mathbb{Z}/(N/b)\mathbb{Z} \oplus \mathbb{Z}/(N/a)\mathbb{Z}.$$

It follows that $\Lambda/\Lambda' \cong \mathbb{Z}/N\mathbb{Z} \iff \Lambda'/N\Lambda \cong \mathbb{Z}/N\mathbb{Z}$.
Step 2: If $I$ is an ideal of $\mathcal{O}$ with $\mathcal{O}/I \cong \mathbb{Z}/N\mathbb{Z}$, then $I \supset N\mathcal{O}$, so $I/N\mathcal{O} \cong \mathbb{Z}/N\mathbb{Z}$ by Step 1. Let $M$ be an $\mathcal{O}$-submodule of $\mathcal{O}/N\mathcal{O}$ with underyling $\mathbb{Z}$-module $\mathbb{Z}/N\mathbb{Z}$. Then $M = I/N\mathcal{O}$ for an ideal $I$ of $\mathcal{O}$, and by Step 1 we have $\mathcal{O}/I \cong \mathbb{Z}/N\mathbb{Z}$.
(ii) $\implies$ (iii): Let $P \in \mathcal{O}/N\mathcal{O}$ have order $N$ such that the subgroup generated by $P$ is an $\mathcal{O}$-submodule. For all $g \in C_N(\mathcal{O})$, $gP = a_gP$ for $a_g \in (\mathbb{Z}/N\mathbb{Z})^\times$. Conversely, since $C_N(\mathcal{O})$ contains all scalar matrices, the orbit of $C_N(\mathcal{O})$ on $P$ has size $\varphi(N)$.
(iii) $\implies$ (ii): Case 1: Suppose $2 \nmid N$ or $\left(\frac{\Delta}{2}\right) \neq 1$. Let $P \in \mathcal{O}/N\mathcal{O}$ be a point of order $N$ with $C_N(\mathcal{O})$-orbit of size $\varphi(N)$. There is a $\mathbb{Z}/N\mathbb{Z}$-basis $e_1, e_2$ of $\mathcal{O}/N\mathcal{O}$ with $e_1 = P$, and our hypothesis gives that with respect to this basis $C_N(\mathcal{O})$ lies in the subalgebra $\left\{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, d \in \mathbb{Z}/N\mathbb{Z}\right\}$ of upper triangular matrices. By Lemma 6.13, $\mathcal{O}/N\mathcal{O}$ also lies in the subalgebra of upper triangular matrices, and thus $\langle P \rangle$ is an $\mathcal{O}$-stable submodule with underlying $\mathbb{Z}$-module $\mathbb{Z}/N\mathbb{Z}$.

Case 2: Suppose $2 \mid N$ and $\left(\frac{\Delta}{2}\right) = 1$, and write $N = 2^aN'$ with $2 \nmid N'$. By Lemma 6.12 and the equivalence of (i) and (ii), there is an ideal $I_1$ in $\mathcal{O}$ with $\mathcal{O}/I_1 \cong \mathbb{Z}/2^a\mathbb{Z}$, and by Case 1 there is an ideal $I_2$ in $\mathcal{O}$ with $\mathcal{O}/I_2 \cong \mathbb{Z}/N'\mathbb{Z}$. By the Chinese Remainder Theorem, $\mathcal{O}/I_1I_2 \cong \mathbb{Z}/N\mathbb{Z}$. Since (i) $\iff$ (ii), this suffices. □

**Theorem 6.15.** *Let $\mathcal{O}$ be an order of discriminant $\Delta$, and let $N \in \mathbb{Z}^+$. The following are equivalent:*
*(i) $H(\mathcal{O}, N)$ holds.*
*(ii) $\Delta$ is a square in $\mathbb{Z}/4N\mathbb{Z}$.*

*Proof.* Using the Chinese Remainder Theorem and Lemma 6.11, we reduce to the case in which $N = \ell^a$ is a power of a prime number $\ell$.
Case 1 ($\ell$ is odd): Since $\gcd(4, \ell^a) = 1$, we may put $D = \frac{\Delta}{4} \in \mathbb{Z}/\ell^a\mathbb{Z}$. Then $\Delta$ is a square in $\mathbb{Z}/4\ell^a\mathbb{Z}$ iff $D$ is a square in $\mathbb{Z}/\ell^a\mathbb{Z}$, and

$$\mathcal{O}/\ell^a\mathcal{O} \cong \mathbb{Z}/\ell^a\mathbb{Z}[t]/(t^2 - D). \tag{6}$$

If there is $s \in \mathbb{Z}/\ell^a\mathbb{Z}$ such that $D = s^2$, then

$$\mathcal{O}/\ell^a\mathcal{O} \cong \mathbb{Z}/\ell^a\mathbb{Z}[t]/((t+s)(t-s)),$$

so if $I$ is the ideal $\langle t + s, \ell^a \rangle$ of $\mathcal{O}$, then $\mathcal{O}/I \cong \mathbb{Z}/\ell^a\mathbb{Z}$. By Lemma 6.14, $H(\mathcal{O}, \ell^a)$ holds. Conversely, suppose $H(\mathcal{O}, \ell^a)$ holds, so by Lemma 6.14 there is an ideal $I$ of $\mathcal{O}$ with $\mathcal{O}/I \cong \mathbb{Z}/\ell^a\mathbb{Z}$. Since $\ell^a \in I$, we may regard $I$ as an ideal of $\mathcal{O}/\ell^a\mathcal{O}$ such that $(\mathcal{O}/\ell^a\mathcal{O})/I \cong \mathbb{Z}/\ell^a\mathbb{Z}$. In other words, we have a $\mathbb{Z}/\ell^a\mathbb{Z}$-algebra homomorphism

$$f : \mathbb{Z}/\ell^a\mathbb{Z}[t]/(t^2 - D) \to \mathbb{Z}/\ell^a\mathbb{Z}.$$

Then $f(t)^2 = D \in \mathbb{Z}/\ell^a\mathbb{Z}$, so $D$ is a square in $\mathbb{Z}/\ell^a\mathbb{Z}$.
Case 2 ($\ell = 2$, $\Delta$ is odd): Then $\left(\frac{\Delta}{\ell}\right) = \pm 1$.
• If $\left(\frac{\Delta}{\ell}\right) = 1$, then $\Delta \equiv 1 \pmod 8$; by Hensel's Lemma, $\Delta$ is a square in $\mathbb{Z}/\ell^a\mathbb{Z}$. On the other hand, by Lemmas 6.12a) and 6.14, $H(\mathcal{O}, \ell^a)$ holds.
• If $\left(\frac{\Delta}{\ell}\right) = -1$, then $\Delta \equiv 5 \pmod 8$, so $\Delta$ is not a square modulo 8 and thus not a square modulo $4 \cdot 2^a$. On the other hand, by Lemma 6.12b) $H(\mathcal{O}, \ell^a)$ does not hold.
Case 3: ($\ell = 2$, $\Delta$ is even): Again we may put $D = \frac{\Delta}{4} \in \mathbb{Z}/\ell^a\mathbb{Z}$, and again (6) holds. The argument of Case 1 shows that $H(\mathcal{O}, \ell^a)$ holds iff $D$ is a square modulo $\mathbb{Z}/\ell^a\mathbb{Z}$ iff $\Delta$ is a square modulo $\mathbb{Z}/4\ell^a\mathbb{Z}$. □

**Proposition 6.16.** *Let $\mathcal{O}$ be an order, and let $N \in \mathbb{Z}^+$. The following are equivalent:*
*(i) $C_N(\mathcal{O})$ acts simply transitively on order $N$ elements of $\mathcal{O}/N\mathcal{O}$.*

*(ii)* $C_N(\mathcal{O})$ *acts transitively on order* $N$ *elements of* $\mathcal{O}/N\mathcal{O}$.
*(iii) For all primes* $\ell \mid N$ *we have* $\left(\frac{\Delta}{\ell}\right) = -1$.

*Proof.* As usual, we may assume $N = \ell^a$ is a prime power. Certainly (i) $\implies$ (ii).
(ii) $\implies$ (iii): We have

$$\#C_{\ell^a}(\mathcal{O}) = \ell^{2a-2}(\ell-1)\left(\ell - (\frac{\Delta}{\ell})\right),$$

whereas the number of elements of order $\ell^a$ in $\mathcal{O}/\ell^a\mathcal{O}$ is

$$N(\mathcal{O},\ell^a) := \#\mathcal{O}/\ell^a\mathcal{O} - \#\ell\mathcal{O}/\ell^a\mathcal{O} = \ell^{2a-2}(\ell-1)(\ell+1).$$

Transitivity of the action implies $\#C_{\ell^a}(\mathcal{O}) \geq N(\mathcal{O},\ell^a)$, which holds iff $\left(\frac{\Delta}{\ell}\right) = -1$.
(iii) $\implies$ (i): Since $\left(\frac{\Delta}{\ell}\right) \neq 0$, we have $\mathcal{O}/\ell^a\mathcal{O} \cong \mathcal{O}_K/\ell^a\mathcal{O}_K$, and thus also $C_{\ell^a}(\mathcal{O}) = (\mathcal{O}/\ell^a\mathcal{O})^\times \cong C_{\ell^a}(\mathcal{O}_K)$. Thus $\mathcal{O}/\ell^a\mathcal{O}$ is a finite local principal ring with maximal ideal $\mathfrak{m} = \langle \ell \rangle$ and unit group $C_{\ell^a}(\mathcal{O}) = \mathcal{O}/\ell^a\mathcal{O} \setminus \mathfrak{m}$. The set of order $\ell^a$ elements of $\mathcal{O}/\ell^a\mathcal{O}$ is $\mathcal{O}/\ell^a\mathcal{O} \setminus \mathfrak{m} = C_{\ell^a}(\mathcal{O})$, so the action of the unit group $C_{\ell^a}(\mathcal{O})$ on this set is the action of $C_{\ell^a}(\mathcal{O})$ on itself, which is simply transitive. □

**Corollary 6.17.** *Let* $\mathcal{O}$ *an order of conductor* $\mathfrak{f}$. *Let* $N = \prod_{i=1}^r \ell_i^{a_i} \in \mathbb{Z}^+$ *be such that* $\left(\frac{\Delta}{\ell_i}\right) = -1$ *for all* $i$. *Let* $F$ *be a number field, and let* $E_{/F}$ *be an* $\mathcal{O}$*-CM elliptic curve such that* $E(F)$ *has a point of order* $N$. *Then*

$$\#\overline{C_N(\mathcal{O})} \mid [FK : K(\mathfrak{f})]. \tag{7}$$

*Moreover, for all* $\mathcal{O}$ *and* $N$ *satisfying the above conditions, equality can occur in (7).*

*Proof.* Replace $F$ by $FK$; then $F \supset K(\mathfrak{f})$. By Proposition 6.16, $C_N(\mathcal{O})$ acts transitively on order $N$ elements of $\mathcal{O}/N\mathcal{O}$, so the $\mathcal{O}$-submodule generated by any one of them is $\mathcal{O}/N\mathcal{O}$. Thus the existence of one $F$-rational point of order $N$ implies that $\rho_N$ is trivial. Applying Theorem 1.1 gives (7). That equality can occur follows from Corollary 1.4. □

6.6. **Torsion over** $K(j)$**: Part I.** Let $\mathcal{O}$ be an order of discriminant $\Delta = \mathfrak{f}^2\Delta_K$. We will give a complete classification of the possible torsion subgroups of $\mathcal{O}$-CM elliptic curves $E_{/K(\mathfrak{f})}$. In this section we will treat the cases $\Delta \neq -3, -4$. For the remaining cases we will make use of Theorem 7.2, so we will come back to those cases in §7.5.

If $E(K(\mathfrak{f}))$ has a point of order $N$, then since $[C_N(\mathcal{O}) : \rho_N(\mathfrak{g}_{K(\mathfrak{f})})] \mid \#\mathcal{O}^\times$, there must be some $P \in \mathcal{O}/N\mathcal{O}$ of order $N$ with a $C_N(\mathcal{O})$-orbit of order dividing $\#\mathcal{O}^\times$.

- By Theorem 6.2, if $E(K(\mathfrak{f}))$ has a point of order $N$, then $\varphi(N) \mid 2$, so

$$N \in \{1,2,3,4,6\}.$$

- Lemma 2.2b) implies that for all $N \geq 3$, we have $\#C_N(\mathcal{O}) \geq 4$ (equality holds if $N = 3$ and $\Delta \equiv 1 \pmod 3$). By Corollary 1.2 we cannot have $E[N] = E[N](K(\mathfrak{f}))$.

Thus $E(K(\mathfrak{f}))[\text{tors}]$ is isomorphic to one of the groups in the following list:

$$\{e\}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}.$$

We will show that all of these groups occur.

**Points of order 2**: By Theorem 1.1, $E(K(\mathfrak{f}))[2]$ has order 4 if 2 splits in $\mathcal{O}$, order 2 if 2 ramifies in $\mathcal{O}$ and order 1 if 2 is inert in $\mathcal{O}$. Thus:

$$E(K(\mathfrak{f}))[2] \cong \begin{cases} \{e\} & \Delta \equiv 5 \pmod 8 \\ \mathbb{Z}/2\mathbb{Z} & \Delta \equiv 0 \pmod 4 \\ \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} & \Delta \equiv 1 \pmod 8 \end{cases}.$$

**Points of order 3, 4, or 6**: Let $E_{/K(\mathfrak{f})}$ be any $\mathcal{O}$-CM elliptic curve. We claim that for $N \in \{3,4,6\}$, there is a quadratic twist $E^D$ of $E$ such that $E^D(K(\mathfrak{f}))$ has a point of order $N$ iff $H(\mathcal{O},N)$ holds. Indeed, as above, since the index of the mod $N$ Galois representation in $C_N(\mathcal{O})$ divides 2, if some $E^D(K(\mathfrak{f}))$ has a point of order $N$, then $\mathcal{O}/N\mathcal{O}$ has a point of order $N$ with a $C_N(\mathcal{O})$-orbit of size 2. Since $\varphi(N) = 2$, there is a Cartan orbit of size 2 iff $H(\mathcal{O},N)$ holds. Conversely, if $H(\mathcal{O},N)$ holds then there is a point of order $N$ with a $C_N(\mathcal{O})$-orbit of size 2, hence on some quadratic twist $E^D$ we have an $F$-rational point of order $N$. Applying Theorem 6.15, we get:

- Some $\mathcal{O}$-CM $E_{/K(\mathfrak{f})}$ has a point of order 3 iff $\Delta \equiv 0,1 \pmod 3$.
- Some $\mathcal{O}$-CM $E_{/K(\mathfrak{f})}$ has a point of order 4 iff $\Delta \equiv 0,1,4,9 \pmod{16}$.
- Some $\mathcal{O}$-CM $E_{/K(\mathfrak{f})}$ has a point of order 6 iff $\Delta \equiv 0,1,2,9,12,16 \pmod{24}$.

Because the only full $N$-torsion we can have is full 2-torsion, and 2-torsion is invariant under quadratic twists, we immediately deduce the complete answer in all cases.

- If $\Delta \equiv 0 \pmod{48}$, then there are twists $E_1, E_2, E_3$ of $E$ with
$$E_1(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/2\mathbb{Z},\ E_2(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/4\mathbb{Z},\ E_3(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/6\mathbb{Z}.$$
- If $\Delta \equiv 1,9,25,33 \pmod{48}$ then there are twists $E_1, E_2$ of $E$ with
$$E_1(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},\ E_2(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}.$$
- If $\Delta \equiv 4,16,36 \pmod{48}$, then there are twists $E_1, E_2, E_3$ of $E$ with
$$E_1(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/2\mathbb{Z},\ E_2(K(\mathfrak{f})) \cong \mathbb{Z}/4\mathbb{Z},\ E_3(K(\mathfrak{f})) \cong \mathbb{Z}/6\mathbb{Z}.$$
- If $\Delta \equiv 5,29 \pmod{48}$, then $E(K(\mathfrak{f}))[\mathrm{tors}] = \{e\}$.
- If $\Delta \equiv 8,44 \pmod{48}$, then $E(K(\mathfrak{f}))[\mathrm{tors}] = \mathbb{Z}/2\mathbb{Z}$.
- If $\Delta \equiv 12,24,28,40 \pmod{48}$, then there are twists $E_1, E_2$ of $E$ with
$$E_1(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/2\mathbb{Z},\ E_2(K(\mathfrak{f})) \cong \mathbb{Z}/6\mathbb{Z}.$$
- If $\Delta \equiv 13,21,37,45 \pmod{48}$, then there are twists $E_1, E_2$ of $E$ with
$$E_1(K(\mathfrak{f}))[\mathrm{tors}] = \{e\},\ E_2(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/3\mathbb{Z}.$$
- If $\Delta \equiv 17,41 \pmod{48}$, then there are twists $E_1, E_2$ of $E$ with
$$E_1(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},\ E_2(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}.$$
- If $\Delta \equiv 20,32 \pmod{48}$, then there are twists $E_1, E_2$ of $E$ with
$$E_1(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/2\mathbb{Z},\ E_2(K(\mathfrak{f})) \cong \mathbb{Z}/4\mathbb{Z}.$$

### 6.7. Isogenies over $K(j)$: Part I.

**Theorem 6.18.** *Let $\mathcal{O}$ be an order of discriminant $\Delta = \mathfrak{f}^2\Delta_K$, and let $N \in \mathbb{Z}^+$.*
*a) If $\Delta \neq -3,-4$, then there is an $\mathcal{O}$-CM elliptic curve $E_{/K(\mathfrak{f})}$ with a $K(\mathfrak{f})$-rational cyclic $N$-isogeny iff $\Delta$ is a square in $\mathbb{Z}/4N\mathbb{Z}$.*
*b) If $\Delta = -4$, then then there is an $\mathcal{O}$-CM elliptic curve $E_{/K(\mathfrak{f})}$ with a $K(\mathfrak{f})$-rational cyclic $N$-isogeny iff $N$ is of the form $2^{\epsilon}\ell_1^{a_1}\cdots\ell_r^{a_r}$ for primes $\ell_i \equiv 1 \pmod 4$ and $\epsilon, a_1, \ldots, a_r \in \mathbb{N}$ with $\epsilon \leq 2$.*
*c) If $\Delta = -3$, then there is an $\mathcal{O}$-CM elliptic curve $E_{/K(\mathfrak{f})}$ with a $K(\mathfrak{f})$-rational cyclic $N$-isogeny iff $N$ is of the form $2^{\epsilon}3^{a}\ell_1^{a_1}\cdots\ell_r^{a_r}$ for primes $\ell_i \equiv 1 \pmod 3$, $\epsilon, a, a_1, \ldots, a_r \in \mathbb{N}$ with $(\epsilon, a) \in \{(0,0),\ (0,1),\ (0,2),\ (1,0),\ (1,1)\}$.*

*Proof.* Step 1: Let $E_{/K(\mathfrak{f})}$ be an $\mathcal{O}$-CM elliptic curve. If $\Delta$ is a square in $\mathbb{Z}/4N\mathbb{Z}$, then by Theorem 6.15 there is a point $P$ of order $N$ in $\mathcal{O}/N\mathcal{O}$ such that $C = \langle P \rangle$ is invariant under $C_N(\mathcal{O})$, so $C$ is $\mathfrak{g}_{K(\mathfrak{f})}$-stable and $E \to E/C$ is a cyclic $N$-isogeny. If $\Delta \notin \{-4,-3\}$, then the projective Galois representation $\mathbb{P}\rho_N : \mathfrak{g}_{K(\mathfrak{f})} \to C_N(\mathcal{O})/(\mathbb{Z}/N\mathbb{Z})^{\times}$ is a quotient of the reduced Galois representation, hence surjective. So $K(\mathfrak{f})$-rational cyclic $N$-isogenies correspond to $C_N(\mathcal{O})$-orbits on $\mathcal{O}/N\mathcal{O}$ of size $\varphi(N)$, which by Theorem 6.15 exist iff $\Delta$ is a square in $\mathbb{Z}/4N\mathbb{Z}$.

Step 2: If $\Delta \in \{-4,-3\}$, then as above the condition that $\Delta$ is a square modulo $4N$ is sufficient for the existence of a $K(\mathfrak{f})$-rational cyclic $N$-isogeny, but it is no longer clear that it is necessary, and in both cases it turns out not to be. The complete analysis will make use of Theorem 7.2, so we defer the end of the proof until §7.6. $\square$

## 7. The Torsion Degree Theorem

### 7.1. **Statement and Preliminary Reduction.**

Throughout this section $\mathcal{O}$ denotes an order of conductor $\mathfrak{f}$ and discriminant $\Delta = \mathfrak{f}^2\Delta_K$.

For $N \in \mathbb{Z}^{\geq 2}$, let $\widetilde{T}(\mathcal{O},N)$ be the least size of an orbit of $C_N(\mathcal{O})$ on an order $N$ point of $\mathcal{O}/N\mathcal{O}$.

**Lemma 7.1.** *We have* $\widetilde{T}(\mathcal{O},2) = \begin{cases} 1 & \text{if } \left(\frac{\Delta}{2}\right) \neq -1 \\ 3 & \text{if } \left(\frac{\Delta}{2}\right) = -1 \end{cases}.$

*Proof.* By Theorem 6.15, we have $\left(\frac{\Delta}{2}\right) \neq -1$ iff there is a $C_2(\mathcal{O})$-orbit of size $\varphi(2) = 1$ on $\mathcal{O}/2\mathcal{O}$ iff $\widetilde{T}(\mathcal{O},2) = 1$. In the remaining case $\left(\frac{\Delta}{2}\right) = -1$ we have $\#C_2(\mathcal{O}) = 3$ and no orbit of size 1, hence $\widetilde{T}(\mathcal{O},2) = 3$. $\square$

**Theorem 7.2.** *(Torsion Degree Theorem) Let $\mathcal{O}$ be an order of conductor $\mathfrak{f}$, and let $N \in \mathbb{Z}^{\geq 3}$.*
*a) There is $T(\mathcal{O},N) \in \mathbb{Z}^+$ such that:*
*(i) if $F \supset K(\mathfrak{f})$ is a number field and $E_{/F}$ is an $\mathcal{O}$-CM elliptic curve with an $F$-rational point of order $N$, then $T(\mathcal{O},N) \mid [F : K(\mathfrak{f})]$, and*
*(ii) there is a number field $F \supset K(\mathfrak{f})$ with $[F : K(\mathfrak{f})] = T(\mathcal{O},N)$ and an $\mathcal{O}$-CM elliptic curve $E_{/F}$ with an $F$-rational point of order $N$.*
*b) If $(\Delta,N) = (-3,3)$, then $T(\mathcal{O},N) = 1$.*
*c) Suppose $(\Delta,N) \neq (-3,3)$. Let $N = \ell_1^{a_1}\cdots\ell_r^{a_r}$ be the prime power decomposition of $N$. Then*

$$T(\mathcal{O},N) = \frac{\prod_{i=1}^r \widetilde{T}(\mathcal{O},\ell_i^{a_i})}{\#\mathcal{O}^\times}.$$

*d) If $\ell^a = 2$, then $\widetilde{T}(\mathcal{O},\ell^a) = 2$ is computed in Lemma 7.1. If $\ell^a > 2$, then $\widetilde{T}(\mathcal{O},\ell^a)$ is as follows, where $k = \operatorname{ord}_\ell(\mathfrak{f})$:*

$$\text{(1) If } \ell \nmid \mathfrak{f}, \text{ then } \widetilde{T}(\mathcal{O},\ell^a) = \begin{cases} \ell^{a-1}(\ell-1) & \text{if } \left(\frac{\Delta}{\ell}\right) = 1, \\ \ell^{2a-2}(\ell-1) & \text{if } \left(\frac{\Delta}{\ell}\right) = 0, \\ \ell^{2a-2}(\ell^2-1) & \text{if } \left(\frac{\Delta}{\ell}\right) = -1. \end{cases}$$

$$\text{(2) If } \ell \mid \mathfrak{f}, \text{ then } \widetilde{T}(\mathcal{O},\ell^a) = \begin{cases} \ell^{a-1}(\ell-1) & \text{if } \left(\frac{\Delta_K}{\ell}\right) = 1, \\ \ell^{a-1}(\ell-1) & \text{if } \left(\frac{\Delta_K}{\ell}\right) = -1 \text{ and } a \leq 2k, \\ \ell^{2a-2k-1}(\ell-1) & \text{if } \left(\frac{\Delta_K}{\ell}\right) = -1 \text{ and } a > 2k, \\ \ell^{a-1}(\ell-1) & \text{if } \left(\frac{\Delta_K}{\ell}\right) = 0 \text{ and } a \leq 2k+1, \\ \ell^{2a-2k-2}(\ell-1) & \text{if } \left(\frac{\Delta_K}{\ell}\right) = 0 \text{ and } a > 2k+1. \end{cases}$$

**Remark 7.3.** *The case $N = 2$ is excluded because of the somewhat anomalous behavior of 2-torsion. But it is easy to see that Theorem 7.2a) remains true when $N = 2$, and moreover:*

- *If $\Delta \in \{-4,-3\}$ then $T(\mathcal{O},2) = 1$.*
- *Otherwise,* $T(\mathcal{O},2) = \begin{cases} 1 & \text{if } \left(\frac{\Delta}{2}\right) \neq -1 \\ 3 & \text{if } \left(\frac{\Delta}{2}\right) = -1 \end{cases}.$

Let $F \supset K(\mathfrak{f})$ be a number field, and let $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve. As usual, we choose an embedding $F \hookrightarrow \mathbb{C}$ such that $j(E) = j(\mathbb{C}/\mathcal{O})$. Let $P \in E[\text{tors}]$ have order $N$. We call the field

$$K(\mathfrak{f})(\mathfrak{h}(P))$$

the **field of moduli** of $P$. It is independent of the chosen model of $E_{/F}$, and there exists an elliptic curve $E'_{/K(\mathfrak{f})(\mathfrak{h}(P))}$ with an isomorphism $\psi : E \to E'$ such that $\psi(P)$ is $K(\mathfrak{f})(\mathfrak{h}(P))$-rational. Further, the pair $(E, P)$ induces a closed point $\mathcal{P}$ on the modular curve $X_1(N)_{/K}$, and $K(\mathfrak{f})(\mathfrak{h}(P))$ is the residue field $K(\mathcal{P})$. Theorem 7.2 concerns the degree $[K(\mathfrak{f})(\mathfrak{h}(P)) : K(\mathfrak{f})]$. Our setup shows that it is no loss of generality to assume $F = K(\mathfrak{f})$.

Let $q_N : \mathcal{O} \to \mathcal{O}/N\mathcal{O}$ be the natural map, and let $q_N^\times : \mathcal{O}^\times \to C_N(\mathcal{O})$ be the induced map on unit groups. As in the introduction, we define the **reduced mod N Cartan subgroup**:

$$\overline{C_N(\mathcal{O})} = C_N(\mathcal{O})/q_N(\mathcal{O}^\times).$$

Let $\overline{E[N]}$ be the set of $\mathcal{O}^\times$-orbits on $E[N]$. Then the action of $C_N(\mathcal{O})$ on $E[N]$ induces an action of $\overline{C_N(\mathcal{O})}$ on $\overline{E[N]}$. The field of moduli $K(\mathfrak{f})(\mathfrak{h}(P))$ depends only on the image $\overline{P}$ of $P$ in $\overline{E[N]}$. By Corollary 1.2, the composite homomorphism

$$\mathfrak{g}_F \xrightarrow{\rho_{E,N}} C_N(\mathcal{O}) \to \overline{C_N(\mathcal{O})}$$

is surjective (and model-independent). Let $H_{\overline{P}} = \{g \in \overline{C_N(\mathcal{O})} \mid g\overline{P} = \overline{P}\}$. It follows that

$$\operatorname{Aut}(K(\mathfrak{f})(\mathfrak{h}(P))/K(\mathfrak{f})) \cong \overline{C_N(\mathcal{O})}/H_{\overline{P}}.$$

Thus $[K(\mathfrak{f})(\mathfrak{h}(P)) : K(\mathfrak{f})]$ is the size of the orbit of the reduced Cartan subgroup $\overline{C_N(\mathcal{O})}$ on $\overline{P}$. (As we will see, in almost every case this is the size of the orbit of $C_N(\mathcal{O})$ on $P$ divided by $\#\mathcal{O}^\times$.) This reduces the proof of Theorem 7.2 to a purely algebraic problem.

7.2. **Generalities.** For an order $N$ point $P \in \mathcal{O}/N\mathcal{O}$, let $M_P = \{xP \mid x \in \mathcal{O}\}$ be the cyclic $\mathcal{O}$-submodule of $\mathcal{O}/N\mathcal{O}$ generated by $P$. If we put $I_P = \{x \in \mathcal{O} \mid xP = 0\}$, then we have

$$M_P \cong_{\mathcal{O}} \mathcal{O}/I_P.$$

The isomorphism is canonical and determined by mapping $P \in M_P$ to $1 + I_P \in \mathcal{O}/I_P$.

**Lemma 7.4.** *a) With notation as above, let*

$$S(I_P) = \{g \in C_N(\mathcal{O}) \mid g \equiv 1 \pmod{I_P}\}.$$

*Then with respect to the $C_N(\mathcal{O})$-action, $S(I_P)$ is the stabilizer of $P$, so as a $C_N(\mathcal{O})$-set the orbit of $C_N(\mathcal{O})$ on $P$ is isomorphic to $C_N(\mathcal{O})/S(I_P)$.*
*b) Moreover, there is a canonical isomorphism of groups $C_N(\mathcal{O})/S(I_P) \xrightarrow{\sim} (\mathcal{O}/I_P)^\times$.*

*Proof.* a) For $g \in C_N(\mathcal{O})$, we have $gP = P \iff (g-1)P = 0 \iff (g-1) \in I_P$, giving the first assertion. The Orbit Stabilizer Theorem gives the second assertion.
b) The ring homomorphism $f : \mathcal{O}/N\mathcal{O} \to \mathcal{O}/I$ induces a homomorphism on unit groups $f^\times : C_N(\mathcal{O}) \to (\mathcal{O}/I_P)^\times$, with kernel $S(I_P)$. Since $\mathcal{O}/N\mathcal{O}$ has finitely many maximal ideals, $f^\times$ is surjective [CA, Thm. 4.32]. □

**Lemma 7.5.** *There is a positive integer $M \mid N$ such that*

$$\mathcal{O}/I_P \cong_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/M\mathbb{Z}.$$

*Proof.* As a $\mathbb{Z}$-module, $\mathcal{O}/I_P$ is a quotient of $\mathcal{O}/N\mathcal{O} \cong_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}$, so

$$\mathcal{O}/I_P \cong_{\mathbb{Z}} \mathbb{Z}/N'\mathbb{Z} \oplus \mathbb{Z}/M\mathbb{Z}$$

with $M \mid N' \mid N$. Since $P$ has order $N$ in $(\mathcal{O}/I_P, +)$, we have $N' = N$. □

The following result computes the size of the reduced Cartan orbit on an order $N$ point of $\mathcal{O}/N\mathcal{O}$ in terms of the size of the Cartan orbit. We recall that we have assumed $N \geq 3$.

**Lemma 7.6.** *a) Suppose $(\Delta, N) \neq (-3, 3)$, and let $P \in \mathcal{O}/N\mathcal{O}$ have order $N$. Then the orbit of $C_N(\mathcal{O})$ on $P$ has size $\#\mathcal{O}^\times$ times the size of the orbit of $\overline{C_N(\mathcal{O})}$ on $\overline{P}$.*
*b) Suppose $(\Delta, N) = (-3, 3)$. Then the order 3 points of $\mathcal{O}/3\mathcal{O}$ lie in two orbits under $C_3(\mathcal{O})$: one of size 2 and one of size 6. The corresponding reduced Cartan orbits each have size 1.*

*Proof.* a) The Cartan orbit has size $\#(\mathcal{O}/I_P)^\times$, and the reduced Cartan orbit is smaller by a factor of the cardinality of the image of $\mathcal{O}^\times \to (\mathcal{O}/I_P)^\times$.
- Suppose $\Delta \notin \{-4,-3\}$. Then $\mathcal{O}^\times = \{\pm 1\}$, and since $N \geq 3$, we have $-1 \not\equiv 1 \pmod{I_P}$.
- Suppose $\Delta = -4$. Since $I_P \not\supset (2)$, by Lemma 2.10 the group $U_{I_P}(K)$ is trivial, and thus the map $\mathcal{O}^\times \to (\mathcal{O}/I_P)^\times$ is injective.
- Suppose $\Delta = -3$. By assumption $N \geq 4$, so $I_P \nmid (\zeta_3 - 1)$ and the map $\mathcal{O}^\times \to (\mathcal{O}/I_P)^\times$ is injective.

b) The assertion about Cartan orbits is a case of [CCS13, Lemma 19]. (And another proof will be given in the next section.) The fact that both reduced Cartan orbits have size 1 follows from the already established fact that there is an $\mathcal{O}$-CM $E_{/\mathbb{Q}(\sqrt{-3})}$ with full 3-torsion. □

In view of Lemma 7.6, to prove Theorem 7.2 it suffices to compute the least size of an orbit of $C_N(\mathcal{O})$ on an order $N$ point of $\mathcal{O}/N\mathcal{O}$ and show that this divides the size of every such orbit. The following result further reduce us to the case of $N$ a prime power.

**Proposition 7.7.** *Let $N \geq 2$ have prime power decomposition $N = \ell_1^{a_1} \cdots \ell_r^{a_r}$. Let $P \in \mathcal{O}/N\mathcal{O}$ have order $N$, and let $I_P = \operatorname{ann} P$. For $1 \leq i \leq r$, let $P_i = \frac{N}{\ell_i^{a_i}} P$, and let $I_{P_i} = \operatorname{ann} P_i$. Then:*
*a) The ideals $I_{P_1}, \ldots, I_{P_r}$ are pairwise comaximal: we have $I_{P_i} + I_{P_j} = \mathcal{O}$ for all $i \neq j$.*
*b) We have $I_P = I_{P_1} \cdots I_{P_r}$.*
*c) We have a canonical isomorphism of rings*

$$\mathcal{O}/I_P \xrightarrow{\sim} \prod_{i=1}^r \mathcal{O}/I_{P_i}$$

*which induces a canonical isomorphism of unit groups*

$$(\mathcal{O}/I_P)^\times \xrightarrow{\sim} \prod_{i=1}^r (\mathcal{O}/I_{P_i})^\times.$$

*d) The Cartan orbit of $P$ is isomorphic, as a $C_N(\mathcal{O})$-set, to the direct product of the $C_{\ell_i^{a_i}}(\mathcal{O})$-orbits of the $P_i$'s.*

*Proof.* a) For $1 \leq i \leq r$, we have $(\mathcal{O}/I_{P_i}, +) \cong \mathbb{Z}/\ell_i^{a_i}\mathbb{Z} \oplus \mathbb{Z}/\ell_i^{b_i}\mathbb{Z}$ with $0 \leq b_i \leq a_i$; in particular it is an $\ell_i$-group. Thus for $i \neq j$, $(\mathcal{O}/(I_i + I_j), +)$ is a homomorphic image of an $\ell_i$-group and an $\ell_j$-group, so it is trivial.
b) By the Chinese Remainder Theorem, we have $I_{P_1} \cdots I_{P_n} = \bigcap_{i=1}^n I_{P_i}$. Since $P_i$ is a multiple of $P$, we have $I_P \subset I_{P_i}$ for all $i$, and thus $I_P \subset \bigcap_{i=1}^r I_{P_i}$. Conversely, choose $y_1, \ldots, y_r \in \mathbb{Z}$ such that $\sum_{i=1}^r y_i \frac{N}{\ell_i^{a_i}} = 1$. If $x \in \bigcap_{i=1}^r I_{P_i}$ then $x \frac{N}{\ell_i^{a_i}} P = 0$ for all $i$, hence

$$0 = \sum_{i=1}^r y_i \frac{N}{\ell_i^{a_i}} xP = xP,$$

so $x \in I_P$. Thus $I_P = \bigcap_{i=1}^n I_{P_i} = I_{P_1} \cdots I_{P_n}$.
c) The Chinese Remainder Theorem gives the first isomorphism; the second follows by passing to unit groups.
d) Apply Lemma 7.4 and part c). □

### 7.3. The Case $\ell \nmid \mathfrak{f}$.

**Theorem 7.8.** *Let $E_{/K(\mathfrak{f})}$ be an $\mathcal{O}$-CM elliptic curve. Let $\ell^a > 2$ be a prime power such that $\ell \nmid \mathfrak{f}$. We will describe all orbits of $C_{\ell^a}(\mathcal{O})$ on order $\ell^a$ points of $\mathcal{O}/\ell^a\mathcal{O}$: their sizes and their multiplicities.*
*a) If $\left(\frac{\Delta}{\ell}\right) = 1$, there are $2a+1$ orbits: two orbits of size $\ell^{a-1}(\ell-1)$, for all $1 \leq i \leq a-1$ two orbits of size $\ell^{a+i-2}(\ell-1)^2$, and one orbit of size $\ell^{2a-2}(\ell-1)^2$.*
*b) If $\left(\frac{\Delta}{\ell}\right) = 0$, there are two orbits: an orbit of size $\ell^{2a-2}(\ell-1)$ and an orbit of size $\ell^{2a-1}(\ell-1)$.*
*c) If $\left(\frac{\Delta}{\ell}\right) = -1$, there is one orbit, of size $\ell^{2a-2}(\ell^2-1)$.*

*Proof.* Step 1: Suppose $\mathcal{O} = \mathcal{O}_K$. Then every $\mathcal{O}$-submodule of $E[N]$ is of the form $E[I]$ for an ideal $I \supset N\mathcal{O}$, and $E[I] \cong_{\mathcal{O}} \mathcal{O}/I$: thus every submodule is of the form $M_P = \langle P \rangle_{\mathcal{O}}$ and is determined by its annihilator ideal $I_P$. Conversely, if $I \supset N\mathcal{O}$ is an ideal, then Lemmas 2.3 and 2.4 give that $E[I]$ is an $\mathcal{O}$-submodule of $E[N]$ with annihilator ideal $I$.

**Split Case $\left(\frac{\Delta}{\ell}\right) = 1$:** Then $\ell\mathcal{O} = \mathfrak{p}_1\mathfrak{p}_2$ for distinct prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$ of norm $\ell$. The ideals containing $\ell^a\mathcal{O}$ are precisely $\mathfrak{p}_1^c\mathfrak{p}_2^d$ with $\max(c,d) \leq a$. We have ring isomorphisms

$$\mathcal{O}/\mathfrak{p}_1^c\mathfrak{p}_2^d \cong \mathcal{O}/\mathfrak{p}_1^c \times \mathcal{O}/\mathfrak{p}_2^d \cong \mathbb{Z}/\ell^c\mathbb{Z} \times \mathbb{Z}/\ell^d\mathbb{Z},$$

hence unit group isomorphisms

$$(\mathcal{O}/\mathfrak{p}_1^c\mathfrak{p}_2^d)^\times \cong (\mathcal{O}/\mathfrak{p}_1^c)^\times \times (\mathcal{O}/\mathfrak{p}_2^c)^\times \cong (\mathbb{Z}/\ell^c\mathbb{Z})^\times \times (\mathbb{Z}/\ell^d\mathbb{Z})^\times,$$

so

$$\#(\mathcal{O}/\mathfrak{p}_1^c\mathfrak{p}_2^d)^\times = \varphi(\ell^c)\varphi(\ell^d).$$

To get points of order $\ell^a$ we impose the condition $\max(c,d) = a$. Thus $\mathcal{O}$-modules generated by the points of order $\ell^a$ are

$$E[\mathfrak{p}_1^a],\ E[\mathfrak{p}_1^a\mathfrak{p}_2], \ldots,\ E[\mathfrak{p}_1^a\mathfrak{p}_2^a] = E[\ell^a],\ E[\mathfrak{p}_1^{a-1}\mathfrak{p}_2^a], \ldots,\ E[\mathfrak{p}_1\mathfrak{p}_2^a],\ E[\mathfrak{p}_2^a].$$

So there are $2a+1$ Cartan orbits, one of size $\varphi(\ell^a)\varphi(\ell^a)$ and, for all $0 \leq i \leq a-1$, two of size $\varphi(\ell^a)\varphi(\ell^i)$. The smallest orbit size is $\ell^{a-1}(\ell-1)$, and all the other orbit sizes are multiples of it.

**Ramified Case $\left(\frac{\Delta}{\ell}\right) = 0$:** Then $\ell\mathcal{O} = \mathfrak{p}^2$ for a prime ideal $\mathfrak{p}$ of norm $\ell$. For any $b \in \mathbb{Z}^+$, the ring $\mathcal{O}/\mathfrak{p}^b$ is local of order $\ell^b$ with residue field $\mathbb{Z}/\ell\mathbb{Z}$, so the maximal ideal has size $\ell^{b-1}$ and thus

$$\#(\mathcal{O}/\mathfrak{p}^b)^\times = \ell^b - \ell^{b-1} = \ell^{b-1}(\ell-1).$$

Since $\mathfrak{p}^2 = (\ell)$, the least $c \in \mathbb{N}$ such that $\ell^c \in \mathfrak{p}^b$ is $c = \lceil \frac{b}{2} \rceil$. It follows that

$$(\mathcal{O}/\mathfrak{p}^b, +) \cong_{\mathbb{Z}} \mathbb{Z}/\ell^{\lceil \frac{b}{2} \rceil}\mathbb{Z} \oplus \mathbb{Z}/\ell^{\lfloor \frac{b}{2} \rfloor}\mathbb{Z}.$$

So the annihilator ideals of points of order $\ell^a$ in $\mathcal{O}/\ell^a\mathcal{O}$ are precisely $\mathfrak{p}^{2a-1}$ and $\mathfrak{p}^{2a}$. We get two Cartan orbits, one of size $\#(\mathcal{O}/\mathfrak{p}^{2a-1})^\times = \ell^{2a-2}(\ell-1)$ and one of size $\#(\mathcal{O}/\mathfrak{p}^{2a})^\times = \ell^{2a-1}(\ell-1)$. The smallest orbit size is $\ell^{2a-2}(\ell-1)$, and the other orbit size is a multiple of it.

**Inert Case $\left(\frac{\Delta}{\ell}\right) = -1$:** Then $\ell\mathcal{O}$ is a prime ideal, so the ideals containing $\ell^a\mathcal{O}$ are precisely $\ell^i\mathcal{O}$ for $i \leq a$. Clearly $\mathcal{O}/\ell^i\mathcal{O}$ has exponent $\ell^a$ iff $i = a$, so the $\mathcal{O}$-module generated by any point of order $\ell^a$ is $E[\ell^a]$. There is a single Cartan orbit, of size $\#(\mathcal{O}/\ell^a\mathcal{O})^\times = \varphi_K(\ell^a) = \ell^{2a-2}(\ell^2-1)$.

Step 2: Now let $\mathcal{O}$ be an order with $\ell \nmid \mathfrak{f}$. The natural maps $\mathcal{O}/\ell^a\mathcal{O} \to \mathcal{O}_K/\ell^a\mathcal{O}_K$ and $C_{\ell^a}(\mathcal{O}) \to C_{\ell^a}(\mathcal{O}_K)$ are isomorphisms, so the sizes and multiplicities of orbits carry over from $\mathcal{O}_K$ to $\mathcal{O}$. □

7.4. **The Case $\ell \mid \mathfrak{f}$.** Now suppose $\ell \mid \mathfrak{f}$. The ring $\mathcal{O}/\ell\mathcal{O}$ is isomorphic to $\mathbb{Z}/\ell\mathbb{Z}[\epsilon]/(\epsilon^2)$ – as one sees, e.g., using the explicit representation of (5) – and is thus a local Artinian ring with maximal ideal $\mathfrak{p}$, say, and residue field $\mathbb{Z}/\ell\mathbb{Z}$. Because $[\mathfrak{p} : \ell O] = \ell$, the only proper nonzero $\mathcal{O}$-submodule of $\mathcal{O}/\ell\mathcal{O}$ is $\mathfrak{p}/\ell$. Thus there are two Cartan orbits on the order $\ell$ elements of $\mathcal{O}/\ell\mathcal{O}$: one of order $\ell-1$ and one of order $\ell^2 - \ell = \#(\mathcal{O}/\ell\mathcal{O})^\times$.

For all $a \in \mathbb{Z}^+$, the ring $\mathcal{O}/\ell^a\mathcal{O}$ is local – for a maximal ideal $\mathfrak{m}$ of $\mathcal{O}$, we have $\ell^a \in \mathfrak{m} \iff \ell \in \mathfrak{m}$ – with residue field $\mathbb{Z}/\ell\mathbb{Z}$. In turn it follows that for any order $\ell^a$ point $P \in \mathcal{O}/\ell^a\mathcal{O}$ and $I_P = \{x \in \mathcal{O} \mid xP = 0\}$, the ring $\mathcal{O}/I_P$ is local with residue field $\mathbb{Z}/\ell\mathbb{Z}$. By Lemma 7.5, we may write

$$M_P = \mathcal{O}/I_P \cong_{\mathbb{Z}} \mathbb{Z}/\ell^a\mathbb{Z} \oplus \mathbb{Z}/\ell^b\mathbb{Z} \tag{8}$$

for some $0 \leq b \leq a$, and then

$$\#(\mathcal{O}/I_P)^\times = \#\mathcal{O}/I_P - \frac{\#\mathcal{O}/I_P}{\ell} = \ell^{a+b-1}(\ell-1).$$

So the size of a Cartan orbit on an order $\ell^a$ element of $\mathcal{O}/\ell^a\mathcal{O}$ is of the form $(\ell-1)\ell^c$ for some $a-1 \leq c \leq 2a-1$. So in this case it is *a priori* clear that the minimal size of a Cartan orbit divides the size of all the Cartan orbits. We want to understand how Cartan orbits grow when we lift a point of order $\ell^a$ to a point of order $\ell^{a+1}$. First observe that $x \mapsto \ell x$ gives an $\mathcal{O}$-module isomorphism

$$\mathcal{O}/\ell^a\mathcal{O} \xrightarrow{\sim} \ell\mathcal{O}/\ell^{a+1}\mathcal{O},$$

so we can view $\mathcal{O}/\ell^a\mathcal{O}$ as an $\mathcal{O}$-submodule of $\mathcal{O}/\ell^{a+1}\mathcal{O}$. With $P$ as in (8), let $Q \in \mathcal{O}/\ell^{a+1}\mathcal{O}$ be such that $\ell Q = P$. Put $M_Q = \{xQ \mid x \in \mathcal{O}\}$ and $I_Q = \{x \in \mathcal{O} \mid xQ = 0\}$, and write

$$M_Q = \mathcal{O}/I_Q \cong_{\mathbb{Z}} \mathbb{Z}/\ell^{a+1}\mathbb{Z} \oplus \mathbb{Z}/\ell^{b'}\mathbb{Z} \tag{9}$$

for $0 \leq b' \leq a+1$. Because $\ell Q = P$, we have $\ell M_Q = M_P$. Thus we find: if $b = 0$, then $b' \in \{0,1\}$, whereas if $b \geq 1$ then necessarily $b' = b+1$. So: if the $C_{\ell^a}(\mathcal{O})$-orbit on $P$ has the smallest possible size $\varphi(\ell^a)$, then the $C_{\ell^{a+1}}(\mathcal{O})$-orbit on $Q$ either has size $\varphi(\ell^{a+1})$ or size $\varphi(\ell^{a+2})$ (as we will see shortly, both possibilities can occur), whereas if the $C_{\ell^a}(\mathcal{O})$-orbit on $P$ has size $\varphi(\ell^{a+b}) > \varphi(\ell^a)$, then the $C_{\ell^{a+1}}(\mathcal{O})$-orbit on $Q$ has size $\varphi(\ell^{a+b+2})$: i.e., upon lifting from $P$ to $Q$ the size grows by a factor of $\ell^2$.

Since $H(\mathcal{O}, \ell^{a+1})$ implies $H(\mathcal{O}, \ell^a)$, for each fixed $\ell$ and $\mathcal{O}$ there are two possibilities.
**Type I**: $H(\mathcal{O}, \ell^a)$ holds for all $a \in \mathbb{Z}^+$.
In Type I, for all $a \in \mathbb{Z}^+$ the least size of a $C_{\ell^a}(\mathcal{O})$-orbit is $\varphi(\ell^a)$.
**Type II:** There is some $A \in \mathbb{Z}^+$ such that $H(\mathcal{O}, \ell^a)$ holds iff $a \leq A$.
In Type II, for $1 \leq a \leq A$, the least size of a $C_{\ell^a}(\mathcal{O})$-orbit is $\varphi(\ell^a)$, but for all $a \geq A$, whenever we lift a point of order $\ell^a$ to a point of order $\ell^{a+1}$ the size of the Cartan orbit grows by a factor of $\ell^2$, so for all $a > A$ the least size of a $C_{\ell^a}(\mathcal{O})$-orbit is $\ell^{a-A}\varphi(\ell^a)$.

We now determine the smallest size of a $C_{\ell^a}(\mathcal{O})$-orbit on an order $\ell^a$ point of $\mathcal{O}/\ell^a\mathcal{O}$ by using Theorem 6.15 to determine the type and compute the value of $A$ in Type II.

**Case 1**: Suppose $\left(\frac{\Delta_K}{\ell}\right) = 1$. Then for all $a \in \mathbb{Z}^+$ $H(\mathcal{O}_K, \ell^a)$ holds, so $\Delta_K$ is a square modulo $4\ell^a$, hence $\Delta = \mathfrak{f}^2\Delta_K$ is also a square modulo $4\ell^a$, so $H(\mathcal{O}, \ell^a)$ holds, and we are in Type I.
**Case 2**: Suppose $\left(\frac{\Delta_K}{\ell}\right) = -1$, and put $k = \operatorname{ord}_\ell(\mathfrak{f})$.

- Let $\ell > 2$. If $a \leq 2k$, then $\ell^a \mid \Delta$, so $\Delta$ is a square mod $\ell^a$ and hence also mod $4\ell^a$: thus $H(\mathcal{O}, \ell^a)$ holds. However, if $a = 2k+1$ then we claim $H(\mathcal{O}, \ell^a)$ does not hold. Indeed, suppose there is $s \in \mathbb{Z}$ such that $\Delta = \mathfrak{f}^2\Delta_K \equiv s^2 \pmod{\ell^a}$. Then $\ell^k \mid s$; taking $S = \frac{s}{\ell^k}$ we have $\frac{\mathfrak{f}^2}{\ell^{2k}}\Delta_K \equiv S^2 \pmod{\ell^{a-2k}}$, which implies that $\Delta_K$ is a square modulo $\ell$: contradiction. So we are in Type II with $A = 2k$.
- Let $\ell = 2$, and write $\mathfrak{f} = 2^k F$. Suppose $a \leq 2k$. Since $4 \mid \Delta_K - 1$, we have

$$2^{a+2} \mid (2^kF)^2(\Delta_K - 1) = \Delta - (2^kF)^2,$$

so $H(\mathcal{O}, 2^a)$ holds. Suppose $a \geq 2k+1$. If $\Delta$ is a square modulo $2^{a+2}$, then we find that $\Delta_K \equiv 1 \pmod 8$, so $\left(\frac{\Delta_K}{2}\right) = 1$: contradiction. So we are in Type II with $A = 2k$.
**Case 3**: Suppose $\left(\frac{\Delta_K}{\ell}\right) = 0$, and put $k = \operatorname{ord}_\ell(\mathfrak{f})$.

- Let $\ell > 2$. If $a \leq 2k+1$, then $\ell^a \mid \Delta$, so $\Delta$ is a square mod $\ell^a$ and hence also mod $4\ell^a$: thus $H\mathcal{O}, \ell^a)$ holds. However, if $a = 2k+2$ then we claim $H(\mathcal{O}, \ell^a)$ does not hold. Indeed, $\operatorname{ord}_\ell(\Delta) = 2k+1 < a$, so if $\Delta \equiv s^2 \pmod{\ell^a}$, then $\operatorname{ord}_\ell(s^2) = 2k+2$: contradiction. So we are in Type II with $A = 2k+1$.
- Let $\ell = 2$, and write $\mathfrak{f} = 2^kF$. Suppose $a \leq 2k+1$. Since $4 \mid \Delta_K$, there is $s \in \mathbb{Z}$ such that $8 \mid \Delta_K - s^2$, so

$$2^{a+2} \mid 2^{2k+3} \mid (2^kF)^2(\Delta_K - s^2) = \Delta - (2^kFs)^2,$$

so $H(\mathcal{O}, 2^a)$ holds. Suppose $a \geq 2k+2$. If $\Delta$ is a square modulo $2^{a+2}$, then $\Delta_K$ is a square modulo $2^{a+2-2k}$, hence modulo 16: contradiction. So we are in Type II with $A = 2k+1$.

7.5. **Torsion over $K(j)$: Part II.** We return to complete the classification of torsion on $\mathcal{O}$-CM elliptic curves $E_{/K(\mathfrak{f})}$ begun in §6.6.

II. Suppose $\Delta = -4$, so $j = 1728$ and $K(\mathfrak{f}) = K = \mathbb{Q}(\sqrt{-1})$.

- By Theorem 6.2, if $E(K)$ has a point of order $N$, then $\varphi(N) \mid 4$, so

$$N \in \{1, 2, 3, 4, 5, 6, 8, 10\}.$$

- Using Theorem 7.2 we get

$$T(\mathcal{O}, 1) = T(\mathcal{O}, 2) = T(\mathcal{O}, 4) = T(\mathcal{O}, 5) = T(\mathcal{O}, 10) = 1,$$

$$T(\mathcal{O}, 3) = T(\mathcal{O}, 6) = 2,\ T(\mathcal{O}, 8) = 4.$$

- We have $C_2(\mathcal{O}) = \mu_4/\{\pm 1\}$. Thus $\# C_2(\mathcal{O}) = 2$ so every $\mathcal{O}$-CM elliptic curve $E_{/K}$ has a $K$-rational point of order 2, and some $\mathcal{O}$-CM elliptic curve $E_{/K}$ has $E[2] = E[2](K)$.
- Because $\widetilde{T}(\mathcal{O}, 5) = 4$, if an $\mathcal{O}$-CM elliptic curve $E_{/K}$ has a $K$-rational point of order 5, the index of the mod 5 Galois representation in $C_5(\mathcal{O})$ is divisible by 4. Because $\# C_2(\mathcal{O}) = 2$, if an $\mathcal{O}$-CM elliptic curve $E_{/K}$ has full 2-torsion then the index of the mod 2 Galois representation in $C_2(\mathcal{O})$ is divisible by 2. Thus if an $\mathcal{O}$-CM elliptic curve $E_{/K}$ had $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z} \hookrightarrow E(K)[\mathrm{tors}]$, the index of the mod 10 Galois representation in $C_{10}(\mathcal{O})$ would be divisible by 8, contradicting Corollary 1.3.
- If $N \geq 3$ then $\# C_N(\mathcal{O}) > \# \mathcal{O}^\times$, so no $\mathcal{O}$-CM elliptic curve $E_{/K}$ has $E[N] = E[N](K)$.

Thus the groups which can occur as $E(K)[\mathrm{tors}]$ are precisely

$$\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/10\mathbb{Z}.$$

III. Suppose $\Delta = -3$, so $j = 0$ and $K(\mathfrak{f}) = K = \mathbb{Q}(\sqrt{-3})$.

- By Theorem 6.2, if $E(K(\mathfrak{f}))$ has a point of order $N$, then $\varphi(N) \mid 6$, so

$$N \in \{1, 2, 3, 4, 6, 7, 9, 14, 18\}.$$

- Using Theorem 7.2 we get

$$T(\mathcal{O}, 1) = T(\mathcal{O}, 2) = T(\mathcal{O}, 3) = T(\mathcal{O}, 6) = T(\mathcal{O}, 7) = 1,$$

$$T(\mathcal{O}, 4) = 2,\ T(\mathcal{O}, 9) = T(\mathcal{O}, 14) = 3,\ T(\mathcal{O}, 18) = 9.$$

- We have $C_2(\mathcal{O}) = \mu_6/\{\pm 1\}$. Thus as we range over all $\mathcal{O}$-CM elliptic curves $E_{/K}$, the group $E(K)[2]$ can be trivial (using Corollary 1.5) or have size 4, but it cannot have size 2.
- We have $C_3(\mathcal{O}) = \mu_6$. Thus there is an $\mathcal{O}$-CM elliptic curve $E_{/K}$ with $E[3] = E[3](K)$.
- If $N \geq 4$ then $\# C_N(\mathcal{O}) > \# \mathcal{O}^\times$, so no $\mathcal{O}$-CM elliptic curve $E_{/K}$ has $E[N] = E[N](K)$.

Thus the groups which can occur as $E(K)[\mathrm{tors}]$ are precisely

$$\{e\}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/7\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

**Remark 7.9.** *a) Case I. of the above calculation is a more detailed and explicit version of one of the main results of* [Pa89]. *Parish offers addenda on Cases II. and III., but without proof, and the possibilities* $E(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/10\mathbb{Z}$ *in Case II. and* $E(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/7\mathbb{Z}$ *and* $E(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ *in Case III are not mentioned.*
*b) In Cases II. and III. a classification of the possibilities for* $E(K(\mathfrak{f}))[\mathrm{tors}]$ apart *from the "Olson groups"* $\{e\}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *was done in* [BCS17, Thm. 1.4] *using computer calculations on degrees of preimages of* $j = 0$ *and* $j = 1728$ *on modular curves* [BCS17, Table 2].

7.6. **Isogenies over $K(j)$: Part II.** We return to complete the classification of $K(j)$-rational cyclic isogenies for elliptic curves with CM by the orders of discriminants $\Delta = -4$ and $\Delta = -3$. Recall that these cases have additional complexity coming from the fact that $\mu_K$ acts nontrivially on the projectivized torsion group $\mathbb{P}E[N]$. In this case, there is an $\mathcal{O}$-CM elliptic curve $(E_0)_{/K}$ for which the projective mod $N$ Galois representation

$$\mathbb{P}\rho_N : \mathfrak{g}_K \to C_N(\mathcal{O})/(\mathbb{Z}/N\mathbb{Z})^\times$$

is surjective. As we vary over the $K$-models of $E_0$, the representation $\mathbb{P}\rho_N$ twists by a character

$$\mathbb{P}\chi : \mathfrak{g}_K \to \mu_K/\{\pm 1\}.$$

Thus the index of $\mathbb{P}\rho_N(\mathfrak{g}_K)$ in $C_N(\mathcal{O})/(\mathbb{Z}/N\mathbb{Z})^\times$ divides 2 when $w_K = 4$ and divides 3 when $w_K = 6$.

We will rule out the existence of $K$-rational cyclic $N$-isogenies for various values of $N$ using the following "$\widetilde{T}$-argument": suppose that $\widetilde{T}(\mathcal{O}, N) > \varphi(N)\frac{w_K}{2}$. Then every $C_N(\mathcal{O})$-orbit on a point of order $N$ in $\mathcal{O}/N\mathcal{O}$ has size a multiple of $\widetilde{T}(\mathcal{O}, N)$, so every $C_N(\mathcal{O})/(\mathbb{Z}/N\mathbb{Z})^\times$-orbit on $\mathbb{P}E[N]$ has size a multiple of $\frac{\widetilde{T}(\mathcal{O},N)}{\varphi(N)}$, which by our hypothesis is greater than $\frac{w_K}{2}$. So after passing to a field extension $L$ of degree $\frac{w_K}{2}$ to trivialize $\mathbb{P}\chi$, we find that $\mathfrak{g}_L$ acts without fixed points on $\mathbb{P}E[N]$, and there is no $L$-rational cyclic $N$-isogeny and thus certainly no $K$-rational cyclic $N$-isogeny.

Let $\mathcal{O}$ be the order of discriminant $\Delta = -4$, so $K(j) = K = \mathbb{Q}(\sqrt{-1})$ and $w_K = 4$.
• If $\ell \equiv 1 \pmod 4$, then for all $a \in \mathbb{Z}^+$ we have that $-4$ is a square in $\mathbb{Z}/4\ell^a\mathbb{Z}$ so there is a $K$-rational cyclic $\ell^a$-isogeny. In fact we get that *every* $\mathcal{O}$-CM elliptic curve $E_{/K}$ has a $K$-rational cyclic $\ell^a$-isogeny.
• If $\ell \equiv 3 \pmod 4$, since $\frac{\widetilde{T}(\mathcal{O},\ell)}{\varphi(\ell)\frac{w_K}{2}} = \frac{\ell^2-1}{2(\ell-1)} = \frac{\ell+1}{2} > 1$, by the $\widetilde{T}$-argument there is no $K$-rational $\ell$-isogeny.
• If $\ell = 2$, then since $T(\mathcal{O}, 4) = 1$, we can have a $K$-rational point of order 4 (as already seen in §7.5), hence a cyclic $K$-rational 4-isogeny. Since $\frac{\widetilde{T}(\mathcal{O},8)}{\varphi(8)\frac{w_K}{2}} = \frac{16}{4\cdot 2} > 1$, by the $\widetilde{T}$-argument there is no cyclic $K$-rational 8-isogeny.
Any elliptic curve over a number field admitting a rational cyclic $N$-isogeny also admits a rational cyclic $M$-isogeny for all $M \mid N$. Moreover, if an elliptic curve $E_{/F}$ admits $F$-rational cyclic $N_1, \ldots, N_r$ isogenies for pairwise coprime $N_1, \ldots, N_r$, then the subgroup generated by the kernels of these isogenies is $F$-rational and cyclic of order $N_1 \cdots N_r$ so $E$ admits an $F$-rational cyclic $N_1 \cdots N_r$-isogeny. The assertion of Theorem 6.18b) now follows.

Let $\mathcal{O}$ be the order of discriminant $\Delta = -3$, so $K(j) = K = \mathbb{Q}(\sqrt{-3})$ and $w_K = 6$.
• If $\ell \equiv 1 \pmod 3$, then similarly to the $\Delta = -4$ case above we get that every $\mathcal{O}$-CM elliptic curve $E_{/K}$ has a $K$-rational cyclic $\ell^a$-isogeny for all $a \in \mathbb{Z}^+$.
• If $\ell \equiv 2 \pmod 3$ and $\ell > 2$, then since $\frac{\widetilde{T}(\mathcal{O},\ell)}{\varphi(\ell)\frac{w_K}{2}} = \frac{\ell^2-1}{3(\ell-1)} = \frac{\ell+1}{3} > 1$, by the $\widetilde{T}$-argument there is no cyclic $K$-rational $\ell$-isogeny.
• If $\ell = 2$, then since $T(\mathcal{O}, 2) = 1$ there is an $\mathcal{O}$-CM elliptic curve $E_{/K}$ with a $K$-rational 2-isogeny.
•. Since $\frac{\widetilde{T}(\mathcal{O},4)}{\varphi(4)\frac{w_K}{2}} = \frac{12}{2\cdot 3} > 1$, by the $\widetilde{T}$-argument there is no cyclic $K$-rational 4-isogeny.
• We claim that there is an $\mathcal{O}$-CM elliptic curve $E_{/K}$ with a $K$-rational cyclic 9-isogeny. Let $\mathfrak{p}$ be the unique prime ideal of $\mathcal{O}$ lying over 3, and let $P$ be a generator of the cyclic $\mathcal{O}$-module $E[\mathfrak{p}^3] \subset E[9]$, so $P$ has order 9. By Lemma 7.4, the $C_9(\mathcal{O})$-orbit on $P$ can be identified with the unit group $(\mathcal{O}/\mathfrak{p}^3)^\times$, of order 18. The $\mathcal{O}$-module generated by $P$ is also isomorphic to $(\zeta_3 - 1)\mathcal{O}/9\mathcal{O}$, and using this representation it is easy to compute that the group $(\mathcal{O}/\mathfrak{p}^3)^\times$ is generated by the images of the scalar matrices $(\mathbb{Z}/9\mathbb{Z})^\times$ and the cube roots of unity. Thus Galois acts on the image of $P$ in $\mathbb{P}E[9]$ via a character $\mathbb{P}\chi$. After twisting by the inverse of this character, the image of $P$ in $\mathbb{P}E[9]$ becomes fixed by Galois and we get a $K$-rational cyclic 9-isogeny.
• Since $\frac{\widetilde{T}(\mathcal{O},18)}{\varphi(18)\frac{w_K}{2}} = \frac{54}{3\cdot 6} > 1$, by the $\widetilde{T}$-argument there is no $K$-rational cyclic 18-isogeny.
• Since $\frac{\widetilde{T}(\mathcal{O},27)}{\varphi(27)\frac{w_K}{2}} = \frac{162}{3\cdot 18} > 1$, by the $\widetilde{T}$-argument there is no $K$-rational cyclic 27-isogeny.

• From §7.5 (or Theorem 7.2) we know there is an $\mathcal{O}$-CM elliptic curve $E_{/K}$ with a rational point of order 6, hence certainly a cyclic $K$-rational 6-isogeny.
Using the same considerations as in the $\Delta = -4$ case above we get the assertion of Theorem 6.18c).

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