TORSION POINTS AND GALOIS REPRESENTATIONS ON CM ELLIPTIC CURVES

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Abstract. We prove three theorems on torsion points and Galois representations for complex multiplication (CM) elliptic curves over number fields. The first theorem is a sharp version of Serre's Open Image Theorem in the CM case. The second theorem determines the degrees in which a CM elliptic curve has a rational point of order N, provided the field of definition contains the CM field. The third theorem bounds the size of the torsion subgroup of an elliptic curve with CM by a nonmaximal order in terms of the torsion subgroup of an elliptic curve with CM by the maximal order. We give several applications.

Contents

1. Introduction

1.1. **Overview.** Let F be a field of characteristic 0, and let $E_{/F}$ be an elliptic curve. We say E has **complex multiplication (CM)** if the endomorphism algebra

$$\mathrm{End}^0 E = \mathrm{End}(E_{/\overline{F}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is strictly larger than $\mathbb{Q}$, in which case it is necessarily an imaginary quadratic field K and $\mathcal{O} = \mathrm{End}(E_{/\overline{F}})$ is a $\mathbb{Z}$-order in K. This paper continues a program of study of torsion points and Galois representations on CM elliptic curves defined over number fields. Contributions have been made by Olson [Ol74], Silverberg [Si88], [Si92], Parish [Pa89], Aoki [Ao95], [Ao06], Ross [Ro94], Kwon [Kw99], Prasad-Yogananda [PY01], Breuer [Br10] and Lombardo [Lo15], and the present authors and our collaborators [CCRS13], [CCRS14], [BCS], [CP15], [BCP], [BP16].

Two long-term goals of this program are on the one hand to completely understand the adelic Galois representation on any CM elliptic curve defined over a number field and on the other hand to determine all degrees of CM points on modular curves associated to congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. These two problems are closely related. An archetypical example is the following case of the **First Main Theorem of Complex Multiplication** (the full statement is reproduced as Theorem 2.9): if K is an imaginary quadratic field $E_{/K(j(E))}$ is an $\mathcal{O}_K$-CM elliptic curve, then for all $N \in \mathbb{Z}^+$ the field obtained by adjoining to $K(j(E))$ the Weber function of the N-torsion subgroup is $K^{(N)}$, the N-ray class field of K. For all $N \geq 3$, we have (see Lemma 2.11)

$$[K^{(N)} : K(j(E))] = \frac{\#(\mathcal{O}_K/N\mathcal{O}_K)^\times}{\#\mathcal{O}_K^\times}.$$

This implies that the mod N Galois representation on an $\mathcal{O}_K$-CM elliptic curve $E_{/K(j(E))}$ is as large as possible *up to twisting*, and we will show there is an $\mathcal{O}_K$-CM elliptic curve $E_{/K(j(E))}$ such that the mod N Galois representation surjects onto the mod N Cartan subgroup $(\mathcal{O}/N\mathcal{O})^\times$ (see Theorem 4.8). This is a sharp version of Serre's Open Image Theorem in the $\mathcal{O}_K$-CM case. The corresponding result on the modular curve side is: the field of moduli of an $\mathcal{O}_K$-CM point on $X(N)_{/K(\zeta_N)}$ is $K^{(N)}$.

The above results restrict to the case of the maximal order $\mathcal{O}_K$, as does most of the classical theory.[1] Here we work in the context of an arbitrary order $\mathcal{O}$, of conductor $\mathfrak{f}$, in an imaginary quadratic field K. Let $F \supset K$ be a number field, and let $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve. For any positive integer N, we define the **reduced mod N Cartan subgroup** to be the quotient of $C_N(\mathcal{O}) = (\mathcal{O}/N\mathcal{O})^\times$ by the image of $\mathcal{O}^\times$ under the natural map $q_N : \mathcal{O} \to \mathcal{O}/N\mathcal{O}$. That is,

$$\overline{C_N(\mathcal{O})} = C_N(\mathcal{O})/q_N(\mathcal{O}^\times).$$

(The map $q_N^\times : \mathcal{O}^\times \to (\mathcal{O}/N\mathcal{O})^\times$ is injective when $N \geq 3$; when $N = 2$ its kernel is $\{\pm 1\}$.) We define the **reduced Galois representation** to be the following composite homomorphism:

$$\overline{\rho_N} : \mathfrak{g}_F \xrightarrow{\rho_N} C_N(\mathcal{O}) \to \overline{C_N(\mathcal{O})}.$$

This representation is independent of the F-rational model of E, and our first result shows that it plays a natural role in the study of Galois representations on CM elliptic curves.

Theorem 1.1. *(Uniform Open Image Theorem)*
a) For all $\mathcal{O}$-CM elliptic curves $E_{/K(j(E))}$, the reduced Galois representation

$$\overline{\rho_N} : \mathfrak{g}_{K(j(E))} \to \overline{C_N(\mathcal{O})}$$

is surjective.
b) For all number fields $F \supset K$ and all $\mathcal{O}$-CM elliptic curves $E_{/F}$ we have

$$[C_N(\mathcal{O}) : \rho_N(\mathfrak{g}_F)] \mid \#\mathcal{O}^\times [F : K(j(E))] \leq 6[F : K(j(E))].$$

[1] The adelic formalism incorporates arbitrary orders, but even so the explicit determination of the class field in the "First Main Theorem" has been given only for the maximal order.

c) For all orders $\mathcal{O}$ and all $N \geq 2$, there is a number field $F \supset K$ and an $\mathcal{O}$-CM elliptic curve $E_{/F}$ such that $E[N] = E[N](F)$ and

$$[F : K(j(E))] = \#\overline{C_N(\mathcal{O})}.$$

The surjectivity of $\overline{\rho_N}$ is deduced from the following result, which is an extension of the classical First Main Theorem of Complex Multiplication from maximal orders to all orders.

Theorem 1.2. *Let $\mathcal{O}$ be an order of conductor $\mathfrak{f}$, let $E_{/\mathbb{C}}$ be an $\mathcal{O}$-CM elliptic curve, and let $N \in \mathbb{Z}^+$. Then the Weber function field $K(j(E))(\mathfrak{h}(E[N]))$ is the compositum of the N-ray class field of K with the $N\mathfrak{f}$-ring class field of K. Moreover, we have $[K(j(E))(\mathfrak{h}(E[N])) : K(j(E))] = \#\overline{C_N(\mathcal{O})}$.*

Let us briefly describe the proof. Building on the classical First Main Theorem, our prior work with Stankewicz [BCS], and work of Parish [Pa89], we show that $K(j(E))(\mathfrak{h}(E[N]))$ contains $K^{(N)}$ and $K(N\mathfrak{f})$. Some observations on Weber functions established in §2.3 imply that the order of the reduced Cartan subgroup $\#\overline{C_N(\mathcal{O})}$ is an upper bound for the degree $[K(j(E))(\mathfrak{h}(E[N])) : K(j(E))]$, so it remains to show that equality holds. To do this, we use both class field theory and an analysis of the Galois representation on an auxiliary $\mathcal{O}_K$-CM elliptic curve.

It follows from Theorem 1.1 that if $E_{/F}$ is an $\mathcal{O}$-CM elliptic curve and $F \supset K$, the index of the image of the adelic Galois representation on E in the Cartan subgroup $\widehat{C} = (\mathcal{O} \otimes \widehat{\mathbb{Z}})^\times$ divides $\#\mathcal{O}^\times[F : K(j(E))]$. If F does not contain K, then the image of the adelic Galois representation has index dividing $[F : \mathbb{Q}(j(E))]\#\mathcal{O}^\times$ in a subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ that contains the adelic Cartan $\widehat{C}$ with index 2. This is close to being a complete description of the adelic Galois representation on any CM elliptic curve defined over a number field. It falls short in two aspects: first, for a fixed $N \geq 3$, to get a mod N Galois representation with index $\#\mathcal{O}^\times$ in the mod N Cartan, our construction takes F to be a proper extension of the minimal possible ground field $K(j(E))$. Second, it shows that at any finite level N the index of the mod N representation in the Cartan can be any divisor of $\#\mathcal{O}^\times$ but does not address whether this can happen for the adelic Galois representation. These do not impact the second aspect of our program, which studies degrees of level N structures of CM elliptic curves and fields of moduli of CM points on modular curves and studies *all* pairs $(E, L_N)_{/F}$ for a level N structure L_N over a number field $F \supset K(j(E))$, not just pairs in which the underlying elliptic curve E arises from base extension of an elliptic curve $E_{/K(j(E))}$.

We propose to use Theorem 1.1 to determine all degrees of CM points on modular curves. To carry this out requires further work of a more algebraic nature: an analysis of orbits of the mod N Cartan subgroup $C_N(\mathcal{O})$ on level N structures. To understand the relevance of this, let $E_{/K(j(E))}$ be an $\mathcal{O}$-CM elliptic curve. If $P \in E[\mathrm{tors}]$ is a point of order N, the field of moduli of the point (E, P) on $X_1(N)$ depends only on the $\mathcal{O}^\times$ orbit $\overline{P}$ of P. Since the reduced Galois representation is surjective, the degree of this field over $K(j(E))$ may be computed by determining the size of the orbit of $\overline{C_N(\mathcal{O})}$ on $\overline{P}$.

We give an analysis of Cartan orbits on $\mathcal{O}/N\mathcal{O}$ in §5 and §6. The algebra is much simpler when $\mathcal{O}$ is maximal, and in this case our analysis is complete. When $\mathcal{O}$ is nonmaximal we give substantial, but not full, information on the structure of the Cartan orbits, enough to yield the following result.

Theorem 1.3. *Let $\mathcal{O}$ be an order in K of conductor $\mathfrak{f}$, and let $N \in \mathbb{Z}^{\geq 2}$.*
There is a positive integer $T(\mathcal{O}, N)$, explicitly computed in §6, such that:
(i) if $F \supset K$ is a number field and $E_{/F}$ is an $\mathcal{O}$-CM elliptic curve with an F-rational point of order N, then $T(\mathcal{O}, N) \mid [F : K(j(E))]$, and
(ii) there is a number field $F \supset K$ and an $\mathcal{O}$-CM elliptic curve $E_{/F}$ such that $[F : K(j(E))] = T(\mathcal{O}, N)$ and $E(F)$ contains a point of order N.

Theorem 1.3 should be compared to Theorem 5.2, a refinement of bounds of Silverberg [Si88], [Si92] and Prasad-Yogananda [PY01]. Theorem 5.2 also gives a divisibility on $[F : K(j(E))]$ imposed by the existence of an F-rational point of order N: in the current notation, Theorem 5.2 asserts

$$\varphi(N) \mid \#\mathcal{O}^\times \cdot T(\mathcal{O}, N).$$

This bound is "homogenous" in the sense that it is a single bound that holds in all cases. Theorem 1.3 gives the optimal divisibility in *all* cases.

We give two other applications of our Cartan orbit analysis: the determination of all possible torsion subgroups of a K-CM elliptic curve $E_{/K(j(E))}$ (§5.7) and the set of $N \in \mathbb{Z}^+$ for which there is a $K(j(E))$-rational cyclic N-isogeny (Theorem 5.18).

Although we seek *results* which treat elliptic curves with CM by a nonmaximal order on an equal footing with the $\mathcal{O}_K$-CM case, in most cases (e.g. in Theorem 1.1) the *proofs* use "change of order" functorialities. Let F be a number field, let $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve, and let $\mathfrak{f}$ be the conductor of $\mathcal{O}$. Then there is an F-rational isogeny $\iota : E \to E'$ such that $\operatorname{End} E' = \mathcal{O}_K$. The induced $\mathfrak{g}_F$-module map $E[N] \to E'[N]$ is an isomorphism iff $\gcd(\mathfrak{f}, N) = 1$; otherwise there is a nontrivial kernel. But nevertheless there are relations between the mod N Galois representations on E and E'. Here is the last main result of this paper:

Theorem 1.4. *(Isogeny Torsion Theorem) Let $\mathcal{O}$ be an order in an imaginary quadratic field K, $F \supset K$ be a number field, $E_{/F}$ an $\mathcal{O}$-CM elliptic curve, and $\iota : E \to E'$ the canonical isogeny, with E' an $\mathcal{O}_K$-CM elliptic curve. Then:*

$$\#E(F)[\text{tors}] \mid \#E'(F)[\text{tors}].$$

We give examples where the exponent of $E'(F)[\text{tors}]$ is strictly smaller than that of $E(F)[\text{tors}]$, showing we cannot hope to view $E(F)[\text{tors}]$ as a subgroup of $E'(F)[\text{tors}]$, and we prove that $\frac{\#E'(F)[\text{tors}]}{\#E(F)[\text{tors}]}$ can be arbitrarily large (see Propositions 5.8 and 5.9). Moreover, the statement is false if we do not require $F \supset K$. Despite the fact that this relationship is not as strong as one might hope, Theorem 1.4 has applications to determining fields of moduli of partial level N structures (§5.2, §5.3).

1.2. **Related work.** Our proof of Theorem 1.1 builds crucially on work of J.L. Parish [Pa89]. Also the classification of torsion over $K(j(E))$ is one of the main results of [Pa89]. Parish's work has minor flaws with regard to the imaginary quadratic fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ – leading in particular to some omitted groups in his classification of torsion over $K(j(E))$ – and at another key point is a bit laconic, so when we want to use results appearing in or motivated by [Pa89] we give complete proofs.

A paper of R. Ross [Ro94] contains a result related to Theorem 1.4: in the notation of Theorem 1.4, Ross's assertion implies that the groups $E(F)[\text{tors}]$ and $E'(F)[\text{tors}]$ have the same exponent. This is false: Proposition 5.8 gives counterexamples. Nevertheless it was Ross's work that led us to the statement of Theorem 1.4.

S. Kwon gave a classification of degrees of cyclic isogenies rational over $\mathbb{Q}(j(E))$ in the CM case [Kw99]. Our Theorem 5.18 is the analogue over $K(j(E))$.

D. Lombardo has recently shown that if $E_{/F}$ is a CM elliptic curve defined over a number field F containing the CM field K, then the index of the adelic Galois representation in the Cartan subgroup divides $\#\mathcal{O}^\times[F : K]$ [Lo15]. This is in general a weaker bound than that of Theorem 1.1; the two coincide when $j(E) \in \mathbb{Q}$. On the other hand, Lombardo establishes largeness of Galois results for all abelian varieties of CM type and then specializes to elliptic curves.

Á. Lozano-Robledo has informed us that he has also done work on the image of the adelic Galois representation in the CM case.

2. Preliminaries

2.1. **Foundations.** We begin by setting some terminology for orders in imaginary quadratic fields. Let K be an imaginary quadratic field and $\mathcal{O}$ a $\mathbb{Z}$-order in K. We put

$$\mathfrak{f} = [\mathcal{O}_K : \mathcal{O}],$$

the **conductor** of $\mathcal{O}$. Then

$$\mathcal{O} = \mathbb{Z} + \mathfrak{f}\mathcal{O}_K, \ \Delta(\mathcal{O}) = \mathfrak{f}^2\Delta_K.$$

Conversely, for fixed K and $\mathfrak{f} \in \mathbb{Z}^+$ there is a unique order $\mathcal{O}(\mathfrak{f})$ in K of conductor $\mathfrak{f}$. Thus an imaginary quadratic order is determined by its discriminant Δ, a negative integer which is 0 or 1 modulo 4. Conversely, for any negative integer Δ which is 0 or 1 modulo 4, we put

$$\tau_\Delta = \frac{\Delta + \sqrt{\Delta}}{2},$$

and then $\mathbb{Z}[\tau_\Delta]$ is an order in K of discriminant Δ.

Throughout this paper we will use the following terminological convention: by "an order $\mathcal{O}$" we always mean a $\mathbb{Z}$-order $\mathcal{O}$ in an imaginary quadratic field, which is determined as the fraction field of $\mathcal{O}$ and denoted by K. We may specify an order $\mathcal{O}$ by giving its discriminant, which also determines K. If K is already given, then we specify an order $\mathcal{O}$ in K by giving the conductor $\mathfrak{f}$.

For any $\mathcal{O}$-CM elliptic curve E we have $K(j(E)) = K(\mathfrak{f})$, the ring class field of K of conductor $\mathfrak{f}$ ([Co89, Thm. 11.1]). We may thus determine $[K(j(E)) : K]$ via the following formula:

Theorem 2.1. *For $N \in \mathbb{Z}^+$, let $K(N)$ denote the N-**ring class field** of K. Then $K(1) = K^{(1)}$ is the Hilbert class field of K, and for all $N \geq 2$ we have*

$$[K(N) : K^{(1)}] = \frac{2}{w_K} N \prod_{p \mid N} \left(1 - \left(\frac{\Delta_K}{p}\right) \frac{1}{p}\right).$$

Proof. See e.g. [Co89, Cor. 7.24]. □

For number field F, a positive integer N, and $E_{/F}$ an elliptic curve, we denote by ρ_N the homomorphism

$$\mathfrak{g}_F \to \operatorname{Aut} E[N] \cong \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}),$$

the **modulo N Galois representation**. If $E_{/F}$ has CM by the order $\mathcal{O}$ in K, then $E[N] \cong_{\mathcal{O}} \mathcal{O}/N\mathcal{O}$ (see [Pa89, Lemma 1], generalized in Lemma 2.5 below), and provided $F \supset K$ we have

$$\rho_N : \mathfrak{g}_F \hookrightarrow \operatorname{Aut}_{\mathcal{O}} E[N] \cong \mathrm{GL}_1(\mathcal{O}/N\mathcal{O}) = (\mathcal{O}/N\mathcal{O})^\times.$$

In other words, the image of the mod N Galois representation lands in the **mod N Cartan subgroup**

$$C_N(\mathcal{O}) = (\mathcal{O}/N\mathcal{O})^\times.$$

Lemma 2.2. *Let $\mathcal{O}$ be an order of discriminant Δ, and let $N = p_1^{a_r} \cdots p_r^{a_r} \in \mathbb{Z}^+$.*
a) We have $C_N(\mathcal{O}) = \prod_{i=1}^r C_{p_i^{a_i}}(\mathcal{O})$ (canonical isomorphism).
b) We have $\#C_N(\mathcal{O}) = N^2 \prod_{p \mid N} \left(1 - \left(\frac{\Delta}{p}\right) \frac{1}{p}\right) \left(1 - \frac{1}{p}\right)$.

Proof. a) It suffices to tensor the Chinese Remainder Theorem isomorphism $\mathbb{Z}/N\mathbb{Z} = \prod_{i=1}^r \mathbb{Z}/p_i^{a_i}\mathbb{Z}$ with the $\mathbb{Z}$-module $\mathcal{O}$ and pass to the unit groups.
b) By [CCRS13], for any prime number p we have

$$\#C_p(\mathcal{O}) = p^2 \left(1 - \left(\frac{\Delta}{p}\right) \frac{1}{p}\right) \left(1 - \frac{1}{p}\right).$$

The natural map $C_{p^a}(\mathcal{O}) \to C_p(\mathcal{O})$ is surjective with kernel of size p^{2a-2} [CP15, p. 3]. Together with part a) this shows that if $N = p_1^{a_1} \cdots p_r^{a_r}$ then

$$\#C_N(\mathcal{O}) = \prod_{i=1}^r p_i^{2a_i - 2} (p_i - 1) \left(p_i - \left(\frac{\Delta}{p_i}\right)\right) = N^2 \prod_{p \mid N} \left(1 - \left(\frac{\Delta}{p}\right) \frac{1}{p}\right) \left(1 - \frac{1}{p}\right).$$ □

Finally, we establish the following relationship between $[K(\mathfrak{f}) : K^{(1)}]$ and $\#C_N(\mathcal{O})$ which will be used in the proof of Theorem 1.1. Here, φ denotes Euler's totient function and $\varphi_K(I)$ the natural generalization for a nonzero ideal I of $\mathcal{O}_K$. That is,

$$\varphi_K(I) = \#(\mathcal{O}_K/I)^\times = |I| \prod_{\mathfrak{p} \mid I} \left(1 - \frac{1}{|\mathfrak{p}|}\right),$$

where $|I| = \#\mathcal{O}_K/I$.

Lemma 2.3. *Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, and let $\mathcal{O}$ be the order in K of conductor $\mathfrak{f}$. Then for $N \in \mathbb{Z}^+$ we have*

$$\frac{\varphi_K(N\mathfrak{f})\varphi(N)}{[K(\mathfrak{f}) : K^{(1)}]\varphi(N\mathfrak{f})} = [\mathcal{O}_K^\times : \mathcal{O}^\times] \cdot \#C_N(\mathcal{O}). \tag{1}$$

Proof. If $\mathfrak{f} = 1$, then (1) reduces to $\varphi_K(N) = \#(\mathcal{O}_K/N\mathcal{O}_K)^\times$, which is true. Suppose $\mathfrak{f} > 1$, so Theorem 2.1 can be applied. Then the left hand side of (1) is

$$\begin{aligned}
&[\mathcal{O}_K^\times : \mathcal{O}^\times]N^2 \, \frac{\prod_{p|N\mathfrak{f}} \left(1 - \left(\frac{\Delta_K}{p}\right)\frac{1}{p}\right)\left(1 - \frac{1}{p}\right) \prod_{p|N} \left(1 - \frac{1}{p}\right)}{\prod_{p|\mathfrak{f}} \left(1 - \left(\frac{\Delta_K}{p}\right)\frac{1}{p}\right) \prod_{p|N\mathfrak{f}} \left(1 - \frac{1}{p}\right)} \\
&= [\mathcal{O}_K^\times : \mathcal{O}^\times]N^2 \prod_{p|N} \left(1 - \frac{1}{p}\right) \prod_{p|N\mathfrak{f},\ p\nmid\mathfrak{f}} \left(1 - \left(\frac{\Delta_K}{p}\right)\frac{1}{p}\right) \\
&= [\mathcal{O}_K^\times : \mathcal{O}^\times]N^2 \prod_{p|N} \left(1 - \left(\frac{\Delta}{p}\right)\frac{1}{p}\right) = [\mathcal{O}_K^\times : \mathcal{O}^\times]\#C_N(\mathcal{O}).
\end{aligned}$$

□

2.2. **Torsion Kernels.** Let $E_{/\mathbb{C}}$ be an $\mathcal{O}$-CM elliptic curve. For a nonzero ideal I of $\mathcal{O}$, we define the **I-torsion kernel**

$$E[I] = \{P \in E \mid \forall \alpha \in I,\ \alpha P = 0\}.$$

There is an invertible ideal $\Lambda \subset \mathcal{O}$ such that

$$E \cong \mathbb{C}/\Lambda.$$

If we put

$$(\Lambda : I) = \{x \in \mathbb{C} \mid xI \subset \Lambda\} = \{x \in K \mid xI \subset \Lambda\}$$

then we have (immediately) that

$$E[I] = \{x \in \mathbb{C}/\Lambda \mid xI \subset \Lambda\} = (\Lambda : I)/\Lambda.$$

Let $|I| = \#\mathcal{O}/I$.

Lemma 2.4. *Let $I, J \subset \mathcal{O}$ be nonzero ideals and $E_{/\mathbb{C}}$ be an $\mathcal{O}$-CM elliptic curve.*
a) If $I \subset J$, then $E[J] \subset E[I]$.
b) We have $E[I] \subset E[|I|]$. In particular

$$\#E[I] \leq |I|^2.$$

Proof. a) This is immediate from the definition. b) By Lagrange's Theorem, every element of $\mathcal{O}/I$ is killed by $|I|$, so $|I| \subset |I|\mathcal{O} \subset I$. Apply part a). □

Lemma 2.5. *If I is an invertible $\mathcal{O}$-ideal, then*

$$E[I] = I^{-1}\Lambda/\Lambda \cong_{\mathcal{O}} \mathcal{O}/I.$$

In particular $\#E[I] = |I| = \#\mathcal{O}/I$.

Proof. An ideal I is invertible iff there is an $\mathcal{O}$-submodule I^{-1} of K such that $II^{-1} = \mathcal{O}$. If so, then for $x \in K$ we have

$$xI \subset \Lambda \iff xII^{-1} = x\mathcal{O} \subset I^{-1}\Lambda \iff x \in I^{-1}\Lambda,$$

giving $E[I] = I^{-1}\Lambda/\Lambda$. Because Λ is a locally free $\mathcal{O}$-module, for all $\mathfrak{p} \in \operatorname{Spec}\mathcal{O}$ we have $\Lambda_\mathfrak{p} \cong \mathcal{O}_\mathfrak{p}$ and thus $(I^{-1}\Lambda/\Lambda)_\mathfrak{p} \cong (I^{-1}/\mathcal{O})_\mathfrak{p} \cong (\mathcal{O}/I)_\mathfrak{p}$. Thus $I^{-1}\Lambda/\Lambda$ is locally free of rank 1 as an $\mathcal{O}/I$-module. But the ring $\mathcal{O}/I$ is semilocal, hence has trivial Picard group: any locally free rank 1 $\mathcal{O}/I$-module is isomorphic to $\mathcal{O}/I$. □

Lemma 2.6. *Let R be a Dedekind domain, and let M be a cyclic torsion R-module, and let $N \subset M$ be an R-submodule. Then:*
a) N is also a cyclic R-module.
b) We have $N \cong R/\operatorname{ann} N$.

Proof. Let $I = \operatorname{ann} M$. Since M is a finitely generated torsion module over a domain, we have $I \neq 0$ and $M \cong R/I$. Thus $N \cong I'/I$ for some ideal $I' \supset I$. The ring R/I is principal Artinian [CA, Thm. 20.11], so the ideal I'/I of R/I is principal. Thus N is a cyclic, torsion R-module, so $N \cong R/\operatorname{ann} N$. □

Theorem 2.7. *Let $E_{/\mathbb{C}}$ be an $\mathcal{O}_K$-CM elliptic curve, and let $M \subset E(\mathbb{C})$ be a finite $\mathcal{O}_K$-submodule. Then $M = E[\operatorname{ann} M] \cong_{\mathcal{O}} \mathcal{O}/\operatorname{ann} M$ and thus $\#M = |\operatorname{ann} M|$.*

Proof. That $M \subset E[\operatorname{ann} M]$ is a tautology. Because $\mathcal{O} = \mathcal{O}_K$ every nonzero $\mathcal{O}$-ideal is invertible, so by Lemma 2.5 we have $\#E[\operatorname{ann} M] = |\operatorname{ann} M|$. On the other hand, let $\mathfrak{t} = \#M$. Then $M \subset E[\mathfrak{t}] \cong_{\mathcal{O}_K} \mathcal{O}_K/\mathfrak{t}\mathcal{O}_K$, a finite cyclic $\mathcal{O}_K$-module. By Lemma 2.6 we have $M \cong \mathcal{O}_K/\operatorname{ann} M$ so $\#M = |\operatorname{ann} M|$. Thus $M = E[\operatorname{ann} M]$, hence Lemma 2.5 gives $M \cong \mathcal{O}/\operatorname{ann} M$ and $\#M = |\operatorname{ann} M|$. □

Remark 2.8. *There is nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$ such that the local ring $\mathcal{O}_{\mathfrak{p}}$ is not a DVR. If $\mathfrak{p} \cap \mathbb{Z} = (\ell)$, then $\mathcal{O}/\mathfrak{p} \cong \mathbb{Z}/\ell\mathbb{Z}$. Since every ideal of $\mathcal{O}$ can be generated by two elements, we have $\dim_{\mathcal{O}/\mathfrak{p}} \mathfrak{p}/\mathfrak{p}^2 = 2$. Thus $\#\mathcal{O}/\mathfrak{p}^2 = \ell^3$ and $(\ell^3) \subset \mathfrak{p}^2$. It follows that in the quotient ring $\mathcal{O}/\ell^3\mathcal{O}$ the maximal ideal $\mathfrak{p} + \ell^3\mathcal{O}$ is not principal. Let $E_{/\mathbb{C}}$ be an $\mathcal{O}$-CM elliptic curve, so $E[\ell^3] \cong_{\mathcal{O}} \mathcal{O}/\ell^3\mathcal{O}$. So the $\mathcal{O}$-submodule $M = \mathfrak{p}E[\ell^3]$ of $E[\ell^3]$ is not cyclic and thus not isomorphic to $\mathcal{O}/\operatorname{ann} M$.*

Now we recall an important classical result.

Theorem 2.9. *(First Main Theorem of Complex Multiplication) Let $E_{/\mathbb{C}}$ be an $\mathcal{O}_K$-CM elliptic curve, and let I be a nonzero ideal of $\mathcal{O}_K$. Let $\mathfrak{h} : E \to \mathbb{P}^1$ be a Weber function. Then:*

$$K^{(1)}(\mathfrak{h}(E[I])) = K^I.$$

Proof. See e.g. [Si94, Thm. II.5.6]. □

Combining Theorems 2.7 and 2.9, we get the class-field theoretic containment corresponding to any finite $\mathcal{O}_K$-submodule of $E(\overline{F})$, for any $\mathcal{O}_K$-CM elliptic curve E defined over a number field $F \supset K$. Theorem 2.9 implies that whenever E is an $\mathcal{O}_K$-CM elliptic curve, $K^{(1)}(\mathfrak{h}(E[N])) = K^{(N)}$. In the case of CM by an arbitrary order in K, we will show the Weber Function Field need not equal $K^{(N)}$ (see Theorem 4.6), but a containment has previously been established.

Theorem 2.10. [BCS, Thm. 3.16] *Let E be a K-CM elliptic curve defined over a number field $F \supset K$. Then we have*

$$F(\mathfrak{h}(E[N])) \supset K^{(N)}. \tag{2}$$

For convenience, we record here the formulas for $[K^I : K^{(1)}]$.

Lemma 2.11. *Let I be a nonzero ideal of K, and let K^I be the I-**ray class field**. We put $U(K) = \mathcal{O}_K^\times$ and $U_I(K) = \{x \in U(K) \mid x - 1 \in I\}$.*
a) We have

$$[K^I : K^{(1)}] = \frac{\varphi_K(I)}{[U(K) : U_I(K)]}.$$

b) If $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$, then

$$[K^I : K^{(1)}] = \begin{cases} \varphi_K(I) & I \mid (2) \\ \frac{\varphi_K(I)}{2} & I \nmid (2) \end{cases}.$$

c) If $K = \mathbb{Q}(\sqrt{-1})$, then

$$[K^I : K^{(1)}] = \begin{cases} \varphi_K(I) & I \mid (1+i) \\ \frac{\varphi_K(I)}{2} & I \nmid (1+i) \text{ and } I \mid (2) \\ \frac{\varphi_K(I)}{4} & I \nmid (2) \end{cases}.$$

d) If $K = \mathbb{Q}(\sqrt{-3})$, then

$$[K^I : K^{(1)}] = \begin{cases} 1 & I = (1) \\ \frac{\varphi_K(I)}{2} & I \neq (1) \text{ and } I \mid (\zeta_3 - 1) \\ \frac{\varphi_K(I)}{3} & I = (2) \\ \frac{\varphi_K(I)}{6} & \text{otherwise} \end{cases}.$$

Proof. Parts b)-d) can be deduced from a), which appears as [Co00, Cor. 3.2.4]. □

2.3. **On Weber Functions.**

Theorem 2.12. *(Weber Function Principle) Let $N \in \mathbb{Z}^{\geq 3}$, let $\mathcal{O}$ be the order of conductor $\mathfrak{f}$ in K, and let $F = K(\mathfrak{f})$. For an $\mathcal{O}$-CM elliptic curve $E_{/F}$, fix an embedding $F \hookrightarrow \mathbb{C}$ such that $j(E) = j(\mathbb{C}/\mathcal{O})$. Define*

$$W(N, \mathcal{O}) = K(\mathfrak{f})(\mathfrak{h}(E[N])).$$

a) $W(N, \mathcal{O})$ is a subfield of $F(E[N])$ and $[F(E[N]) : W(N, \mathcal{O})] \mid \#\mathcal{O}^\times$.
b) There is an elliptic curve $E_{/F}$ such that

$$[F(E[N]) : W(N, \mathcal{O})] = \#\mathcal{O}^\times.$$

c) As we range over all elliptic curves $E_{/F}$ with $j(E) = j(\mathbb{C}/\mathcal{O})$, we have

$$\bigcap_E F(E[N]) = W(N, \mathcal{O}).$$

Proof. a) Let $w = \#\mathcal{O}^\times$. The field $F(E[N])/F$ is Galois with Galois group $\rho_N(\mathfrak{g}_F) \subset C_N(\mathcal{O})$. Because $N \geq 3$, the homomorphism $\mathcal{O}^\times \to C_N(\mathcal{O})$ is injective. Since $\mathfrak{h}(P) = \mathfrak{h}(Q)$ for points P, Q on E if and only if there exists $\xi \in \mathcal{O}^\times$ such that $\xi(P) = Q$ (e.g. [La87, Thm. I.7]), it follows that

$$W(N, \mathcal{O}) = F(E[N])^{\rho_N(\mathfrak{g}_F) \cap \mathcal{O}^\times}.$$

Thus

$$[F(E[N]) : W(N, \mathcal{O})] \mid w.$$

b), c) If $E_{/F}$, $E'_{/F}$ with $j(E) = j(E')$, then $K(\mathfrak{f})(\mathfrak{h}(E[N])) = K(\mathfrak{f})(\mathfrak{h}(E'[N]))$ by the model independence of the Weber function. Thus $W(N, \mathcal{O}) \subset \bigcap_E F(E[N])$. To see that equality holds, let $E_{/F}$ have $j(E) = j(\mathbb{C}/\mathcal{O})$. Let $\mathfrak{p}$ be a prime of $\mathcal{O}_F$ which is unramified in $F' = F(E[N])$. By weak approximation, there is $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$. Put $L = F(\pi^{\frac{1}{w}})$, and let $\chi : \mathfrak{g}_F \to \mu_w$ be a character with splitting field $\overline{F}^{\ker \chi} = L$. Then L/F is totally ramified over $\mathfrak{p}$, so F' and L are linearly disjoint over F. It follows that

$$\rho_{N, E^\chi}(\mathfrak{g}_{F'}) = (\rho_{N, E_{/F'}} \otimes \chi)(\mathfrak{g}_{F'}) = \chi(\mathfrak{g}_{F'}) = \mu_w.$$

Thus

$$w = [F(E^\chi[N]) : F(E[N]) \cap F(E^\chi[N])] \mid [F(E^\chi[N]) : W(N, \mathcal{O})] \mid w,$$

so $F(E^\chi[N])$ has degree w over $W(N, \mathcal{O}) = F(E[N]) \cap F(E^\chi[N])$. □

Remark 2.13. *Theorem 2.12 holds for $N = 2$ with $\#\mathcal{O}^\times$ replaced by $\frac{\#\mathcal{O}^\times}{2}$. See §4.5 and §4.6.*

3. The Isogeny Torsion Theorem

3.1. **Proof of the Isogeny Torsion Theorem.** Let Δ_K be the discriminant of K and Δ the discriminant of $\mathcal{O}$, so $\Delta = \mathfrak{f}^2 \Delta_K$, where $\mathfrak{f}$ is the conductor of $\mathcal{O}$. There is an F-rational isogeny $\iota : E \to E'$ and a field embedding $F \hookrightarrow \mathbb{C}$ such that after extending the base to $\mathbb{C}$ we have $E \cong_{\mathbb{C}} \mathbb{C}/\mathcal{O}$, $E' \cong_{\mathbb{C}} \mathbb{C}/\mathcal{O}_K$, and after adjusting the source and target of $\iota_{\mathbb{C}}$ by these isomorphisms $\iota_{/\mathbb{C}}$ becomes the quotient map $\mathbb{C}/\mathcal{O} \to \mathbb{C}/\mathcal{O}_K$. The kernel of ι is cyclic of order $\mathfrak{f}$. Let $\tau_K = \frac{\Delta_K + \sqrt{\Delta_K}}{2}$ so $\mathcal{O}_K = \mathbb{Z}[\tau_K]$ and $\mathcal{O} = \mathbb{Z}[\mathfrak{f}\tau_K]$. Identifying $E[\text{tors}]$ with $\mathbb{C}/\mathcal{O}[\text{tors}]$ as above, for any $N \in \mathbb{Z}^+$ we have that $e_1 = \frac{1}{N} + \mathcal{O}$, $e_2 = \frac{\mathfrak{f}\tau_K}{N} + \mathcal{O}$

is a $\mathbb{Z}/N\mathbb{Z}$-basis for $E[N]$, and similarly $e_1' = \frac{1}{N} + \mathcal{O}_K, e_2' = \frac{\tau_K}{N} + \mathcal{O}_K$ is a $\mathbb{Z}/N\mathbb{Z}$-basis for $E'[N]$. With respect to this basis the image of the mod N Galois representation consists of matrices of the form

$$\left[\begin{array}{cc} a & b\mathfrak{f}^2 \frac{\Delta_K - \Delta_K^2}{4} \\ b & a + b\mathfrak{f}\Delta_K \end{array} \right] \mid a, b \in \mathbb{Z}/N\mathbb{Z}. \tag{3}$$

For finite commutative groups T and T', we have $\#T \mid \#T'$ if and only if $\#T[\ell^\infty] \mid \#T'[\ell^\infty]$ for all prime numbers ℓ. So we fix ℓ and show $\#E(F)[\ell^\infty] \mid \#E'(F)[\ell^\infty]$. If $\ell \nmid \mathfrak{f}$ then ι induces an isomorphism $E(F)[\ell^\infty] \to E'(F)[\ell^\infty]$, so we may assume that $\mathrm{ord}_\ell \mathfrak{f} \geq 1$. By (e.g.) the Mordell-Weil Theorem there is $0 \leq m \leq n$ such that

$$E(F)[\ell^\infty] \cong \mathbb{Z}/\ell^m\mathbb{Z} \oplus \mathbb{Z}/\ell^n\mathbb{Z}.$$

There is nothing to show unless $n \geq 1$, so we assume so. Put $N = \ell^n$, so $E(F)[\ell^\infty] \subset E[N]$, and let $\{e_1, e_2\}$ be the basis for $E[\ell^n]$ and $\{e_1', e_2'\}$ be the basis for $E'[\ell^n]$ as above. Put $k = \min(\mathrm{ord}_\ell \mathfrak{f}, n)$.

By assumption, there exists a point $P \in E(F)$ of order ℓ^n. Then $P' = \iota(P)$ has order ℓ^d for some $n - k \leq d \leq n$. If $d = n$, then $E'(F)[\ell^n]$ has exponent ℓ^n and full ℓ^m-torsion since $\iota(\ell^{n-m}e_1) = \ell^{n-m}e_1' \in E'(F)$ generates $E'[\ell^m]$ as an $\mathcal{O}_K$-module. Thus $E'(F)[\ell^n]$ has size at least ℓ^{m+n} and we are done. So we may assume $d < n$. There are $\alpha, \beta \in \mathbb{Z}/\ell^n\mathbb{Z}$ such that $P = \alpha e_1 + \beta e_2$, so we have

$$0 = \ell^d \iota(P) = \iota(\ell^d P) = \iota(\ell^d \alpha e_1) + \iota(\ell^d \beta e_2) = \ell^d \alpha e_1' + \ell^d \mathfrak{f} \beta e_2' = \ell^d \alpha e_1'$$

since $\ell^k \mid \mathfrak{f}$. This implies $\ell^{n-d} \mid \alpha$, so we may write $\alpha = \ell^{n-d}\alpha'$. In addition, we conclude $\ell \nmid \beta$ since $\ell^d P = \ell^d \beta e_2$ has order ℓ^{n-d}.

Put $\delta = \min(m + n - d, n)$. Since $\delta \leq m + n - d \leq m + k$ and E has full ℓ^m-torsion, the mod ℓ^δ Galois representation takes a restricted form:

$$\rho_{\ell^\delta}(\mathfrak{g}_F) \subset \left\{ \left[\begin{array}{cc} 1 + \ell^m A & 0 \\ \ell^m B & 1 + \ell^m A \end{array} \right] \mid A, B \in \mathbb{Z}/\ell^\delta\mathbb{Z} \right\}.$$

Since $\ell^{n-\delta}P = \alpha\ell^{n-\delta}e_1 + \beta\ell^{n-\delta}e_2$ is rational, all such matrices in the image of Galois satisfy

$$\left[\begin{array}{cc} 1 + \ell^m A & 0 \\ \ell^m B & 1 + \ell^m A \end{array} \right] \left[\begin{array}{c} \ell^{n-d}\alpha' \\ \beta \end{array} \right] = \left[\begin{array}{c} \ell^{n-d}\alpha' \\ \beta \end{array} \right],$$

which gives the condition

$$\ell^{n+m-d} B\alpha' + \beta\ell^m A \equiv 0 \pmod{\ell^\delta}.$$

But $\delta \leq n + m - d$ and $\ell \nmid \beta$, so this implies $\ell^{\delta - m} \mid A$. Thus the image of the mod ℓ^δ Galois representation consists of matrices of the form

$$\left[\begin{array}{cc} 1 & 0 \\ \ell^m B & 1 \end{array} \right].$$

It follows that $\iota(\ell^{n-\delta}e_1) \in E'[\mathrm{tors}]$ is F-rational. Indeed, for all $\sigma \in \mathfrak{g}_F$ we have

$$\begin{aligned} \sigma(\iota(\ell^{n-\delta}e_1)) &= \iota(\sigma(\ell^{n-\delta}e_1)) \\ &= \iota(\ell^{n-\delta}e_1 + \ell^m B\ell^{n-\delta}e_2) \\ &= \iota(\ell^{n-\delta}e_1) + \ell^{m+n-\delta} B\mathfrak{f}e_2' \\ &= \iota(\ell^{n-\delta}e_1), \end{aligned}$$

since $\ell^k \mid \mathfrak{f}$ and $m + n + k - \delta \geq n$. So $\iota(\ell^{n-\delta}e_1) = \ell^{n-\delta}e_1'$ is an F-rational point of E' of order ℓ^δ which generates $E'[\ell^\delta]$ as an $\mathcal{O}_K$-module. If $\delta = n$, then

$$\#E(F)[\ell^\infty] = \ell^{m+n} \leq \ell^{2n} = \#E'(F)[\ell^n] \leq \#E'(F)[\ell^\infty].$$

Otherwise, $\delta = m + n - d$ and E' has full ℓ^δ-torsion and a point of order ℓ^d. Thus $E'(F)[\ell^n]$ has size at least $\ell^{\delta+d} = \ell^{m+n}$.

4. The Uniform Open Image Theorem

4.1. **The Projective Torsion Field.** Let F be a field. For a positive integer N not divisible by the characteristic of F and $E_{/F}$ an elliptic curve, we define the **projective modulo N Galois representation** as the composite map

$$\mathbb{P}\rho_N : \mathfrak{g}_F \xrightarrow{\rho_N} \operatorname{Aut} E[N] \cong \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to \mathrm{PGL}_2(\mathbb{Z}/N\mathbb{Z}) \coloneqq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/(\mathbb{Z}/N\mathbb{Z})^\times.$$

The **projective torsion field** is

$$F(\mathbb{P}E[N]) = \overline{F}^{\ker \mathbb{P}\rho_N}.$$

Thus $F(\mathbb{P}E[N])$ is the unique minimal field extension of F on which the image of ρ_N consists of scalar matrices. It follows that $F(E[N])/F(\mathbb{P}E[N])$ is a Galois extension with automorphism group a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$.

Observe that the projective Galois representation and thus the projective torsion field are unchanged by *quadratic* twists. If $E_{/F}$ has CM by an order of discriminant $\Delta = \mathfrak{f}^2\Delta_K \neq -3, -4$ and $F \supset K$, then the projective N-torsion field is a well-defined abelian extension of $K(\mathfrak{f})$. An important result of J.L. Parish identifies this projective torsion field with a suitable ring class field. When $\Delta = -4$ (resp. $\Delta = -3$) we have quartic twists (resp. sextic twists) which can change the projective Galois representation and the projective torsion field.

Theorem 4.1. *Let $\mathcal{O}$ be an order of discriminant $\Delta = \mathfrak{f}^2\Delta_K$. Let E be an $\mathcal{O}$-CM elliptic curve defined over $F = K(\mathfrak{f})$. Let $N \geq 2$.*
a) We have $F(\mathbb{P}E[N]) \supset K(N\mathfrak{f})$. Thus we may put

$$d(E, N) = [F(\mathbb{P}E[N]) : K(N\mathfrak{f})].$$

b) If $\Delta \notin \{-3, -4\}$, then $d(E, N) = 1$, i.e., $F(\mathbb{P}E[N]) = K(N\mathfrak{f})$.
c) If $\Delta = -4$, then $d(E, N) \mid 2$.
d) If $\Delta = -3$, then $d(E, N) \mid 3$.

Proof. For $N \in \mathbb{Z}^+$, let $\mathcal{O}(N)$ be the order of conductor N in K. Thus $\mathcal{O} = \mathcal{O}(\mathfrak{f})$.
Step 1: We show that $F(\mathbb{P}E[N]) \supset K(N\mathfrak{f})$ in all cases.
There is a field embedding $F \hookrightarrow \mathbb{C}$ such that $E_{/\mathbb{C}} \cong \mathbb{C}/\mathcal{O}$. The $\mathbb{C}$-linear map $z \mapsto Nz$ carries $\mathcal{O}(\mathfrak{f})$ into $\mathcal{O}(N\mathfrak{f})$ and induces a cyclic N-isogeny $\mathbb{C}/\mathcal{O}(\mathfrak{f}) \to \mathbb{C}/\mathcal{O}(N\mathfrak{f})$. Let C be the kernel of this isogeny, viewed as a finite étale subgroup scheme of $E_{/\mathbb{C}}$. Then C has a (unique) minimal field of definition $F(C) \subset F(E[N])$, hence of finite degree over F. The field $F(\mathbb{P}E[N])$ is precisely the compositum of the minimal fields of definition of all order N cyclic subgroup schemes $C \subset E_{/\mathbb{C}}$, so $F(C) \subset F(\mathbb{P}E[N])$. Since C is $F(\mathbb{P}E[N])$-rational, the elliptic curve E/C has a model over this field, and thus

$$F(\mathbb{P}E[N]) \supset K(j(E/C)) = K(N\mathfrak{f}).$$

Step 2: In view of Step 1, we have $F(\mathbb{P}E[N]) \supset K(N\mathfrak{f}) \supset K(\mathfrak{f}) = K(j(E))$, so we have $F(\mathbb{P}E[N]) = K(N\mathfrak{f})$ iff $[F(\mathbb{P}E[N]) : K(\mathfrak{f})] \leq [K(N\mathfrak{f}) : K(\mathfrak{f})]$. We have

$$[F(\mathbb{P}E[N]) : K(\mathfrak{f})] = \#\mathbb{P}\rho_N(\mathfrak{g}_F) \leq \#(\mathcal{O}/N\mathcal{O})^\times/(\mathbb{Z}/N\mathbb{Z})^\times = N\prod_{p \mid N}\left(1 - \left(\frac{\Delta}{p}\right)\frac{1}{p}\right).$$

• Suppose $\mathfrak{f} > 1$. Using Theorem 2.1 to compute $[K(N\mathfrak{f}) : K^{(1)}]$ and $[K(\mathfrak{f}) : K^{(1)}]$ gives

$$[K(N\mathfrak{f}) : K(\mathfrak{f})] = \frac{[K(N\mathfrak{f}) : K^{(1)}]}{[K(\mathfrak{f}) : K^{(1)}]} = N\prod_{p \mid N,\ p \nmid \mathfrak{f}}\left(1 - \left(\frac{\Delta}{p}\right)\frac{1}{p}\right) = N\prod_{p \mid N}\left(1 - \left(\frac{\Delta}{p}\right)\frac{1}{p}\right),$$

because $1 - \left(\frac{\Delta}{p}\right)\frac{1}{p} = 1$ for all $p \mid \mathfrak{f}$. Thus $d(E, N) = 1$ in this case.
• Suppose $\mathfrak{f} = 1$, so $\Delta = \Delta_K$. Then

$$[K(N\mathfrak{f}) : K(\mathfrak{f})] = [K(N) : K^{(1)}] = \frac{2}{w_K}N\prod_{p \mid N}\left(1 - \left(\frac{\Delta}{p}\right)\frac{1}{p}\right).$$

If $\Delta \notin \{-3,-4\}$ then $\frac{2}{w_K} = 1$, and again we get $d(E,N) = 1$. If $\Delta = -4$ then $\frac{2}{w_K} = \frac{1}{2}$, so the calculation shows $d(E,N) \in \{1,2\}$, and if $\Delta = -3$ then $\frac{2}{w_K} = \frac{1}{3}$, so the calculation shows $d(E,N) \in \{1,3\}$. □

The following result is an an analogue of [BCS, Thm. 5.6] for higher twists.

Proposition 4.2. *(Higher Twisting at the Bottom)*
For $M \in \mathbb{Z}^+$, we denote the mod M cyclotomic character by χ_M.
a) Let $K = \mathbb{Q}(\sqrt{-1})$ and let $\ell \equiv 5 \pmod 8$ be a prime number. There is a character $\Psi : \mathfrak{g}_K \to (\mathbb{Z}/\ell\mathbb{Z})^\times$ of order $\frac{\ell-1}{4}$ and an $\mathcal{O}_K$-CM elliptic curve $E_{/K}$ such that the mod ℓ Galois representation is

$$\sigma \mapsto \rho_\ell(\sigma) = \begin{bmatrix} \Psi(\sigma) & 0 \\ 0 & \Psi^{-1}(\sigma)\chi_\ell(\sigma) \end{bmatrix}.$$

b) Let $K = \mathbb{Q}(\sqrt{-3})$ and let $\ell \equiv 7, 31 \pmod{36}$ be a prime number. There is a character $\Psi : \mathfrak{g}_K \to (\mathbb{Z}/\ell\mathbb{Z})^\times$ of order $\frac{\ell-1}{6}$ and an $\mathcal{O}_K$-CM elliptic curve $E_{/K}$ such that the mod ℓ Galois representation is

$$\sigma \mapsto \rho_\ell(\sigma) = \begin{bmatrix} \Psi(\sigma) & 0 \\ 0 & \Psi^{-1}(\sigma)\chi_\ell(\sigma) \end{bmatrix}.$$

Proof. a) Because $\ell \equiv 1 \pmod 4$, the Cartan subgroup $C_\ell(\mathcal{O})$ is split, and for an $\mathcal{O}_K$-CM elliptic curve $(E_1)_{/K}$, the mod ℓ Galois representation has the form

$$\sigma \mapsto \rho_\ell(\sigma) = \begin{bmatrix} \Psi_1(\sigma) & 0 \\ 0 & \Psi_1^{-1}(\sigma)\chi_\ell(\sigma) \end{bmatrix}$$

for a character $\Psi_1 : \mathfrak{g}_K \to (\mathbb{Z}/\ell\mathbb{Z})^\times$. Under this isomorphism, the matrix representation of $i \in \mathcal{O}_K$ is a diagonal matrix $\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$, where z is a primitive 4th root of unity in $\mathbb{Z}/\ell\mathbb{Z}$. A general $\mathcal{O}_K$-CM elliptic curve over K is of the form E_1^ψ for a character $\psi : \mathfrak{g}_K \to \mu_4 \subset (\mathbb{Z}/\ell\mathbb{Z})^\times$. Let $Q_4(\ell) = (\mathbb{Z}/\ell\mathbb{Z})^\times/(\mathbb{Z}/\ell\mathbb{Z})^{\times 4}$. Then the image of z in $Q_4(\ell)$ has order 4: if not, there is $w \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ such that $z = w^2$, and then w has order 8 in $(\mathbb{Z}/\ell\mathbb{Z})^\times$, contradicting the assumption that $\ell \equiv 5 \pmod 8$. Thus the natural map $\mu_4 \to Q_4(\ell)$ given by $i \mapsto z \pmod{(\mathbb{Z}/\ell\mathbb{Z})^{\times 4}}$ is an isomorphism; we denote the inverse isomorphism $Q_4(\ell) \to \mu_4$ by ι. Now take

$$\psi : \mathfrak{g}_K \overset{\Psi_1^{-1}}{\to} (\mathbb{Z}/\ell\mathbb{Z})^\times \overset{q}{\to} Q_4(\ell) \overset{\iota}{\to} \mu_4.$$

Let $\Psi_2 = \psi\Psi_1$. Then the twist E_1^ψ has mod ℓ Galois representation

$$\sigma \mapsto \rho_\ell(\sigma) = \begin{bmatrix} \Psi_2(\sigma) & 0 \\ 0 & \Psi_2^{-1}(\sigma)\chi_\ell(\sigma) \end{bmatrix}.$$

The composite $\Psi_2 : \mathfrak{g}_K \to (\mathbb{Z}/\ell\mathbb{Z})^\times \to Q_4(\ell)$ is trivial, so $\Psi_2(\mathfrak{g}_K)$ has order $c \mid \frac{\ell-1}{4}$. Thus

$$\#\rho_{\ell,E_1^\psi}(\mathfrak{g}_K) \mid c(\ell-1) \mid \frac{(\ell-1)^2}{4} = [K^{(\ell)} : K^{(1)}] = [K^{(\ell)} : K].$$

Because $K(E_1^\psi[\ell]) \supset K^{(\ell)}$, we have $\#\rho_{\ell,E_1^\psi}(\mathfrak{g}_K) = \frac{(\ell-1)^2}{4}$ and $c = \frac{\ell-1}{4}$.
b) Since $\ell \equiv 1 \pmod 3$, we have a primitive 6th root of unity z in $\mathbb{Z}/\ell\mathbb{Z}$. Since $\ell \equiv 7, 31 \pmod{36}$, we have $4, 9 \nmid \ell - 1$, so z has order 6 in $Q_6(\ell) = (\mathbb{Z}/\ell\mathbb{Z})^\times/(\mathbb{Z}/\ell\mathbb{Z})^{\times 6}$. Also $\frac{(\ell-1)^2}{6} = [K^{(\ell)} : K^{(1)}]$. The argument of part a) carries over. □

Example 4.3. *a) Let $K = \mathbb{Q}(\sqrt{-1})$, and let $\ell \equiv 5 \pmod 8$. Let $E_{/K}$ be an $\mathcal{O}_K$-CM elliptic curve with mod ℓ Galois representation as in Proposition 4.2a). Then since $\chi_\ell(\mathfrak{g}_K) = (\mathbb{Z}/\ell\mathbb{Z})^\times$, $[K(\mathbb{P}E[\ell]) : K] = \ell - 1$, whereas $[K(\ell) : K] = \frac{\ell-1}{2}$. So $d(E,\ell) = 2$.*
b) Let $K = \mathbb{Q}(\sqrt{-3})$, and let $\ell \equiv 7, 31 \pmod{36}$. Let $E_{/K}$ be an $\mathcal{O}$-CM elliptic curve with mod ℓ Galois representation as in Proposition 4.2b). As in part a), we have $[K(\mathbb{P}E[\ell]) : K] = \ell - 1$ and $[K(\ell) : K] = \frac{\ell-1}{3}$. So $d(E,\ell) = 3$.

Remark 4.4. *Parts a) and b) of Theorem 4.1 are due to J.L. Parish* [Pa89, Prop. 3]*. However, Parish alludes to a calculation of the above sort rather than explicitly carrying it out. Since Theorem 4.1 will play an important role in the proof of Theorem 1.1, we have given a complete proof.*

In [Pa89, Prop. 3]*, Parish assumes* $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$*. In an "addendum"* [Pa89, p. 263]*, he claims:*
- *If* $\Delta = -4$ *then* $F(\mathbb{P}E[N]) = K(N)$ *for all* $N \geq 3$*, and*
- *If* $\Delta = -3$ *then* $F(\mathbb{P}E[N]) = K(N)$ *for all* $N \geq 4$.

As Example 4.3 shows, both claims are false.

Proposition 4.5. *Let* $\mathcal{O}$ *be an order of discriminant* $\Delta = \mathfrak{f}^2\Delta_K$*, and let* $N \in \mathbb{Z}^+$*. Then there is an* $\mathcal{O}$*-CM elliptic curve* $E_{/K(N\mathfrak{f})}$ *such that the mod* N *Galois representation consists of scalar matrices.*

Proof. When $\Delta \notin \{-3, -4\}$, this is immediate from Theorem 4.1b): in that case, the elliptic curve has a model defined over $K(\mathfrak{f})$. Thus we may assume that $\Delta \in \{-3, -4\}$, so $\mathfrak{f} = 1$. Let $\zeta \in \mathcal{O}_K^\times$ be a primitive w_Kth root of unity. Let $\mathcal{O}$ be the order in K of conductor N, let $\tilde{E}_{/K(N)}$ be an $\mathcal{O}$-CM elliptic curve, and let $\iota : \tilde{E} \to E$ be the canonical $K(N)$-rational isogeny to an $\mathcal{O}_K$-CM elliptic curve E, let $\iota^\vee : E \to \tilde{E}$ be the dual isogeny, and let C be the kernel of $\iota^\vee$. Identifying $E[N]$ with $N^{-1}\mathcal{O}_K/\mathcal{O}_K \subset \mathbb{C}/\mathcal{O}_K$, $\iota^\vee : \mathbb{C}/\mathcal{O}_K \to \mathbb{C}/\mathcal{O}$ is the map $z + \mathcal{O}_K \mapsto Nz + \mathcal{O}$, so C is the $\mathbb{Z}$-submodule of $\mathbb{C}/\mathcal{O}_K$ generated by $P_1 = \frac{1}{N} + \mathcal{O}_K$. Because C is stable under the action of $\mathfrak{g}_{K(N)}$, this action is given by an isogeny character, say

$$\sigma(P_1) = \Psi(\sigma)P_1.$$

Let $P_2 = \zeta P_1$. Then $\{P_1, P_2\}$ is a $\mathbb{Z}/N\mathbb{Z}$-basis for $E[N]$. Moreover, for $\sigma \in \mathfrak{g}_{K(N)}$,

$$\sigma P_2 = \sigma\zeta P_1 = \zeta\sigma P_1 = \zeta\Psi(\sigma)P_1 = \Psi(\sigma)\zeta P_1 = \Psi(\sigma)P_2.$$

It follows that $\sigma \in \mathfrak{g}_{K(N)}$ acts on $E[N]$ via the scalar matrix $\Psi(\sigma)$. □

4.2. **Proof of Theorem 1.1b) when** $F = K(\mathfrak{f})$**.** In this section we prove Theorem 1.1 b) in the case $F = K(\mathfrak{f})$. The general case $F \supset K(\mathfrak{f})$ is treated in the next section.

Step 1: Let $\mathcal{O}$ be an order in K of conductor $\mathfrak{f}$, let $F = K(\mathfrak{f})$ and let $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve. Let $N \in \mathbb{Z}^+$. Identifying $\rho_N(\mathfrak{g}_F)$ with a subgroup of $C_N(\mathcal{O})$, put

$$\mathcal{I}_N = \mathcal{I}_N(E_{/K(\mathfrak{f})}) = [C_N(\mathcal{O}) : \rho_N(\mathfrak{g}_F)].$$

Our task is to show that as we vary over all imaginary quadratic fields K, all $\mathfrak{f} \in \mathbb{Z}^+$, and all elliptic curves E defined over $F = K(\mathfrak{f})$ with $\operatorname{End} E \cong \mathcal{O}(\mathfrak{f})$ in K and all $N \in \mathbb{Z}^+$, we have $\mathcal{I}_N \mid \#\mathcal{O}^\times$, or equivalently,

$$\frac{\#C_N(\mathcal{O})}{\#\mathcal{O}^\times} \mid [F(E[N]) : F].$$

Because the ρ_N form an inverse system, we have $N \mid N' \implies \mathcal{I}_N \mid \mathcal{I}_{N'}$. So we may – and shall – assume $3 \mid N$ and thus

$$[K^{(N)} : K^{(1)}] = \frac{\varphi_K(N)}{w_K}.$$

Put $L = K(N\mathfrak{f})$. By Theorems 2.10 and 4.1 we have

$$F(E[N]) \supset K^{(N)}L,$$

so it is enough to show that

$$\frac{\#C_N(\mathcal{O})}{\#\mathcal{O}^\times} \mid [K^{(N)}L : K(\mathfrak{f})]. \tag{4}$$

Although (4) is purely class-field theoretic, we will show it using CM elliptic curves.

Step 2: Suppose first that $\mathfrak{f} = 1$, so $\mathcal{O} = \mathcal{O}_K$. Then since $3 \mid N$ we have

$$[K^{(N)}L : K(\mathfrak{f})] = [K^{(N)} : K^{(1)}] = \frac{\varphi_K(N)}{w_K},$$

and we are done. Now suppose that $\mathfrak{f} > 1$. Then

$$[K^{(N)}L : K(\mathfrak{f})] = \frac{[K^{(N\mathfrak{f})} : K^{(1)}]}{[K(\mathfrak{f}) : K^{(1)}][K^{(N\mathfrak{f})} : K^{(N)}L]}. \tag{5}$$

Combining equation (5) with Lemmas 2.11 and 2.3 we get

$$[K^{(N)}L : K(\mathfrak{f})] = \frac{\varphi_K(N\mathfrak{f})/w_K}{[K(\mathfrak{f}) : K^{(1)}][K^{(N\mathfrak{f})} : K^{(N)}L]}$$
$$= \frac{\#C_N(\mathcal{O})}{\#\mathcal{O}^\times} \cdot \frac{\varphi(N\mathfrak{f})/\varphi(N)}{[K^{(N\mathfrak{f})} : K^{(N)}L]}.$$

It now suffices to show that

$$[K^{(N\mathfrak{f})} : K^{(N)}L] \mid \frac{\varphi(N\mathfrak{f})}{\varphi(N)}.$$

Step 3: By Proposition 4.5 there is an $\mathcal{O}_K$-CM elliptic curve $(E_0)_{/L}$ for which the mod $N\mathfrak{f}$ Galois representation has scalar image. Since $3 \mid N\mathfrak{f}$, there is a character $\Psi : \mathfrak{g}_L \to (\mathbb{Z}/3\mathbb{Z})^\times = \{\pm 1\}$ such that

$$\rho_3(\sigma) = \begin{bmatrix} \Psi(\sigma) & 0 \\ 0 & \Psi(\sigma) \end{bmatrix}.$$

Thus the quadratic twist E_1 of E_0 by Ψ has trivial mod 3 Galois representation, so

$$L(E_1[3]) = L = L(\mathfrak{h}(E_1[3])).$$

We CLAIM that this implies that for all $3 \mid M \in \mathbb{Z}^+$, we have

$$L(E_1[M]) = L(\mathfrak{h}(E_1[M])) = LK^{(M)}.$$

PROOF OF CLAIM: The Galois group $\mathrm{Aut}(L(E_1[M])/L)$ is naturally identified with a subgroup $G(M)$ of $C_M(\mathcal{O}_K)$. Because $M \geq 3$, the composite homomorphism

$$\mathcal{O}_K^\times \to C_M(\mathcal{O}_K) \to C_3(\mathcal{O}_K) \tag{6}$$

is injective. We have

$$L(\mathfrak{h}(E_1[M])) = L(E_1[M])^{G(M)\cap\mathcal{O}_K^\times}.$$

Since

$$G(3) \cap \mathcal{O}_K^\times = (G(M) \cap \mathcal{O}_K^\times) \pmod 3,$$

the injectivity of (6) means that if $G(M) \cap \mathcal{O}_K^\times \supsetneq \{e\}$, then $G(3) \cap \mathcal{O}_K^\times \supsetneq \{e\}$. But $G(3) = \{e\}$, so that $G(M) \cap \mathcal{O}_K^\times = \{e\}$, establishing the claim.

Let $G = \mathrm{Aut}(K^{(N\mathfrak{f})}/L)$, $H = \mathrm{Aut}(K^{(N\mathfrak{f})}/K^{(N)}L)$. Since $K^{(N\mathfrak{f})} = L(E_1[N\mathfrak{f}])$ and $K^{(N)}L = L(E_1[N])$, we may identify G with a subgroup of scalar matrices of $C_{N\mathfrak{f}}(\mathcal{O}_K)$, and H is the subgroup of matrices which are 1 mod N. So $\#H \mid \frac{\varphi(N\mathfrak{f})}{\varphi(N)}$.

4.3. **End of the Proof of Theorem 1.1b).** Let $\mathcal{O}$ be the order of conductor $\mathfrak{f}$ in an imaginary quadratic field K, let $F \supset K(\mathfrak{f})$ be a number field, and let $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve: we may choose the embedding $F \hookrightarrow \mathbb{C}$ such that $j(E) = j(\mathbb{C}/\mathcal{O})$. We want to to show that the index of the image of the Galois representation on $E_{/F}$ in $C_N(\mathcal{O})$ divides $\#\mathcal{O}^\times[F : K(\mathfrak{f})]$. Equivalently, we want to establish this divisibility on the index of the mod N Galois representation for all sufficiently divisible $N \in \mathbb{Z}^+$, so we may (and shall) assume that $3 \mid N$. Let $(E_1)_{/K(\mathfrak{f})}$ be an elliptic curve with $j(E_1) = j(E)$. Put

$$W(N, \mathcal{O}) = K(\mathfrak{f})(\mathfrak{h}(E_1[N])).$$

We saw above that $K^{(N)}K(N\mathfrak{f}) \subset K(\mathfrak{f})(E_1[N])$. Since this holds for all E_1 with $j(E_1) = j(\mathbb{C}/\mathcal{O})$, the Weber Function Principle (Theorem 2.12) gives

$$K^{(N)}K(N\mathfrak{f}) \subset W(N, \mathcal{O}).$$

By part a) of the Weber Function Principle and (4) we have

$$[W(N,\mathcal{O}) : K(\mathfrak{f})] \mid \frac{\#C_N(\mathcal{O})}{\#\mathcal{O}^\times} \mid [K^{(N)}K(N\mathfrak{f}) : K(\mathfrak{f})],$$

so we deduce

$$W(N,\mathcal{O}) = K^{(N)}K(N\mathfrak{f}),\ [W(N,\mathcal{O}) : K(\mathfrak{f})] = \frac{\#C_N(\mathcal{O})}{\#\mathcal{O}^\times}. \tag{7}$$

It follows from (7) and the Weber Function Principle that we may choose $(E_1)_{/K(\mathfrak{f})}$ so that $\rho_{E_1,N}(\mathfrak{g}_{K(\mathfrak{f})}) = C_N(\mathcal{O})$. By the standard theory of twists, there is an extension L/F of degree $\#\mathcal{O}^\times$ such that $E_{/L} \cong (E_1)_{/L}$, and thus

$$[C_N(\mathcal{O}) : \rho_{E,N}(\mathfrak{g}_F)] \mid [C_N(\mathcal{O}) : \rho_{E_1,N}(\mathfrak{g}_L)] \mid [L : K(\mathfrak{f})] = \#\mathcal{O}^\times[F : K(\mathfrak{f})].$$

4.4. **Proof of Theorem 1.1c).** Let $\mathcal{O}$ be the order of conductor $\mathfrak{f}$ in an imaginary quadratic field K, let $w = \#\mathcal{O}^\times$, and let $N \geq 3$; the last assumption implies that $\mu_w \hookrightarrow C_N(\mathcal{O})$. Let $E_{/K(\mathfrak{f})}$ be any $\mathcal{O}$-CM elliptic curve. Again we may view $G = \mathrm{Aut}(K(\mathfrak{f})(E[N])/K(\mathfrak{f}))$ as a subgroup of $C_N(\mathcal{O})$. Let $H = G \cap \mu_w$ and $L = (K(\mathfrak{f})(E[N]))^H$. Then $[L : K(\mathfrak{f})] \mid \frac{\#C_N(\mathcal{O})}{w}$ and $\rho_{E,N}(\mathfrak{g}_L) \subset \mu_w$, so a suitable twist of $E_{/L}$ has trivial mod N Galois representation. By Theorem 1.1b) we must in fact have

$$[L : K(\mathfrak{f})] = \frac{\#C_N(\mathcal{O})}{w}.$$

The excluded case $N = 2$ will be treated in Corollary 4.7.

4.5. **Proof of Theorem 1.2.** Let $\mathcal{O}$ be the order of conductor $\mathfrak{f}$ in an imaginary quadratic field K. For $N \in \mathbb{Z}^+$, put

$$W(N,\mathcal{O}) = K(\mathfrak{f})(\mathfrak{h}(E[N])),$$

where $E_{/K(\mathfrak{f})}$ is an elliptic curve with $j(E) = j(\mathbb{C}/\mathcal{O})$; the field is independent of the choice of E. In the course of proving Theorem 1.1 we showed that

$$W(N,\mathcal{O}) \supset K^{(N)}K(N\mathfrak{f}).$$

When $3 \mid N$, we showed (7) that equality holds and that $[W(N,\mathcal{O}) : K(\mathfrak{f})] = \frac{\#C_N(\mathcal{O})}{\#\mathcal{O}^\times}$. Conversely, these statements about $W(N,\mathcal{O})$ imply Theorem 1.1: immediately when $F = K(\mathfrak{f})$, and by an easy twisting argument when $F \supset K(\mathfrak{f})$, as in §4.3. This leaves open the question of whether (7) holds when $3 \nmid N$. The next result shows that it holds for all $N \geq 3$ and gives a suitable analogue when $N = 2$.

Theorem 4.6. *Let $\mathcal{O}$ be an order of conductor $\mathfrak{f}$.*
a) For all $N \geq 3$, we have

$$W(N,\mathcal{O}) = K^{(N)}K(N\mathfrak{f}),\ [W(N,\mathcal{O}) : K(\mathfrak{f})] = \frac{\#C_N(\mathcal{O})}{\#\mathcal{O}^\times}. \tag{8}$$

b) We have

$$W(2,\mathcal{O}) = K^{(2)}K(2\mathfrak{f}) = K(2\mathfrak{f}),\ [W(2,\mathcal{O}) : K(\mathfrak{f})] = \frac{2\#C_2(\mathcal{O})}{\#\mathcal{O}^\times}. \tag{9}$$

Proof. a) Step 0: When $\mathfrak{f} = 1$ this reduces to known results: $W_N = K^{(N)}$ and (since $N \geq 3$) $[K^{(N)} : K^{(1)}] = \frac{\varphi_K(N)}{\#\mathcal{O}^\times}$. Thus we may assume $\mathfrak{f} > 1$, so $\mathcal{O}^\times = \{\pm 1\}$.
Step 1: Let $M = K^{(N)}K(N\mathfrak{f})$. We already know

$$M \subset W(N,\mathcal{O})$$

and (by the Weber Function Principle)

$$[W(N,\mathcal{O}) : K(\mathfrak{f})] \mid \frac{\#C_N(\mathcal{O})}{\#\mathcal{O}^\times}.$$

So it suffices to show

$$\frac{\#C_N(\mathcal{O})}{\#\mathcal{O}^\times} \mid [M : K(\mathfrak{f})]. \tag{10}$$

In turn, for this it is sufficient to construct an $\mathcal{O}$-CM elliptic curve $E_{/M}$ with trivial mod N Galois representation, for then Theorem 1.1 gives (10).

Step 2: By Proposition 4.5, there is an $\mathcal{O}$-CM elliptic curve $E_{/K(N\mathfrak{f})}$ for which the mod N Galois representation consists of scalar matrices. We extend the base to get $E_{/M}$. Let $\iota : E \to E'$ be the canonical isogeny to an $\mathcal{O}_K$-CM elliptic curve. Since $W(N, \mathcal{O}_K) = K^{(N)} \subset M$, we have $\rho_{E',N}(\mathfrak{g}_M) \subset \mu_K$. Let $\iota^\vee : E' \to E$ be the dual isogeny. Since ι and $\iota^\vee$ are cyclic isogenies, there is a point $P \in E'(\overline{M})$ of order N such that $Q = \iota^\vee(P)$ has order N. If $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ then $\mu_K = \{\pm 1\}$, so the $\mathfrak{g}_M$-orbit of P is contained in $\{\pm P\}$, hence the $\mathfrak{g}_M$-orbit of Q is contained in $\{\pm Q\}$. Thus Q is an M-rational point on a suitable quadratic twist E^D, and since quadratic twists do not change whether the Galois representation is given by scalar matrices, the mod N Galois representation on $E^D_{/M}$ is trivial.

Now suppose $K = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$ and let ζ be a primitive w_Kth root of unity, so $\mathcal{O} = \mathbb{Z}[\mathfrak{f}\zeta]$. Then we can take $P = \frac{\zeta}{N}$. Explicitly, the dual isogeny is

$$\iota^\vee : z + \mathcal{O}_K \mapsto \mathfrak{f}z + \mathcal{O},$$

so $Q = \iota^\vee(P) = \frac{\mathfrak{f}\zeta}{N} + \mathcal{O}$, which has order N in $\mathbb{C}/\mathcal{O}$. Suppose for some $1 \leq k < w_K$, the point $\iota^\vee(\zeta^k P) = \frac{\mathfrak{f}\zeta^{k+1}}{N} + \mathcal{O}$ is a scalar multiple of $Q = \frac{\mathfrak{f}\zeta}{N} + \mathcal{O}$. Then $\frac{\mathfrak{f}\zeta^{k+1}}{N} + \mathcal{O} = \alpha\frac{\mathfrak{f}\zeta}{N} + \mathcal{O}$ for some $\alpha \in (\mathbb{Z}/N\mathbb{Z})^\times$. Since $\frac{\mathfrak{f}\zeta^{k+1}}{N} - \alpha\frac{\mathfrak{f}\zeta}{N} \in \mathcal{O}$ only if $\alpha \in \{\pm 1\}$, again the $\mathfrak{g}_M$-orbit of Q is contained in $\{\pm Q\}$ and we can make a quadratic twist as above. (In fact this argument shows that $\rho_{N,E'}(\mathfrak{g}_M) \subset \{\pm 1\}$.)

b) Again, when $\mathfrak{f} = 1$, this reduces to known results: $W(2, \mathcal{O}) = K^{(2)}$ and $[K^{(2)} : K^{(1)}] = \frac{2\varphi_K(2)}{\#\mathcal{O}^\times}$ (Lemma 2.11). So suppose $\mathfrak{f} > 1$. It follows from Theorem 2.1 and Lemma 2.11 that $K^{(2)} = K(2)$ and thus $K^{(2)}K(2\mathfrak{f}) = K(2\mathfrak{f})$. Further, from Theorem 2.1 and Lemma 2.2 we get

$$[K(2\mathfrak{f}) : K(\mathfrak{f})] = \#C_2(\mathcal{O}) = \frac{2\#C_2(\mathcal{O})}{\#\mathcal{O}^\times}.$$

Since $N = 2$ and $\mathcal{O}^\times = \{\pm 1\}$, Theorem 4.1 implies that for any $\mathcal{O}$-CM elliptic curve $E_{/K(\mathfrak{f})}$ we have

$$W(2, \mathcal{O}) = K(\mathfrak{f})(\mathfrak{h}(E[2])) = K(\mathfrak{f})(x(E[2])) = K(\mathfrak{f})(E[2])) \supset K(\mathfrak{f})(\mathbb{P}E[2])) \supset K(2\mathfrak{f}).$$

Since $[W(2, \mathcal{O}) : K(\mathfrak{f})] = [K(\mathfrak{f})(E[2]) : K(\mathfrak{f})] \leq \#C_2(\mathcal{O})$, we conclude $W(2, \mathcal{O}) = K(2\mathfrak{f})$. □

Corollary 4.7. *For all orders $\mathcal{O}$, there is a number field $F \supset K$ and an $\mathcal{O}$-CM elliptic curve $E_{/F}$ such that $E[2] = E[2](F)$ and $[F : K(j(E))] = \frac{2\#C_2(\mathcal{O})}{\#\mathcal{O}^\times}$.*

Proof. Let $\mathcal{O}$ be the order of conductor $\mathfrak{f}$ in K. By Proposition 4.5, there is an $\mathcal{O}$-CM elliptic curve $E_{/K(2\mathfrak{f})}$ such that the mod 2 Galois representation consists of scalar matrices. Since the only scalar matrix in $C_2(\mathcal{O})$ is the identity, we have $E[2] = E[2](K(2\mathfrak{f}))$. So we may take $F = K(2\mathfrak{f})$. □

4.6. **Proof of Theorem 1.1a).** From Theorem 4.6 we deduce that the image of Galois is as large as possible, up to twisting.

Theorem 4.8. *Let $\mathcal{O}$ be the order of conductor $\mathfrak{f}$ in an imaginary quadratic field K, and let $E_{/K(\mathfrak{f})}$ be an $\mathcal{O}$-CM elliptic curve. For any $N \in \mathbb{Z}^+$, there is a twist E' of $E_{/K(\mathfrak{f})}$ such that $\rho_{E',N}(\mathfrak{g}_{K(\mathfrak{f})}) = C_N(\mathcal{O})$.*

Proof. If $N \geq 3$, this follows from part b) of the Weber Function Principle and part a) of Theorem 4.6. If $N = 2$ and $j \neq 0, 1728$, then this is Theorem 4.6b). We now address the remaining cases.

Let $K = \mathbb{Q}(\sqrt{-3})$, and let $E_{/K}$ be an $\mathcal{O}_K$-CM elliptic curve. If $[K(E[2]) : K] = 3$, we are done, so suppose $K(E[2]) = K$. As in the proof of the Weber Function Principle, let $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$ for some prime $\mathfrak{p}$ of $\mathcal{O}_K$. Then let $L = K(\pi^{\frac{1}{6}})$, and let $\chi : \mathfrak{g}_K \to \mu_6$ be a character with splitting field $\overline{F}^{\ker\chi} = L$. Then the twist $E^\chi_{/K}$ has $[K(E^\chi[2]) : K] = 3$, as desired. The case $K = \mathbb{Q}(\sqrt{-1})$ is similar: if $K(E[2]) = K$,

we take $\chi : \mathfrak{g}_K \to \mu_4$ to be a character corresponding to $L = K(\pi^{\frac{1}{4}})$. Then we will have $E^{\chi}_{/K}$ with $[K(E^{\chi}[2]) : K] = 2$. □

Corollary 4.9. *For all $\mathcal{O}$-CM elliptic curves $E_{/K(\mathfrak{f})}$, the reduced Galois representation*

$$\overline{\rho_N} : \mathfrak{g}_{K(\mathfrak{f})} \to \overline{C_N(\mathcal{O})}$$

is surjective.

Proof. Let $E_{/K(\mathfrak{f})}$ be an $\mathcal{O}$-CM elliptic curve. By Theorem 4.8, there is a character $\chi : \mathfrak{g}_{K(\mathfrak{f})} \to \mu_w$ and a twist E^{χ} of $E_{/K(\mathfrak{f})}$ such that $\rho_{E^{\chi},N}(\mathfrak{g}_{K(\mathfrak{f})}) = C_N(\mathcal{O})$. Since $\rho_{E^{\chi},N}(\sigma) = \chi(\sigma)\rho_{E,N}(\sigma)$, the result follows. □

5. Applications

5.1. **SPY Divisibilities.**

Lemma 5.1. *Let H, K be subgroups of a group G. If H is normal and $H \cap K = \{1\}$, then $\#K \mid [G : H]$.*

Proof. The composite homomorphism $K \hookrightarrow G \to G/H$ is an injection. □

Theorem 5.2. *Let $\mathcal{O}$ be an order in an imaginary quadratic field K, and let E be an $\mathcal{O}$-CM elliptic curve defined over a number field $F \supset K$. If $E(F)$ has a point of order $N \in \mathbb{Z}^+$, then*

$$\varphi(N) \mid \frac{\#\mathcal{O}^{\times}}{2} \frac{[F : \mathbb{Q}]}{\#\operatorname{Pic} \mathcal{O}}.$$

Proof. Let $\mathcal{I}_N = [C_N(\mathcal{O}) : \rho_N(\mathfrak{g}_F)]$ be the index of the mod N Galois representation in the Cartan subgroup. By Theorem 1.1 we have

$$\mathcal{I}_N \mid \#\mathcal{O}^{\times}[F : K(j(E))] = \frac{\#\mathcal{O}^{\times}}{2} \frac{[F : \mathbb{Q}]}{\#\operatorname{Pic} \mathcal{O}}.$$

Since there is a rational point of order N, $\rho_N(\mathfrak{g}_F)$ contains no scalar matrices other than the identity, so by Lemma 5.1 we have $\varphi(N) \mid \mathcal{I}_N$, and we're done. □

5.2. **A Theorem of Franz.** Let $\mathcal{O}$ be an order in K, of conductor $\mathfrak{f}$, and let $E_{/K(\mathfrak{f})}$ be an $\mathcal{O}$-CM elliptic curve. Choose a field embedding $K(\mathfrak{f}) \hookrightarrow \mathbb{C}$ such that $j(E) = j(\mathbb{C}/\mathcal{O})$ and an isomorphism $E_{/\mathbb{C}} \xrightarrow{\sim} \mathbb{C}/\mathcal{O}$. This induces an isomorphism $E(\overline{K(\mathfrak{f})})[\text{tors}] \xrightarrow{\sim} \mathbb{C}/\mathcal{O}[\text{tors}]$, which we use to view (the image in $\mathbb{C}/\mathcal{O}$ of) $\tau_K = \frac{\Delta_K + \sqrt{\Delta_K}}{2}$ as a point of $E(\overline{K(\mathfrak{f})})[\text{tors}]$ of order $\mathfrak{f}$.

Theorem 5.3. *(Franz* [Fr35]*) With notation as above, we have*

$$K(\mathfrak{f})(\mathfrak{h}(\tau_K)) = K^{(\mathfrak{f})}.$$

Proof. As in the proof of Theorem 1.4, over $\mathbb{C}$ we may view the canonical isogeny as $\iota : \mathbb{C}/\mathcal{O} \to \mathbb{C}/\mathcal{O}_K$. We take $e_1 = \frac{1}{\mathfrak{f}} + \mathcal{O}$ and $e_2 = \tau_K + \mathcal{O}$ as a basis for $E[\mathfrak{f}]$. Then e_2 generates $\ker(\iota)$, a $K(\mathfrak{f})$-rational cyclic subgroup of order $\mathfrak{f}$, and there is a character $\Psi : \mathfrak{g}_F \to (\mathbb{Z}/\mathfrak{f}\mathbb{Z})^{\times}$ such that

$$\rho_{E,\mathfrak{f}}(\sigma) = \begin{bmatrix} \Psi(\sigma) & 0 \\ * & \Psi(\sigma) \end{bmatrix}.$$

If $\mathfrak{f} \leq 2$, then $K(\mathfrak{f})(\mathfrak{h}(\tau_K)) = K(\mathfrak{f}) = K^{(\mathfrak{f})}$ and the result holds. Thus we may assume $\mathfrak{f} \geq 3$. Let $L \coloneqq K(\mathfrak{f})(\mathfrak{h}(e_2))$. Since $j(E) \neq 0, 1728$, the restriction $\Psi|_{\mathfrak{g}_L} : \mathfrak{g}_L \to \{\pm 1\}$ defines a quadratic character χ, and on the twist E^{χ} of $E_{/L}$ the point e_2 becomes L-rational. As in the proof of Theorem 5.5 of [BCS], let $\Psi^{\pm} : \mathfrak{g}_{K(\mathfrak{f})} \to (\mathbb{Z}/\mathfrak{f}\mathbb{Z}^{\times})/\{\pm 1\}$ denote the composition of Ψ with the natural map $(\mathbb{Z}/\mathfrak{f}\mathbb{Z})^{\times} \to (\mathbb{Z}/\mathfrak{f}\mathbb{Z})^{\times}/\pm 1$. Then $L \subset (\overline{K(\mathfrak{f})})^{\ker \Psi^{\pm}}$, so $[L : K(\mathfrak{f})] \mid \frac{\varphi(\mathfrak{f})}{2}$. If $\iota : E^{\chi} \to E'$ is the canonical isogeny, then the proof of Theorem 1.4 shows that $\iota(e_1)$ is an element of $E'(L)$ which generates $E'[\mathfrak{f}]$ as an $\mathcal{O}_K$-module. Thus E' has full $\mathfrak{f}$-torsion over L, so by Theorem 2.9, $K^{(\mathfrak{f})} \subset L$. So

$$[L : K(\mathfrak{f})] \geq [K^{(\mathfrak{f})} : K(\mathfrak{f})] = \frac{\varphi(\mathfrak{f})}{2} \geq [L : K(\mathfrak{f})],$$

and thus $K(\mathfrak{f})(\mathfrak{h}(e_2)) = L = K^{(\mathfrak{f})}$. □

5.3. **The Field of Moduli of a Point of Prime Order.** In the introduction, we discussed a program to determine fields of moduli of all CM points on modular curves. Theorem 4.6 carries out this program for the curves $X(N)$. In this section we will obtain a result on the curves $X_1(N)$ – not definitive, but enough to illustrate where we are going and to showcase the tools we've developed.

Let $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ be an imaginary quadratic field, and let $\mathcal{O} \subset K$ be the order of conductor $\mathfrak{f}$. Here we use Theorem 1.4 to determine the smallest field $F \supset K$ for which there exists an $\mathcal{O}$-CM elliptic curve $E_{/F}$ with an F-rational point of order $\ell > 2$.

Lemma 5.4. *Let K be an imaginary quadratic field, let $\mathfrak{f} \in \mathbb{Z}^+$, and let $\ell > 2$ be prime. Then $K^{(\ell)} \cap K(\ell\mathfrak{f}) = K(\ell)$.*

Proof. Let $\Delta = \mathfrak{f}^2\Delta_K$. The statement is immediate if $\mathfrak{f} = 1$, so suppose $\mathfrak{f} > 1$. By Theorem 2.1,

$$[K(\ell\mathfrak{f}) : K(\mathfrak{f})] = \ell - \left(\frac{\Delta}{\ell}\right).$$

Since $[K^{(\ell)}K(\ell\mathfrak{f}) : K(\mathfrak{f})] = \#C_\ell(\mathcal{O})/2$ by Theorem 4.6, we have in both cases that

$$[K^{(\ell)}K(\ell\mathfrak{f}) : K(\ell\mathfrak{f})] = \frac{\#C_\ell(\mathcal{O})}{2[K(\ell\mathfrak{f}) : K(\mathfrak{f})]} = \frac{1}{2}(\ell - 1).$$

Thus $[K^{(\ell)} : K^{(\ell)} \cap K(\ell\mathfrak{f})] = [K^{(\ell)}K(\ell\mathfrak{f}) : K(\ell\mathfrak{f})] = \frac{1}{2}(\ell - 1)$. As we have $K(\ell) \subset K^{(\ell)} \cap K(\ell\mathfrak{f})$ and $[K^{(\ell)} : K(\ell)] = \frac{1}{2}(\ell - 1)$, the result follows. □

Theorem 5.5. *Let $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ be an imaginary quadratic field, and let $\mathcal{O}$ be the order of conductor $\mathfrak{f}$ in K. Let $F \supset K$.*
a) Let $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve such that $E(F)$ contains a point of prime order $\ell > 2$. Then there is a prime $\mathfrak{p}$ of $\mathcal{O}_K$ lying over ℓ such that $K(\mathfrak{f})K^{\mathfrak{p}} \subset F$.
b) If $\left(\frac{\Delta}{\ell}\right) \neq -1$, then there is a prime $\mathfrak{p}$ is a prime of $\mathcal{O}_K$ lying over ℓ and an $\mathcal{O}$-CM elliptic curve $E_{/K(\mathfrak{f})K^{\mathfrak{p}}}$ such that $E(K(\mathfrak{f})K^{\mathfrak{p}})$ has a point of order ℓ.

If $\left(\frac{\Delta}{\ell}\right) = -1$, then an $\mathcal{O}$-CM elliptic curve $E_{/F}$ with an F-rational point of order ℓ must have full ℓ-torsion (see [BCS, Thm. 4.8] or Lemma 5.12). In this case, $K(\ell\mathfrak{f})K^{(\ell)} \subset F$ by Theorem 4.6. The existence of an elliptic curve $E_{/K(\ell\mathfrak{f})K^{(\ell)}}$ with full ℓ-torsion is guaranteed by Theorem 1.1c).

Proof. a) Let $F \supset K$ and $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve with an F-rational point of order ℓ. By Theorem 1.4, there is an $\mathcal{O}_K$-CM elliptic curve $E'_{/F}$ with an F-rational point P of order ℓ. If M is the $\mathcal{O}_K$-submodule of $E'(F)$ generated by P, then $M = E'[\operatorname{ann} M]$ and $\#M = |\operatorname{ann} M|$ by Theorem 2.7. Since $\ell \mid \#M$, we must have $\mathfrak{p} \mid \operatorname{ann} M$ for some prime $\mathfrak{p}$ of $\mathcal{O}_K$ above ℓ. By Theorem 2.9 we have

$$K(\mathfrak{f})K^{\mathfrak{p}} \subset K(\mathfrak{f})K^{\operatorname{ann} M} = K(j(E))K^{(1)}(\mathfrak{h}(E'[\operatorname{ann} M])) \subset F.$$

b) If $\left(\frac{\Delta}{\ell}\right) \neq -1$, then an $\mathcal{O}$-CM elliptic curve $E_{/K(\mathfrak{f})}$ possesses a $K(\mathfrak{f})$-rational cyclic subgroup of order ℓ. (See e.g. [CCRS13, p.13]. This is also a special case of Theorem 5.18.) By [BCS, Thm. 5.5], there is an extension $L/K(\mathfrak{f})$ of degree $(\ell - 1)/2$ and a quadratic twist $(E_1)_{/L}$ such that $E_1(L)$ has a point of order ℓ. By part a), there is a prime $\mathfrak{p}$ of $\mathcal{O}_K$ lying over ℓ such that $K(\mathfrak{f})K^{\mathfrak{p}} \subset L$, so it will suffice to show that $[K(\mathfrak{f})K^{\mathfrak{p}} : K(\mathfrak{f})] \geq \frac{\ell-1}{2}$.

If $\ell \nmid \mathfrak{f}$, then ℓ is unramified in $K(\mathfrak{f})$. Thus $K(\mathfrak{f})$, $K^{\mathfrak{p}}$ are linearly disjoint over $K^{(1)}$, and we have $[K(\mathfrak{f})K^{\mathfrak{p}} : K(\mathfrak{f})] = [K^{\mathfrak{p}} : K^{(1)}] = \frac{1}{2}(\ell - 1)$ since $\left(\frac{\Delta_K}{\ell}\right) = \left(\frac{\Delta}{\ell}\right) \neq -1$. If $\ell \mid \mathfrak{f}$, then by Lemma 5.4 we have

$$K^{\mathfrak{p}} \cap K(\mathfrak{f}) \subset K^{(\ell)} \cap K(\mathfrak{f}) = K(\ell).$$

Thus $K^{\mathfrak{p}} \cap K(\mathfrak{f}) = K^{\mathfrak{p}} \cap K(\ell)$, so

$$[K(\mathfrak{f})K^{\mathfrak{p}} : K(\mathfrak{f})] = [K^{\mathfrak{p}} : K^{\mathfrak{p}} \cap K(\mathfrak{f})] = [K^{\mathfrak{p}} : K^{\mathfrak{p}} \cap K(\ell)] = [K(\ell)K^{\mathfrak{p}} : K(\ell)]$$

and it is enough to show that $[K(\ell)K^{\mathfrak{p}} : K(\ell)] \geq \frac{\ell-1}{2}$.

- $\left(\frac{\Delta_K}{\ell}\right) = 1$: We will prove that $K^{\mathfrak{p}} \cap K(\ell) = K^{(1)}$ using CM elliptic curves. Let $(E_0)_{/K^{(1)}}$ be an $\mathcal{O}_K$-CM elliptic curve. Then $E_0[\mathfrak{p}]$ is stable under the action of $\mathfrak{g}_{K^{(1)}}$ and generated by a point P of order ℓ. By [BCS, Thm. 5.5], there is an extension $L/K^{(1)}$ of degree $(\ell-1)/2$ and a quadratic twist $(E_1)_{/L}$ such that P becomes L-rational. By Theorem 2.9 we have $K^{\mathfrak{p}} \subset L$, and $K^{\mathfrak{p}} = L$ since $[K^{\mathfrak{p}} : K^{(1)}] = \frac{1}{2}(\ell-1)$. Over $K(\ell)K^{\mathfrak{p}}$, the curve E_1 has a rational point of order ℓ, and the mod ℓ Galois representation is scalar by Theorem 4.1. Thus E_1 has full ℓ-torsion over $K(\ell)K^{\mathfrak{p}}$, and $K^{(\ell)} \subset K(\ell)K^{\mathfrak{p}}$. This implies $\frac{1}{2}(\ell-1) \mid [K(\ell)K^{\mathfrak{p}} : K(\ell)] = [K^{\mathfrak{p}} : K^{\mathfrak{p}} \cap K(\ell)]$. Since $[K^{\mathfrak{p}} : K^{(1)}] = \frac{1}{2}(\ell-1)$, we have $K^{\mathfrak{p}} \cap K(\ell) = K^{(1)}$, and $[K(\mathfrak{f})K^{\mathfrak{p}} : K(\mathfrak{f})] = [K^{\mathfrak{p}} : K^{(1)}] = \frac{1}{2}(\ell-1)$.
- $\left(\frac{\Delta_K}{\ell}\right) = -1$: In this case, $K^{\mathfrak{p}} = K^{(\ell)}$, so $K^{\mathfrak{p}} \cap K(\ell) = K(\ell)$. This implies $[K(\mathfrak{f})K^{\mathfrak{p}} : K(\mathfrak{f})] = [K^{\mathfrak{p}} : K(\ell)] = \frac{1}{2}(\ell-1)$.
- $\left(\frac{\Delta_K}{\ell}\right) = 0$: Since $[K(\ell) : K^{(1)}] = \ell$ and $[K^{\mathfrak{p}} : K^{(1)}] = \frac{1}{2}(\ell-1)$, we have $K^{\mathfrak{p}} \cap K(\ell) = K^{(1)}$. Thus $[K(\mathfrak{f})K^{\mathfrak{p}} : K(\mathfrak{f})] = [K^{\mathfrak{p}} : K^{(1)}] = \frac{1}{2}(\ell-1)$. □

Remark 5.6. *Assume the setup of Theorem 5.5 but take $K = \mathbb{Q}(\sqrt{-1})$ or $K = \mathbb{Q}(\sqrt{-3})$. Then the assertion of Theorem 5.5b) is false. Indeed, if $\ell \geq 5$ and $\left(\frac{\Delta}{\ell}\right) \neq -1$, we have $[K(\mathfrak{f})K^{\mathfrak{p}} : K(\mathfrak{f})] \mid \frac{1}{w_K}(\ell-1)$. (See Lemma 2.11.) Suppose $F \supset K$, and let $E_{/F}$ be an elliptic curve with CM by the order in K of conductor $\mathfrak{f}$. If $E(F)$ contains a rational point of order ℓ, then Theorem 5.2 implies $\frac{1}{2}(\ell-1) \mid [F : K(\mathfrak{f})]$. Thus F must properly contain $K(\mathfrak{f})K^{\mathfrak{p}}$.*

5.4. **Sharpness in the Isogeny Torsion Theorem.**

Lemma 5.7. *Let E be an $\mathcal{O}$-CM elliptic curve defined over a number field F containing the CM field K, and let $\iota : E \to E'$ be the canonical F-rational isogeny to an $\mathcal{O}_K$-CM elliptic curve $E'_{/F}$. Write*

$$E(F)[\mathrm{tors}] = \mathbb{Z}/s\mathbb{Z} \times \mathbb{Z}/e\mathbb{Z}, \ E'(F)[\mathrm{tors}] = \mathbb{Z}/s'\mathbb{Z} \times \mathbb{Z}/e'\mathbb{Z},$$

where $s \mid e$ and $s' \mid e'$. Then $s \mid s'$.

Proof. There is an $\mathcal{O}_K$-CM elliptic curve $E'_{/F}$ and a canonical F-rational isogeny $\iota : E \to E'$. Once again, there is a field embedding $F \hookrightarrow \mathbb{C}$ such that the base change of ι to $\mathbb{C}$ is, up to isomorphisms on the source and target, given by the canonical map $\mathbb{C}/\mathcal{O} \to \mathbb{C}/\mathcal{O}_K$. It follows that if

$$P = \frac{1}{s} + \mathcal{O} \in E[s], \ P' = \frac{1}{s} + \mathcal{O}_K \in E'[s],$$

then $\iota(P) = P'$ and $\langle P' \rangle_{\mathcal{O}_K} = E'[s]$. As $P \in E(F)$, we have $P' = \iota(P) \in E'(F)$. □

In [Ro94, §4], Ross claims that a CM elliptic curve E defined over a number field F containing the CM field, then the exponent of the finite group $E(F)[\mathrm{tors}]$ is an invariant of the F-rational isogeny class. In the setting of Lemma 5.7, this would give $e = e'$, and combining this with the conclusion of Lemma 5.7 we would get an injective group homomorphism $E(F)[\mathrm{tors}] \hookrightarrow E(F')[\mathrm{tors}]$. This conclusion is stronger than that of Theorem 1.4.

Unfortunately Ross's claim is false: in the setup of Lemma 5.7 one can have $e' < e$ (in which case there is no injective group homomorphism $E(F)[\mathrm{tors}] \hookrightarrow E'(F)[\mathrm{tors}]$), as the following result shows.

Proposition 5.8. *Let $\ell > 3$ be a prime number, let $K = \mathbb{Q}(\sqrt{-\ell})$, let $n \in \mathbb{Z}^{\geq 3}$, let $\mathcal{O}$ be the order in K of conductor $\mathfrak{f} = \ell^{\lfloor \frac{n}{2} \rfloor}$, and let $F = K(\mathfrak{f})$. For any $\mathcal{O}$-CM elliptic curve $E_{/F}$, there is an extension L/F of degree $\varphi(\ell^n)$ such that $E(L)$ has a point of order ℓ^n, and no $\mathcal{O}_K$-CM elliptic curve has an L-rational point of order ℓ^k for $k > \frac{1}{2}\left(n + 1 + \lfloor \frac{n}{2} \rfloor\right)$ (hence no L-rational point of order ℓ^n).*

Proof. Let $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve. As in (3) we may choose a basis $\{e_1, e_2\}$ for $E[\ell^n]$ so that the image of the mod ℓ^n Galois representation consists of matrices

$$\left[\begin{array}{cc} a & b\mathfrak{f}^2 \frac{\Delta_K - \Delta_K^2}{4} \\ b & a + b\mathfrak{f}\Delta_K \end{array} \right] \mid a, b \in \mathbb{Z}/\ell^n\mathbb{Z}.$$

Since ℓ ramifies in K and $\mathfrak{f} = \ell^{\lfloor \frac{n}{2} \rfloor}$, we have $\mathrm{ord}_\ell(b\mathfrak{f}^2 \frac{\Delta_K - \Delta_K^2}{4}) = 1 + 2\lfloor \frac{n}{2} \rfloor \geq n$, so the matrices have the form

$$\left[\begin{array}{cc} a & 0 \\ b & a + b\mathfrak{f}\Delta_K \end{array} \right] \mid a, b \in \mathbb{Z}/\ell^n\mathbb{Z}.$$

The action of $\mathfrak{g}_F$ on $\langle e_2 \rangle$ gives a character $\Phi : \mathfrak{g}_F \to (\mathbb{Z}/\ell^n\mathbb{Z})^\times$. Take $M = (\overline{F})^{\ker \Phi}$. Then $[M : F] \mid \varphi(\ell^n)$ and $\Phi|_{\mathfrak{g}_M}$ is trivial. Thus there exists an extension L/F with $[L : F] = \varphi(\ell^n)$ such that $E(L)$ contains e_2.

Let $E'_{/L}$ be an $\mathcal{O}_K$-CM elliptic curve, and suppose $E'(L)$ contains a point P of order ℓ^k. Let $\mathfrak{p}$ be the prime ideal of $\mathcal{O}_K$ such that $\ell\mathcal{O}_K = \mathfrak{p}^2$. We claim that the $\mathcal{O}_K$-submodule $M = \langle P \rangle_{\mathcal{O}_K}$ of $E'(L)$ generated by P contains $E[\mathfrak{p}^{2k-1}]$ and thus, by Theorem 2.9, that $K^{\mathfrak{p}^{2k-1}} \subset L$. Indeed, by Theorem 2.7, we have $M = E[I]$ for some ideal I of $\mathcal{O}_K$ such that $(\mathcal{O}_K/I, +)$ has ℓ-power order and exponent ℓ^k. Since ℓ ramifies in $\mathcal{O}_K$, this forces I to be of the form $\mathfrak{p}^a$ for some $a \in \mathbb{Z}^+$, and the smallest a such that $(\mathcal{O}_K/\mathfrak{p}^a, +)$ has exponent ℓ^k is $a = 2k - 1$, establishing the claim. Thus

$$\mathrm{ord}_\ell([K^{\mathfrak{p}^{2k-1}} : K^{(1)}]) = 2k - 2 \leq \mathrm{ord}_\ell([L : K^{(1)}]) = \left\lfloor \frac{n}{2} \right\rfloor + n - 1,$$

so $k \leq \frac{1}{2}(n + 1 + \lfloor \frac{n}{2} \rfloor)$. □

In the setting of Theorem 1.4, one wonders whether $\#E(F)[\mathrm{tors}] = \#E'(F)[\mathrm{tors}]$. In fact $\frac{\#E'(F)[\mathrm{tors}]}{\#E(F)[\mathrm{tors}]}$ can be arbitrarily large:

Proposition 5.9. *Let ℓ be an odd prime, let $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ be an imaginary quadratic field, let $\mathcal{O}$ be the order in K of conductor ℓ, and let $F = K(\ell)$. For any $\mathcal{O}$-CM elliptic curve $E_{/F}$ there is an extension L/F such that if $\iota : E \to E'$ is the canonical isogeny to an $\mathcal{O}_K$-CM elliptic curve E, then*

$$\ell \mid \frac{\#E'(L)[\mathrm{tors}]}{\#E(L)[\mathrm{tors}]}.$$

Proof. Let $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve. As above, there is a basis $\{e_1, e_2\}$ for $E[\ell]$ such that

$$\rho_\ell(\mathfrak{g}_F) \subset \left\{ \left[\begin{array}{cc} a & 0 \\ b & a \end{array} \right] \mid a, b \in \mathbb{Z}/\ell\mathbb{Z} \right\}$$

and there is an extension L/F with $[L : F] = \ell - 1$ such that $E(L)$ contains e_2. In fact, $E(L)[\ell^\infty] \cong \mathbb{Z}/\ell\mathbb{Z}$. Indeed, E does not have full ℓ-torsion over L since Theorem 4.6 would imply $K^{(\ell)}K(\ell^2) \subset L$ and $\frac{1}{2}\ell(\ell - 1) = [K^{(\ell)}K(\ell^2) : K(\ell)]$. In addition, E has no point of order ℓ^2 by Theorem 5.2.

Let $\iota : E \to E'$ be the canonical L-rational isogeny from $E_{/L}$ to $E'_{/L}$, where E' has $\mathcal{O}_K$-CM. Since $e_2 \in E(L)$, the proof of Theorem 1.4 shows $\iota(e_1) \in E'(L)$, and $\iota(e_1)$ generates $E'[\ell]$ as an $\mathcal{O}_K$-module. In other words, $\mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z} \hookrightarrow E'(L)[\mathrm{tors}]$. It follows that $\ell \mid \frac{\#E'(L)[\mathrm{tors}]}{\#E(L)[\mathrm{tors}]}$. □

Finally, Theorem 1.4 requires $K \subset F$. This hypothesis cannot be omitted:

Proposition 5.10. *Let $\ell > 3$ be a prime with $\ell \equiv 3 \pmod 4$ and let $n \in \mathbb{Z}^{\geq 3}$. Let $K = \mathbb{Q}(\sqrt{-\ell})$, and let $\mathcal{O}$ be the order in K of conductor $\mathfrak{f} = \ell^{\lfloor \frac{n}{2} \rfloor}$. Let $F = \mathbb{Q}(j(\mathbb{C}/\mathcal{O}))$. There is an elliptic curve $E_{/F}$ and an extension L/F of degree $\frac{\varphi(\ell^n)}{2}$ such that:*
(i) $L \not\supset K$,
(ii) $E(L)$ has a point of order ℓ^n, and
(iii) for every $\mathcal{O}_K$-CM elliptic curve $E'_{/L}$ we have $\ell^n \nmid \#E'(L)[\mathrm{tors}]$.

Proof. Let $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve. By [Kw99, Corollary 4.2], E has an F-rational subgroup which is cyclic of order ℓ^n. It follows from [BCS, Theorem 5.6] that there is a twist E_1 of $E_{/F}$ and an extension L/F of degree $\varphi(\ell^n)/2$ such that $E_1(L)$ has a point of order ℓ^n. Note $[L : \mathbb{Q}] = h_K \ell^{\lfloor \frac{n}{2} \rfloor} \frac{\varphi(\ell^n)}{2}$ is odd (see [Co89, Proposition 3.11]) , so $K \not\subset L$.

Let $E'_{/L}$ be an $\mathcal{O}_K$-CM elliptic curve. Since $[L : \mathbb{Q}]$ is odd, $E'(L)[\ell^\infty]$ must be cyclic, as full ℓ^k-torsion would imply $\mathbb{Q}(\zeta_{\ell^k}) \subset L$ by the Weil pairing. As in the proof of Proposition 5.8, $E'(LK)$ contains no point of order ℓ^n. Hence $E'(L)$ contains no point of order ℓ^n, and $\ell^n \nmid \#E'(L)[\mathrm{tors}]$. □

5.5. **Minimal and Maximal Cartan Orbits.** Let $\mathcal{O}$ be an order, let $N \in \mathbb{Z}^+$, and let $P \in \mathcal{O}/N\mathcal{O}$ be a point of order N. Since $C_N(\mathcal{O})$ contains all scalar matrices, if $P \in \mathcal{O}/N\mathcal{O}$ has order N, then the orbit of $C_N(\mathcal{O})$ on P has size at least $\varphi(N)$. On the other hand, the orbit of $C_N(\mathcal{O})$ on P is certainly no larger than the number of order N points of $\mathcal{O}/N\mathcal{O}$.

In this section we will find all pairs $(\mathcal{O}, N)$ for which there exists a Cartan orbit of this smallest possible size and also all pairs for which there exists a Cartan orbit of this largest possible size.

We introduce the shorthand $H(\mathcal{O}, N)$ to mean: *there is a point P of order N in $\mathcal{O}/N\mathcal{O}$ such that the $C_N(\mathcal{O})$-orbit of P has size $\varphi(N)$.*

Lemma 5.11. *Let $\mathcal{O}$ be an order, and let $N = \ell_1^{a_1} \cdots \ell_r^{a_r} \in \mathbb{Z}^+$. Then $H(\mathcal{O}, N)$ holds iff $H(\mathcal{O}, \ell_i^{a_i})$ holds for all $1 \leq i \leq r$.*

Proof. This is an easy consequence of the Chinese Remainder Theorem. □

Lemma 5.12. *Let $\mathcal{O}$ be the order of discriminant Δ, ℓ a prime number and $a \in \mathbb{Z}^+$.*
a) If $\left(\frac{\Delta}{\ell}\right) = 1$, there is an $\mathcal{O}$-submodule of $\mathcal{O}/\ell^a\mathcal{O}$ with underyling $\mathbb{Z}$-module $\mathbb{Z}/\ell^a\mathbb{Z}$.
b) If $\left(\frac{\Delta}{\ell}\right) = -1$, then $C_{\ell^a}(\mathcal{O})$ acts simply transitively on the order ℓ^a elements of $\mathcal{O}/\ell^a\mathcal{O}$.

Proof. a) if $\left(\frac{\Delta}{\ell}\right) = 1$, then $\mathcal{O}/\ell\mathcal{O} = \mathcal{O}_K/\ell\mathcal{O}_K \cong \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$, so $\mathcal{O} \otimes \mathbb{Z}_\ell$ is isomorphic as a ring to $\mathbb{Z}_\ell \times \mathbb{Z}_\ell$ (see e.g. [Ei, Cor. 7.5]) and thus $\mathcal{O}/\ell^a\mathcal{O}$ is isomorphic as a ring to $\mathbb{Z}/\ell^a\mathbb{Z} \times \mathbb{Z}/\ell^a\mathbb{Z}$.
b) If $\left(\frac{\Delta}{\ell}\right) = -1$, then $\mathcal{O} \otimes \mathbb{Z}_\ell = \mathcal{O}_K \otimes \mathbb{Z}_\ell$ is a complete DVR with uniformizer ℓ, so the ring $\mathcal{O}/\ell^a\mathcal{O}$ is finite, local and principal with maximal ideal $\langle \ell \rangle$. An element of $\mathcal{O}/\ell^a\mathcal{O}$ has order ℓ^a iff it lies in the unit group $C_{\ell^a}(\mathcal{O})$. □

Lemma 5.13. *Let $\mathcal{O}$ be the order of discriminant Δ, and let $N \in \mathbb{Z}^+$. The following are equivalent:*
(i) If $2 \mid N$, then $\left(\frac{\Delta}{2}\right) \neq 1$.
(ii) The $\mathbb{Z}/N\mathbb{Z}$-subalgebra of $\mathcal{O}/N\mathcal{O}$ generated by $C_N(\mathcal{O})$ is $\mathcal{O}/N\mathcal{O}$.

Proof. Using the Chinese Remainder Theorem we reduce to the case of $N = \ell^a$ a power of a prime number ℓ. Let B be the $\mathbb{Z}/\ell^a\mathbb{Z}$-subalgebra generated by $C_{\ell^a}(\mathcal{O})$, so $\#B = \ell^b$ for some $b \leq 2a$.
(i) $\implies$ (ii): Since $0 \in B \setminus C_{\ell^a}(\mathcal{O})$, we have

$$\#B \geq \#C_{\ell^a}(\mathcal{O}) + 1$$

$$= \ell^{2a}\left(1 - \frac{1}{\ell}\right)\left(1 - \left(\frac{\Delta}{\ell}\right)\frac{1}{\ell}\right) + 1 \geq \begin{cases} \frac{4}{9}\ell^{2a} + 1 > \ell^{2a-1}, & \text{if } \ell \geq 3 \\ \frac{1}{2}\ell^{2a} + 1 > \ell^{2a-1}, & \text{if } \ell = 2 \text{ and } \left(\frac{\Delta}{2}\right) \neq 1 \end{cases}.$$

Thus $b = 2a$ and $B = \mathcal{O}/\ell^a\mathcal{O}$.
$\neg$ (i) $\implies$ $\neg$ (ii): If $\ell = 2$ and $\left(\frac{\Delta}{2}\right) = 1$, then

$$\mathcal{O}/2^a\mathcal{O} \cong \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \mid \alpha, \beta \in \mathbb{Z}/2^a\mathbb{Z} \right\}$$

and $C_{2^a}(\mathcal{O})$ consists of the set of such matrices with $\alpha, \beta \in (\mathbb{Z}/2^a\mathbb{Z})^\times$. Thus $C_{2^a}(\mathcal{O})$ is contained in the subalgebra

$$\mathcal{B} = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \mid \alpha, \beta \in \mathbb{Z}/2^a\mathbb{Z} \text{ and } \alpha \equiv \beta \pmod 2 \right\}$$

of order 2^{2a-1}, so $B \subset \mathcal{B} \subsetneq \mathcal{O}/2^a\mathcal{O}$.[2] □

Lemma 5.14. *For an order $\mathcal{O}$ and $N \in \mathbb{Z}^+$, the following are equivalent:*
(i) There is an ideal I of $\mathcal{O}$ with $\mathcal{O}/I \cong \mathbb{Z}/N\mathbb{Z}$.
(ii) There is an $\mathcal{O}$-submodule of $\mathcal{O}/N\mathcal{O}$ with underlying commutative group $\mathbb{Z}/N\mathbb{Z}$.
(iii) $H(\mathcal{O}, N)$ holds.

[2]Since $\#B \geq \#C_{2^a}(\mathcal{O}) + 1 = 2^{2a-2} + 1 > 2^{2a-2}$, in fact we have $B = \mathcal{B}$.

Proof. (i) $\iff$ (ii):
Step 1: Let Λ be a free, rank 2 $\mathbb{Z}$-module, and let Λ' be a $\mathbb{Z}$-submodule of Λ containing $N\Lambda$. By the structure theory of modules over a PID, there is a $\mathbb{Z}$-basis e_1, e_2 for Λ and positive integers $a \mid b$ such that ae_1, be_2 is a $\mathbb{Z}$-basis for Λ'. Thus

$$\Lambda/\Lambda' \cong \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}, \ \Lambda'/N\Lambda \cong \mathbb{Z}/(N/b)\mathbb{Z} \oplus \mathbb{Z}/(N/a)\mathbb{Z}.$$

It follows that $\Lambda/\Lambda' \cong \mathbb{Z}/N\mathbb{Z} \iff \Lambda'/N\Lambda \cong \mathbb{Z}/N\mathbb{Z}$.
Step 2: If I is an ideal of $\mathcal{O}$ with $\mathcal{O}/I \cong \mathbb{Z}/N\mathbb{Z}$, then $I \supset N\mathcal{O}$, so $I/N\mathcal{O} \cong \mathbb{Z}/N\mathbb{Z}$ by Step 1. Let M be an $\mathcal{O}$-submodule of $\mathcal{O}/N\mathcal{O}$ with underyling $\mathbb{Z}$-module $\mathbb{Z}/N\mathbb{Z}$. Then $M = I/N\mathcal{O}$ for an ideal I of $\mathcal{O}$, and by Step 1 we have $\mathcal{O}/I \cong \mathbb{Z}/N\mathbb{Z}$.
(ii) $\implies$ (iii): Let $P \in \mathcal{O}/N\mathcal{O}$ have order N such that the subgroup generated by P is an $\mathcal{O}$-submodule. For all $g \in C_N(\mathcal{O})$, $gP = a_g P$ for $a_g \in (\mathbb{Z}/N\mathbb{Z})^\times$. Conversely, since $C_N(\mathcal{O})$ contains all scalar matrices, the orbit of $C_N(\mathcal{O})$ on P has size $\varphi(N)$.
(iii) $\implies$ (ii): Case 1: Suppose $2 \nmid N$ or $\left(\frac{\Delta}{2}\right) \neq 1$. Let $P \in \mathcal{O}/N\mathcal{O}$ be a point of order N with $C_N(\mathcal{O})$-orbit of size $\varphi(N)$. There is a $\mathbb{Z}/N\mathbb{Z}$-basis e_1, e_2 of $\mathcal{O}/N\mathcal{O}$ with $e_1 = P$, and our hypothesis gives that with respect to this basis $C_N(\mathcal{O})$ lies in the subalgebra $\left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, d \in \mathbb{Z}/N\mathbb{Z} \right\}$ of upper triangular matrices. By Lemma 5.13, $\mathcal{O}/N\mathcal{O}$ also lies in the subalgebra of upper triangular matrices, and thus $\langle P \rangle$ is an $\mathcal{O}$-stable submodule with underlying $\mathbb{Z}$-module $\mathbb{Z}/N\mathbb{Z}$.

Case 2: Suppose $2 \mid N$ and $\left(\frac{\Delta}{2}\right) = 1$, and write $N = 2^a N'$ with $2 \nmid N'$. By Lemma 5.12 and the equivalence of (i) and (ii), there is an ideal I_1 in $\mathcal{O}$ with $\mathcal{O}/I_1 \cong \mathbb{Z}/2^a\mathbb{Z}$, and by Case 1 there is an ideal I_2 in $\mathcal{O}$ with $\mathcal{O}/I_2 \cong \mathbb{Z}/N'\mathbb{Z}$. By the Chinese Remainder Theorem, $\mathcal{O}/I_1I_2 \cong \mathbb{Z}/N\mathbb{Z}$. Since (i) $\iff$ (ii), this suffices. □

Theorem 5.15. *Let $\mathcal{O}$ be an order of discriminant Δ, and let $N \in \mathbb{Z}^+$. The following are equivalent:*
(i) $H(\mathcal{O}, N)$ holds.
(ii) Δ is a square in $\mathbb{Z}/4N\mathbb{Z}$.

Proof. Using the Chinese Remainder Theorem and Lemma 5.11, we reduce to the case in which $N = \ell^a$ is a power of a prime number ℓ.
Case 1 (ℓ is odd): Since $\gcd(4, \ell^a) = 1$, we may put $D = \frac{\Delta}{4} \in \mathbb{Z}/\ell^a\mathbb{Z}$. Then Δ is a square in $\mathbb{Z}/4\ell^a\mathbb{Z}$ iff D is a square in $\mathbb{Z}/\ell^a\mathbb{Z}$, and

$$\mathcal{O}/\ell^a\mathcal{O} \cong \mathbb{Z}/\ell^a\mathbb{Z}[t]/(t^2 - D). \tag{11}$$

If there is $s \in \mathbb{Z}/\ell^a\mathbb{Z}$ such that $D = s^2$, then

$$\mathcal{O}/\ell^a\mathcal{O} \cong \mathbb{Z}/\ell^a\mathbb{Z}[t]/((t+s)(t-s)),$$

so if I is the ideal $\langle t + s, \ell^a \rangle$ of $\mathcal{O}$, then $\mathcal{O}/I \cong \mathbb{Z}/\ell^a\mathbb{Z}$. By Lemma 5.14, $H(\mathcal{O}, \ell^a)$ holds. Conversely, suppose $H(\mathcal{O}, \ell^a)$ holds, so by Lemma 5.14 there is an ideal I of $\mathcal{O}$ with $\mathcal{O}/I \cong \mathbb{Z}/\ell^a\mathbb{Z}$. Since $\ell^a \in I$, we may regard I as an ideal of $\mathcal{O}/\ell^a\mathcal{O}$ such that $(\mathcal{O}/\ell^a\mathcal{O})/I \cong \mathbb{Z}/\ell^a\mathbb{Z}$. In other words, we have a $\mathbb{Z}/\ell^a\mathbb{Z}$-algebra homomorphism

$$f : \mathbb{Z}/\ell^a\mathbb{Z}[t]/(t^2 - D) \to \mathbb{Z}/\ell^a\mathbb{Z}.$$

Then $f(t)^2 = D \in \mathbb{Z}/\ell^a\mathbb{Z}$, so D is a square in $\mathbb{Z}/\ell^a\mathbb{Z}$.
Case 2 ($\ell = 2$, Δ is odd): Then $\left(\frac{\Delta}{\ell}\right) = \pm 1$.
• If $\left(\frac{\Delta}{\ell}\right) = 1$, then $\Delta \equiv 1 \pmod 8$; by Hensel's Lemma, Δ is a square in $\mathbb{Z}/\ell^a\mathbb{Z}$. On the other hand, by Lemmas 5.12a) and 5.14, $H(\mathcal{O}, \ell^a)$ holds.
• If $\left(\frac{\Delta}{\ell}\right) = -1$, then $\Delta \equiv 5 \pmod 8$, so Δ is not a square modulo 8 and thus not a square modulo $4 \cdot 2^a$. On the other hand, by Lemma 5.12b) $H(\mathcal{O}, \ell^a)$ does not hold.
Case 3: ($\ell = 2$, Δ is even): Again we may put $D = \frac{\Delta}{4} \in \mathbb{Z}/\ell^a\mathbb{Z}$, and again (11) holds. The argument of Case 1 shows that $H(\mathcal{O}, \ell^a)$ holds iff D is a square modulo $\mathbb{Z}/\ell^a\mathbb{Z}$ iff Δ is a square modulo $\mathbb{Z}/4\ell^a\mathbb{Z}$. □

Proposition 5.16. *Let $\mathcal{O}$ be an order, and let $N \in \mathbb{Z}^+$. The following are equivalent:*
(i) $C_N(\mathcal{O})$ acts simply transitively on order N elements of $\mathcal{O}/N\mathcal{O}$.

(ii) $C_N(\mathcal{O})$ *acts transitively on order* N *elements of* $\mathcal{O}/N\mathcal{O}$.
(iii) For all primes $\ell \mid N$ *we have* $\left(\frac{\Delta}{\ell}\right) = -1$.

Proof. As usual, we may assume $N = \ell^a$ is a prime power. Certainly (i) $\implies$ (ii).
(ii) $\implies$ (iii): We have

$$\#C_{\ell^a}(\mathcal{O}) = \ell^{2a-2}(\ell-1)\left(\ell - (\frac{\Delta}{\ell})\right),$$

whereas the number of elements of order ℓ^a in $\mathcal{O}/\ell^a\mathcal{O}$ is

$$N(\mathcal{O}, \ell^a) := \#\mathcal{O}/\ell^a\mathcal{O} - \#\ell\mathcal{O}/\ell^a\mathcal{O} = \ell^{2a-2}(\ell-1)(\ell+1).$$

Transitivity of the action implies $\#C_{\ell^a}(\mathcal{O}) \geq N(\mathcal{O}, \ell^a)$, which holds iff $\left(\frac{\Delta}{\ell}\right) = -1$.
(iii) $\implies$ (i): Since $\left(\frac{\Delta}{\ell}\right) \neq 0$, we have $\mathcal{O}/\ell^a\mathcal{O} \cong \mathcal{O}_K/\ell^a\mathcal{O}_K$, and thus also $C_{\ell^a}(\mathcal{O}) = (\mathcal{O}/\ell^a\mathcal{O})^\times \cong C_{\ell^a}(\mathcal{O}_K)$. Thus $\mathcal{O}/\ell^a\mathcal{O}$ is a finite local principal ring with maximal ideal $\mathfrak{m} = \langle \ell \rangle$ and unit group $C_{\ell^a}(\mathcal{O}) = \mathcal{O}/\ell^a\mathcal{O} \setminus \mathfrak{m}$. The set of order ℓ^a elements of $\mathcal{O}/\ell^a\mathcal{O}$ is $\mathcal{O}/\ell^a\mathcal{O} \setminus \mathfrak{m} = C_{\ell^a}(\mathcal{O})$, so the action of the unit group $C_{\ell^a}(\mathcal{O})$ on this set is the action of $C_{\ell^a}(\mathcal{O})$ on itself, which is simply transitive. □

Corollary 5.17. *Let* $\mathcal{O}$ *an order of conductor* $\mathfrak{f}$. *Let* $N = \prod_{i=1}^r \ell_i^{a_i} \in \mathbb{Z}^+$ *be such that* $\left(\frac{\Delta}{\ell_i}\right) = -1$ *for all* i. *Let* F *be a number field, and let* $E_{/F}$ *be an* $\mathcal{O}$*-CM elliptic curve such that* $E(F)$ *has a point of order* N. *Then*

$$\#\overline{C_N(\mathcal{O})} \mid [FK : K(\mathfrak{f})]. \tag{12}$$

Moreover, for all $\mathcal{O}$ *and* N *satisfying the above conditions, equality can occur in (12).*

Proof. Replace F by FK; then $F \supset K(\mathfrak{f})$. By Proposition 5.16, $C_N(\mathcal{O})$ acts transitively on order N elements of $\mathcal{O}/N\mathcal{O}$, so the $\mathcal{O}$-submodule generated by any one of them is $\mathcal{O}/N\mathcal{O}$. Thus the existence of one F-rational point of order N implies that ρ_N is trivial, and thus also $\overline{\rho_{E,N}}$ is trivial. Applying Theorem 1.1 gives (12). That equality can occur follows from Theorem 1.1c). □

5.6. **Torsion over** $K(j)$**: Part I.** Let $\mathcal{O}$ be an order of discriminant $\Delta = \mathfrak{f}^2\Delta_K$. We will give a complete classification of the possible torsion subgroups of $\mathcal{O}$-CM elliptic curves $E_{/K(\mathfrak{f})}$. In this section we will treat the cases $\Delta \neq -3, -4$. For the remaining cases we will make use of Theorem 6.2, so we will come back to those cases in §6.5.

If $E(K(\mathfrak{f}))$ has a point of order N, then since $[C_N(\mathcal{O}) : \rho_N(\mathfrak{g}_{K(\mathfrak{f})})] \mid \#\mathcal{O}^\times$, there must be some $P \in \mathcal{O}/N\mathcal{O}$ of order N with a $C_N(\mathcal{O})$-orbit of order dividing $\#\mathcal{O}^\times$.

• By Theorem 5.2, if $E(K(\mathfrak{f}))$ has a point of order N, then $\varphi(N) \mid 2$, so

$$N \in \{1, 2, 3, 4, 6\}.$$

• Lemma 2.2b) implies that for all $N \geq 3$, we have $\#C_N(\mathcal{O}) \geq 4$ (equality holds if $N = 3$ and $\Delta \equiv 1 \pmod 3$). By Theorem 1.1 we cannot have $E[N] = E[N](K(\mathfrak{f}))$.

Thus $E(K(\mathfrak{f}))[\mathrm{tors}]$ is isomorphic to one of the groups in the following list:

$$\{e\}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}.$$

We will show that all of these groups occur.

Points of order 2: By Theorem 4.6b), $E(K(\mathfrak{f}))[2]$ has order 4 if 2 splits in $\mathcal{O}$, order 2 if 2 ramifies in $\mathcal{O}$ and order 1 if 2 is inert in $\mathcal{O}$. Thus:

$$E(K(\mathfrak{f}))[2] \cong \begin{cases} \{e\} & \Delta \equiv 5 \pmod 8 \\ \mathbb{Z}/2\mathbb{Z} & \Delta \equiv 0 \pmod 4 \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \Delta \equiv 1 \pmod 8 \end{cases}.$$

Points of order 3, 4, or 6: Let $E_{/K(\mathfrak{f})}$ be any $\mathcal{O}$-CM elliptic curve. We claim that for $N \in \{3,4,6\}$, there is a quadratic twist E^D of E such that $E^D(K(\mathfrak{f}))$ has a point of order N iff $H(\mathcal{O},N)$ holds. Indeed, as above, since the index of the mod N Galois representation in $C_N(\mathcal{O})$ divides 2, if some $E^D(K(\mathfrak{f}))$ has a point of order N, then $\mathcal{O}/N\mathcal{O}$ has a point of order N with a $C_N(\mathcal{O})$-orbit of size 2. Since $\varphi(N) = 2$, there is a Cartan orbit of size 2 iff $H(\mathcal{O},N)$ holds. Conversely, if $H(\mathcal{O},N)$ holds then there is a point of order N with a $C_N(\mathcal{O})$-orbit of size 2, hence on some quadratic twist E^D we have an F-rational point of order N. Applying Theorem 5.15, we get:

- Some $\mathcal{O}$-CM $E_{/K(\mathfrak{f})}$ has a point of order 3 iff $\Delta \equiv 0,1 \pmod 3$.
- Some $\mathcal{O}$-CM $E_{/K(\mathfrak{f})}$ has a point of order 4 iff $\Delta \equiv 0,1,4,9 \pmod{16}$.
- Some $\mathcal{O}$-CM $E_{/K(\mathfrak{f})}$ has a point of order 6 iff $\Delta \equiv 0,1,2,9,12,16 \pmod{24}$.

Because the only full N-torsion we can have is full 2-torsion, and 2-torsion is invariant under quadratic twists, we immediately deduce the complete answer in all cases.

- If $\Delta \equiv 0 \pmod{48}$, then there are twists E_1, E_2, E_3 of E with
$$E_1(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/2\mathbb{Z},\ E_2(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/4\mathbb{Z},\ E_3(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/6\mathbb{Z}.$$
- If $\Delta \equiv 1,9,25,33 \pmod{48}$ then there are twists E_1, E_2 of E with
$$E_1(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},\ E_2(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}.$$
- If $\Delta \equiv 4,16,36 \pmod{48}$, then there are twists E_1, E_2, E_3 of E with
$$E_1(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/2\mathbb{Z},\ E_2(K(\mathfrak{f})) \cong \mathbb{Z}/4\mathbb{Z},\ E_3(K(\mathfrak{f})) \cong \mathbb{Z}/6\mathbb{Z}.$$
- If $\Delta \equiv 5,29 \pmod{48}$, then $E(K(\mathfrak{f}))[\mathrm{tors}] = \{e\}$.
- If $\Delta \equiv 8,44 \pmod{48}$, then $E(K(\mathfrak{f}))[\mathrm{tors}] = \mathbb{Z}/2\mathbb{Z}$.
- If $\Delta \equiv 12,24,28,40 \pmod{48}$, then there are twists E_1, E_2 of E with
$$E_1(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/2\mathbb{Z},\ E_2(K(\mathfrak{f})) \cong \mathbb{Z}/6\mathbb{Z}.$$
- If $\Delta \equiv 13,21,37,45 \pmod{48}$, then there are twists E_1, E_2 of E with
$$E_1(K(\mathfrak{f}))[\mathrm{tors}] = \{e\},\ E_2(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/3\mathbb{Z}.$$
- If $\Delta \equiv 17,41 \pmod{48}$, then there are twists E_1, E_2 of E with
$$E_1(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},\ E_2(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}.$$
- If $\Delta \equiv 20,32 \pmod{48}$, then there are twists E_1, E_2 of E with
$$E_1(K(\mathfrak{f}))[\mathrm{tors}] \cong \mathbb{Z}/2\mathbb{Z},\ E_2(K(\mathfrak{f})) \cong \mathbb{Z}/4\mathbb{Z}.$$

5.7. **Isogenies over $K(j)$: Part I.**

Theorem 5.18. *Let $\mathcal{O}$ be an order of discriminant $\Delta = \mathfrak{f}^2\Delta_K$, and let $N \in \mathbb{Z}^+$.*
a) If $\Delta \neq -3,-4$, then there is an $\mathcal{O}$-CM elliptic curve $E_{/K(\mathfrak{f})}$ with a $K(\mathfrak{f})$-rational cyclic N-isogeny iff Δ is a square in $\mathbb{Z}/4N\mathbb{Z}$.
b) If $\Delta = -4$, then then there is an $\mathcal{O}$-CM elliptic curve $E_{/K(\mathfrak{f})}$ with a $K(\mathfrak{f})$-rational cyclic N-isogeny iff N is of the form $2^\epsilon \ell_1^{a_1} \cdots \ell_r^{a_r}$ for primes $\ell_i \equiv 1 \pmod 4$ and $\epsilon, a_1, \ldots, a_r \in \mathbb{N}$ with $\epsilon \leq 2$.
c) If $\Delta = -3$, then there is an $\mathcal{O}$-CM elliptic curve $E_{/K(\mathfrak{f})}$ with a $K(\mathfrak{f})$-rational cyclic N-isogeny iff N is of the form $2^\epsilon 3^a \ell_1^{a_1} \cdots \ell_r^{a_r}$ for primes $\ell_i \equiv 1 \pmod 3$, $\epsilon, a, a_1, \ldots, a_r \in \mathbb{N}$ with $(\epsilon, a) \in \{(0,0),\ (0,1),\ (0,2),\ (1,0),\ (1,1)\}$.

Proof. Step 1: Let $E_{/K(\mathfrak{f})}$ be an $\mathcal{O}$-CM elliptic curve. If Δ is a square in $\mathbb{Z}/4N\mathbb{Z}$, then by Theorem 5.15 there is a point P of order N in $\mathcal{O}/N\mathcal{O}$ such that $C = \langle P \rangle$ is invariant under $C_N(\mathcal{O})$, so C is $\mathfrak{g}_{K(\mathfrak{f})}$-stable and $E \to E/C$ is a cyclic N-isogeny. If $\Delta \notin \{-4,-3\}$, then the projective Galois representation $\mathbb{P}\rho_N : \mathfrak{g}_{K(\mathfrak{f})} \to C_N(\mathcal{O})/(\mathbb{Z}/N\mathbb{Z})^\times$ is a quotient of the reduced Galois representation, hence surjective. So $K(\mathfrak{f})$-rational cyclic N-isogenies correspond to $C_N(\mathcal{O})$-orbits on $\mathcal{O}/N\mathcal{O}$ of size $\varphi(N)$, which by Theorem 5.15 exist iff Δ is a square in $\mathbb{Z}/4N\mathbb{Z}$.

Step 2: If $\Delta \in \{-4, -3\}$, then as above the condition that Δ is a square modulo $4N$ is sufficient for the existence of a $K(\mathfrak{f})$-rational cyclic N-isogeny, but it is no longer clear that it is necessary, and in both cases it turns out not to be. The complete analysis will make use of Theorem 6.2, so we defer the end of the proof until §6.6. □

6. The Torsion Degree Theorem

6.1. **Statement and Preliminary Reduction.** Throughout this section $\mathcal{O}$ denotes an order of conductor $\mathfrak{f}$ and discriminant $\Delta = \mathfrak{f}^2\Delta_K$.

For $N \in \mathbb{Z}^{\geq 2}$, let $\widetilde{T}(\mathcal{O}, N)$ be the least size of an orbit of $C_N(\mathcal{O})$ on an order N point of $\mathcal{O}/N\mathcal{O}$.

Lemma 6.1. *We have* $\widetilde{T}(\mathcal{O}, 2) = \begin{cases} 1 & \text{if } \left(\frac{\Delta}{2}\right) \neq -1 \\ 3 & \text{if } \left(\frac{\Delta}{2}\right) = -1 \end{cases}$.

Proof. By Theorem 5.15, we have $\left(\frac{\Delta}{2}\right) \neq -1$ iff there is a $C_2(\mathcal{O})$-orbit of size $\varphi(2) = 1$ on $\mathcal{O}/2\mathcal{O}$ iff $\widetilde{T}(\mathcal{O}, 2) = 1$. In the remaining case $\left(\frac{\Delta}{2}\right) = -1$ we have $\#C_2(\mathcal{O}) = 3$ and no orbit of size 1, hence $\widetilde{T}(\mathcal{O}, 2) = 3$. □

Theorem 6.2. *(Torsion Degree Theorem) Let $\mathcal{O}$ be an order of conductor $\mathfrak{f}$, and let $N \in \mathbb{Z}^{\geq 3}$.*
a) There is a positive integer $T(\mathcal{O}, N)$ such that:
(i) if $F \supset K(\mathfrak{f})$ is a number field and $E_{/F}$ is an $\mathcal{O}$-CM elliptic curve with an F-rational point of order N, then $T(\mathcal{O}, N) \mid [F : K(\mathfrak{f})]$, and
(ii) there is a number field $F \supset K(\mathfrak{f})$ with $[F : K(\mathfrak{f})] = T(\mathcal{O}, N)$ and an $\mathcal{O}$-CM elliptic curve $E_{/F}$ with an F-rational point of order N.
b) If $(\Delta, N) = (-3, 3)$, then $T(\mathcal{O}, N) = 1$.
c) Suppose $(\Delta, N) \neq (-3, 3)$. Let $N = \ell_1^{a_1} \cdots \ell_r^{a_r}$ be the prime power decomposition of N. Then

$$T(\mathcal{O}, N) = \frac{\prod_{i=1}^r \widetilde{T}(\mathcal{O}, \ell_i^{a_i})}{\#\mathcal{O}^\times}.$$

d) If $\ell^a = 2$, then $\widetilde{T}(\mathcal{O}, \ell^a) = 2$ is computed in Lemma 6.1. If $\ell^a > 2$, then $\widetilde{T}(\mathcal{O}, \ell^a)$ is as follows, where $k = \mathrm{ord}_\ell(\mathfrak{f})$:

(1) *If $\ell \nmid \mathfrak{f}$, then* $\widetilde{T}(\mathcal{O}, \ell^a) = \begin{cases} \ell^{a-1}(\ell - 1) & \text{if } \left(\frac{\Delta}{\ell}\right) = 1, \\ \ell^{2a-2}(\ell - 1) & \text{if } \left(\frac{\Delta}{\ell}\right) = 0, \\ \ell^{2a-2}(\ell^2 - 1) & \text{if } \left(\frac{\Delta}{\ell}\right) = -1. \end{cases}$

(2) *If $\ell \mid \mathfrak{f}$, then* $\widetilde{T}(\mathcal{O}, \ell^a) = \begin{cases} \ell^{a-1}(\ell - 1) & \text{if } \left(\frac{\Delta_K}{\ell}\right) = 1, \\ \ell^{a-1}(\ell - 1) & \text{if } \left(\frac{\Delta_K}{\ell}\right) = -1 \text{ and } a \leq 2k, \\ \ell^{2a-2k-1}(\ell - 1) & \text{if } \left(\frac{\Delta_K}{\ell}\right) = -1 \text{ and } a > 2k, \\ \ell^{a-1}(\ell - 1) & \text{if } \left(\frac{\Delta_K}{\ell}\right) = 0 \text{ and } a \leq 2k + 1, \\ \ell^{2a-2k-2}(\ell - 1) & \text{if } \left(\frac{\Delta_K}{\ell}\right) = 0 \text{ and } a > 2k + 1. \end{cases}$

Remark 6.3. *The case $N = 2$ is excluded because of the somewhat anomalous behavior of 2-torsion. But it is easy to see that Theorem 6.2a) remains true when $N = 2$, and moreover:*
- *If $\Delta \in \{-4, -3\}$ then $T(\mathcal{O}, 2) = 1$.*
- *Otherwise,* $T(\mathcal{O}, 2) = \begin{cases} 1 & \text{if } \left(\frac{\Delta}{2}\right) \neq -1 \\ 3 & \text{if } \left(\frac{\Delta}{2}\right) = -1 \end{cases}$.

Let $F \supset K(\mathfrak{f})$ be a number field, and let $E_{/F}$ be an $\mathcal{O}$-CM elliptic curve. As usual, we choose an embedding $F \hookrightarrow \mathbb{C}$ such that $j(E) = j(\mathbb{C}/\mathcal{O})$. Let $P \in E[\mathrm{tors}]$ have order N. We call the field

$$K(\mathfrak{f})(\mathfrak{h}(P))$$

the **field of moduli** of P. It is independent of the chosen model of $E_{/F}$, and on some twist E^χ of $E_{/K(\mathfrak{f})(\mathfrak{h}(P))}$ the point P is $K(\mathfrak{f})(\mathfrak{h}(P))$-rational. Further, the pair (E,P) induces a closed point $\mathcal{P}$ on the modular curve $X_1(N)_{/K}$, and $K(\mathfrak{f})(\mathfrak{h}(P))$ is the residue field $K(\mathcal{P})$. Theorem 6.2 concerns the degree $[K(\mathfrak{f})(\mathfrak{h}(P)) : K(\mathfrak{f})]$. Our setup shows that it is no loss of generality to assume $F = K(\mathfrak{f})$.

Let $q_N : \mathcal{O} \to \mathcal{O}/N\mathcal{O}$ be the natural map, and let $q_N^\times : \mathcal{O}^\times \to C_N(\mathcal{O})$ be the induced map on unit groups. As in the introduction, we define the **reduced mod N Cartan subgroup**:

$$\overline{C_N(\mathcal{O})} = C_N(\mathcal{O})/q_N(\mathcal{O}^\times).$$

Let $\overline{E[N]}$ be the set of $\mathcal{O}^\times$-orbits on $E[N]$. Then the action of $C_N(\mathcal{O})$ on $E[N]$ induces an action of $\overline{C_N(\mathcal{O})}$ on $\overline{E[N]}$. The field of moduli $K(\mathfrak{f})(\mathfrak{h}(P))$ depends only on the image $\overline{P}$ of P in $\overline{E[N]}$. By Theorem 1.1, the composite homomorphism

$$\mathfrak{g}_F \stackrel{\rho_{E,N}}{\longrightarrow} C_N(\mathcal{O}) \to \overline{C_N(\mathcal{O})}$$

is surjective (and model-independent). Let $H_{\overline{P}} = \{g \in \overline{C_N(\mathcal{O})} \mid g\overline{P} = \overline{P}\}$. It follows that

$$\mathrm{Aut}(K(\mathfrak{f})(\mathfrak{h}(P))/K(\mathfrak{f})) \cong \overline{C_N(\mathcal{O})}/H_{\overline{P}}.$$

Thus $[K(\mathfrak{f})(\mathfrak{h}(P)) : K(\mathfrak{f})]$ is the size of the orbit of the reduced Cartan subgroup $\overline{C_N(\mathcal{O})}$ on $\overline{P}$. (As we will see, in almost every case this is the size of the orbit of $C_N(\mathcal{O})$ on P divided by $\#\mathcal{O}^\times$.) This reduces the proof of Theorem 6.2 to a purely algebraic problem.

6.2. **Generalities.** For an order N point $P \in \mathcal{O}/N\mathcal{O}$, let $M_P = \{xP \mid x \in \mathcal{O}\}$ be the cyclic $\mathcal{O}$-submodule of $\mathcal{O}/N\mathcal{O}$ generated by P. If we put $I_P = \{x \in \mathcal{O} \mid xP = 0\}$, then we have

$$M_P \cong_{\mathcal{O}} \mathcal{O}/I_P.$$

The isomorphism is canonical and determined by mapping $P \in M_P$ to $1 + I_P \in \mathcal{O}/I_P$.

Lemma 6.4. *a) With notation as above, let*

$$S(I_P) = \{g \in C_N(\mathcal{O}) \mid g \equiv 1 \pmod{I_P}\}.$$

Then with respect to the $C_N(\mathcal{O})$-action, $S(I_P)$ is the stabilizer of P, so as a $C_N(\mathcal{O})$-set the orbit of $C_N(\mathcal{O})$ on P is isomorphic to $C_N(\mathcal{O})/S(I_P)$.
b) Moreover, there is a canonical isomorphism of groups $C_N(\mathcal{O})/S(I_P) \stackrel{\sim}{\to} (\mathcal{O}/I_P)^\times$.

Proof. a) For $g \in C_N(\mathcal{O})$, we have $gP = P \iff (g-1)P = 0 \iff (g-1) \in I_P$, giving the first assertion. The Orbit Stabilizer Theorem gives the second assertion.
b) The ring homomorphism $f : \mathcal{O}/N\mathcal{O} \to \mathcal{O}/I$ induces a homomorphism on unit groups $f^\times : C_N(\mathcal{O}) \to (\mathcal{O}/I_P)^\times$, with kernel $S(I_P)$. Since $\mathcal{O}/N\mathcal{O}$ has finitely many maximal ideals, $f^\times$ is surjective [CA, Thm. 4.32]. □

Lemma 6.5. *There is a positive integer $M \mid N$ such that*

$$\mathcal{O}/I_P \cong_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/M\mathbb{Z}.$$

Proof. As a $\mathbb{Z}$-module, $\mathcal{O}/I_P$ is a quotient of $\mathcal{O}/N\mathcal{O} \cong_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}$, so

$$\mathcal{O}/I_P \cong_{\mathbb{Z}} \mathbb{Z}/N'\mathbb{Z} \oplus \mathbb{Z}/M\mathbb{Z}$$

with $M \mid N' \mid N$. Since P has order N in $(\mathcal{O}/I_P, +)$, we have $N' = N$. □

The following result computes the size of the reduced Cartan orbit on an order N point of $\mathcal{O}/N\mathcal{O}$ in terms of the size of the Cartan orbit. We recall that we have assumed $N \geq 3$.

Lemma 6.6. *a) Suppose $(\Delta, N) \neq (-3, 3)$, and let $P \in \mathcal{O}/N\mathcal{O}$ have order N. Then the orbit of $C_N(\mathcal{O})$ on P has size $\#\mathcal{O}^\times$ times the size of the orbit of $\overline{C_N(\mathcal{O})}$ on $\overline{P}$.*
b) Suppose $(\Delta, N) = (-3, 3)$. Then the order 3 points of $\mathcal{O}/3\mathcal{O}$ lie in two orbits under $C_3(\mathcal{O})$: one of size 2 and one of size 6. The corresponding reduced Cartan orbits each have size 1.

Proof. a) The Cartan orbit has size $\#(\mathcal{O}/I_P)^\times$, and the reduced Cartan orbit is smaller by a factor of the cardinality of the image of $\mathcal{O}^\times \to (\mathcal{O}/I_P)^\times$.
• Suppose $\Delta \notin \{-4, -3\}$. Then $\mathcal{O}^\times = \{\pm 1\}$, and since $N \geq 3$, we have $-1 \not\equiv 1 \pmod{I_P}$.
• Suppose $\Delta = -4$. Since $I_P \not\supset (2)$, by Lemma 2.11 the group $U_{I_P}(K)$ is trivial, and thus the map $\mathcal{O}^\times \to (\mathcal{O}/I_P)^\times$ is injective.
• Suppose $\Delta = -3$. By assumption $N \geq 4$, so $I_P \nmid (\zeta_3 - 1)$ and the map $\mathcal{O}^\times \to (\mathcal{O}/I_P)^\times$ is injective.
b) The assertion about Cartan orbits is a case of [CCRS13, Lemma 19]. (And another proof will be given in the next section.) The fact that both reduced Cartan orbits have size 1 follows from the already established fact that there is an $\mathcal{O}$-CM $E_{/\mathbb{Q}(\sqrt{-3})}$ with full 3-torsion. □

In view of Lemma 6.6, to prove Theorem 6.2 it suffices to compute the least size of an orbit of $C_N(\mathcal{O})$ on an order N point of $\mathcal{O}/N\mathcal{O}$ and show that this divides the size of every such orbit. The following result further reduce us to the case of N a prime power.

Proposition 6.7. *Let $N \geq 2$ have prime power decomposition $N = \ell_1^{a_1} \cdots \ell_r^{a_r}$. Let $P \in \mathcal{O}/N\mathcal{O}$ have order N, and let $I_P = \operatorname{ann} P$. For $1 \leq i \leq r$, let $P_i = \frac{N}{\ell_i^{a_i}} P$, and let $I_{P_i} = \operatorname{ann} P_i$. Then:*
a) The ideals $I_{P_1}, \ldots, I_{P_r}$ are pairwise comaximal: we have $I_{P_i} + I_{P_j} = \mathcal{O}$ for all $i \neq j$.
b) We have $I_P = I_{P_1} \cdots I_{P_r}$.
c) We have a canonical isomorphism of rings

$$\mathcal{O}/I_P \xrightarrow{\sim} \prod_{i=1}^r \mathcal{O}/I_{P_i}$$

which induces a canonical isomorphism of unit groups

$$(\mathcal{O}/I_P)^\times \xrightarrow{\sim} \prod_{i=1}^r (\mathcal{O}/I_{P_i})^\times.$$

d) The Cartan orbit of P is isomorphic, as a $C_N(\mathcal{O})$-set, to the direct product of the $C_{\ell_i^{a_i}}(\mathcal{O})$-orbits of the P_i's.

Proof. a) For $1 \leq i \leq r$, we have $(\mathcal{O}/I_{P_i}, +) \cong \mathbb{Z}/\ell_i^{a_i}\mathbb{Z} \oplus \mathbb{Z}/\ell_i^{b_i}\mathbb{Z}$ with $0 \leq b_i \leq a_i$; in particular it is an ℓ_i-group. Thus for $i \neq j$, $(\mathcal{O}/(I_i + I_j), +)$ is a homomorphic image of an ℓ_i-group and an ℓ_j-group, so it is trivial.
b) By the Chinese Remainder Theorem, we have $I_{P_1} \cdots I_{P_n} = \bigcap_{i=1}^n I_{P_i}$. Since P_i is a multiple of P, we have $I_P \subset I_{P_i}$ for all i, and thus $I_P \subset \bigcap_{i=1}^r I_{P_i}$. Conversely, choose $y_1, \ldots, y_r \in \mathbb{Z}$ such that $\sum_{i=1}^r y_i \frac{N}{\ell_i^{a_i}} = 1$. If $x \in \bigcap_{i=1}^r I_{P_i}$ then $x \frac{N}{\ell_i^{a_i}} P = 0$ for all i, hence

$$0 = \sum_{i=1}^r y_i \frac{N}{\ell_i^{a_i}} xP = xP,$$

so $x \in I_P$. Thus $I_P = \bigcap_{i=1}^n I_{P_i} = I_{P_1} \cdots I_{P_n}$.
c) The Chinese Remainder Theorem gives the first isomorphism; the second follows by passing to unit groups.
d) Apply Lemma 6.4 and part c). □

6.3. The Case $\ell \nmid \mathfrak{f}$.

Theorem 6.8. *Let $E_{/K(\mathfrak{f})}$ be an $\mathcal{O}$-CM elliptic curve. Let $\ell^a > 2$ be a prime power such that $\ell \nmid \mathfrak{f}$. We will describe all orbits of $C_{\ell^a}(\mathcal{O})$ on order ℓ^a points of $\mathcal{O}/\ell^a\mathcal{O}$: their sizes and their multiplicities.*
a) If $\left(\frac{\Delta}{\ell}\right) = 1$, there are $2a+1$ orbits: two orbits of size $\ell^{a-1}(\ell-1)$, for all $1 \leq i \leq a-1$ two orbits of size $\ell^{a+i-2}(\ell-1)^2$, and one orbit of size $\ell^{2a-2}(\ell-1)^2$.
b) If $\left(\frac{\Delta}{\ell}\right) = 0$, there are two orbits: an orbit of size $\ell^{2a-2}(\ell-1)$ and an orbit of size $\ell^{2a-1}(\ell-1)$.
c) If $\left(\frac{\Delta}{\ell}\right) = -1$, there is one orbit, of size $\ell^{2a-2}(\ell^2-1)$.

Proof. Step 1: Suppose $\mathcal{O} = \mathcal{O}_K$. Then every $\mathcal{O}$-submodule of $E[N]$ is of the form $E[I]$ for an ideal $I \supset N\mathcal{O}$, and $E[I] \cong_{\mathcal{O}} \mathcal{O}/I$: thus every submodule is of the form $M_P = \langle P \rangle_{\mathcal{O}}$ and is determined by its annhilator ideal I_P. Conversely, if $I \supset N\mathcal{O}$ is an ideal, then Lemmas 2.4 and 2.5 give that $E[I]$ is an $\mathcal{O}$-submodule of $E[N]$ with annihilator ideal I.
Split Case $\left(\frac{\Delta}{\ell}\right) = 1$**:** Then $\ell\mathcal{O} = \mathfrak{p}_1\mathfrak{p}_2$ for distinct prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$ of norm ℓ. The ideals containing $\ell^a\mathcal{O}$ are precisely $\mathfrak{p}_1^c\mathfrak{p}_2^d$ with $\max(c,d) \leq a$. We have ring isomorphisms

$$\mathcal{O}/\mathfrak{p}_1^c\mathfrak{p}_2^d \cong \mathcal{O}/\mathfrak{p}_1^c \times \mathcal{O}/\mathfrak{p}_2^d \cong \mathbb{Z}/\ell^c\mathbb{Z} \times \mathbb{Z}/\ell^d\mathbb{Z},$$

hence unit group isomorphisms

$$(\mathcal{O}/\mathfrak{p}_1^c\mathfrak{p}_2^d)^\times \cong (\mathcal{O}/\mathfrak{p}_1^c)^\times \times (\mathcal{O}/\mathfrak{p}_2^c)^\times \cong (\mathbb{Z}/\ell^c\mathbb{Z})^\times \times (\mathbb{Z}/\ell^d\mathbb{Z})^\times,$$

so

$$\#(\mathcal{O}/\mathfrak{p}_1^c\mathfrak{p}_2^d)^\times = \varphi(\ell^c)\varphi(\ell^d).$$

To get points of order ℓ^a we impose the condition $\max(c,d) = a$. Thus $\mathcal{O}$-modules generated by the points of order ℓ^a are

$$E[\mathfrak{p}_1^a],\ E[\mathfrak{p}_1^a\mathfrak{p}_2], \ldots,\ E[\mathfrak{p}_1^a\mathfrak{p}_2^a] = E[\ell^a],\ E[\mathfrak{p}_1^{a-1}\mathfrak{p}_2^a], \ldots,\ E[\mathfrak{p}_1\mathfrak{p}_2^a],\ E[\mathfrak{p}_2^a].$$

So there are $2a+1$ Cartan orbits, one of size $\varphi(\ell^a)\varphi(\ell^a)$ and, for all $0 \leq i \leq a-1$, two of size $\varphi(\ell^a)\varphi(\ell^i)$. The smallest orbit size is $\ell^{a-1}(\ell-1)$, and all the other orbit sizes are multiples of it.
Ramified Case $\left(\frac{\Delta}{\ell}\right) = 0$**:** Then $\ell\mathcal{O} = \mathfrak{p}^2$ for a prime ideal $\mathfrak{p}$ of norm ℓ. For any $b \in \mathbb{Z}^+$, the ring $\mathcal{O}/\mathfrak{p}^b$ is local of order ℓ^b with residue field $\mathbb{Z}/\ell\mathbb{Z}$, so the maximal ideal has size ℓ^{b-1} and thus

$$\#(\mathcal{O}/\mathfrak{p}^b)^\times = \ell^b - \ell^{b-1} = \ell^{b-1}(\ell-1).$$

Since $\mathfrak{p}^2 = (\ell)$, the least $c \in \mathbb{N}$ such that $\ell^c \in \mathfrak{p}^b$ is $c = \lceil \frac{b}{2} \rceil$. It follows that

$$(\mathcal{O}/\mathfrak{p}^b, +) \cong_{\mathbb{Z}} \mathbb{Z}/\ell^{\lceil \frac{b}{2} \rceil}\mathbb{Z} \oplus \mathbb{Z}/\ell^{\lfloor \frac{b}{2} \rfloor}\mathbb{Z}.$$

So the annihilator ideals of points of order ℓ^a in $\mathcal{O}/\ell^a\mathcal{O}$ are precisely $\mathfrak{p}^{2a-1}$ and $\mathfrak{p}^{2a}$. We get two Cartan orbits, one of size $\#(\mathcal{O}/\mathfrak{p}^{2a-1})^\times = \ell^{2a-2}(\ell-1)$ and one of size $\#(\mathcal{O}/\mathfrak{p}^{2a})^\times = \ell^{2a-1}(\ell-1)$. The smallest orbit size is $\ell^{2a-2}(\ell-1)$, and the other orbit size is a multiple of it.
Inert Case $\left(\frac{\Delta}{\ell}\right) = -1$: Then $\ell\mathcal{O}$ is a prime ideal, so the ideals containing $\ell^a\mathcal{O}$ are precisely $\ell^i\mathcal{O}$ for $i \leq a$. Clearly $\mathcal{O}/\ell^i\mathcal{O}$ has exponent ℓ^a iff $i = a$, so the $\mathcal{O}$-module generated by any point of order ℓ^a is $E[\ell^a]$. There is a single Cartan orbit, of size $\#(\mathcal{O}/\ell^a\mathcal{O})^\times = \varphi_K(\ell^a) = \ell^{2a-2}(\ell^2-1)$.
Step 2: Now let $\mathcal{O}$ be an order with $\ell \nmid \mathfrak{f}$. The natural maps $\mathcal{O}/\ell^a\mathcal{O} \to \mathcal{O}_K/\ell^a\mathcal{O}_K$ and $C_{\ell^a}(\mathcal{O}) \to C_{\ell^a}(\mathcal{O}_K)$ are isomorphisms, so the sizes and multiplicities of orbits carry over from $\mathcal{O}_K$ to $\mathcal{O}$. □

6.4. **The Case** $\ell \mid \mathfrak{f}$**.** Now suppose $\ell \mid \mathfrak{f}$. The ring $\mathcal{O}/\ell\mathcal{O}$ is isomorphic to $\mathbb{Z}/\ell\mathbb{Z}[\epsilon]/(\epsilon^2)$ – as one sees, e.g., using the explicit representation of (3) – and is thus a local Artinian ring with maximal ideal $\mathfrak{p}$, say, and residue field $\mathbb{Z}/\ell\mathbb{Z}$. Because $[\mathfrak{p} : \ell O] = \ell$, the only proper nonzero $\mathcal{O}$-submodule of $\mathcal{O}/\ell\mathcal{O}$ is $\mathfrak{p}/\ell$. Thus there are two Cartan orbits on the order ℓ elements of $\mathcal{O}/\ell\mathcal{O}$: one of order $\ell-1$ and one of order $\ell^2 - \ell = \#(\mathcal{O}/\ell\mathcal{O})^\times$.

For all $a \in \mathbb{Z}^+$, the ring $\mathcal{O}/\ell^a\mathcal{O}$ is local – for a maximal ideal $\mathfrak{m}$ of $\mathcal{O}$, we have $\ell^a \in \mathfrak{m} \iff \ell \in \mathfrak{m}$ – with residue field $\mathbb{Z}/\ell\mathbb{Z}$. It turn follows that for any order ℓ^a point $P \in \mathcal{O}/\ell^a\mathcal{O}$ and $I_P = \{x \in \mathcal{O} \mid xP = 0\}$, the ring $\mathcal{O}/I_P$ is local with residue field $\mathbb{Z}/\ell\mathbb{Z}$. By Lemma 6.5, we may write

$$M_P = \mathcal{O}/I_P \cong_{\mathbb{Z}} \mathbb{Z}/\ell^a\mathbb{Z} \oplus \mathbb{Z}/\ell^b\mathbb{Z} \tag{13}$$

for some $0 \leq b \leq a$, and then

$$\#(\mathcal{O}/I_P)^\times = \#\mathcal{O}/I_P - \frac{\#\mathcal{O}/I_P}{\ell} = \ell^{a+b-1}(\ell-1).$$

So the size of a Cartan orbit on an order ℓ^a element of $\mathcal{O}/\ell^a\mathcal{O}$ is of the form $(\ell-1)\ell^c$ for some $a-1 \leq c \leq 2a-1$. So in this case it is *a priori* clear that the minimal size of a Cartan orbit divides

the size of all the Cartan orbits. We want to understand how Cartan orbits grow when we lift a point of order ℓ^a to a point of order ℓ^{a+1}. First observe that $x \mapsto \ell x$ gives an $\mathcal{O}$-module isomorphism

$$\mathcal{O}/\ell^a\mathcal{O} \xrightarrow{\sim} \ell\mathcal{O}/\ell^{a+1}\mathcal{O},$$

so we can view $\mathcal{O}/\ell^a\mathcal{O}$ as an $\mathcal{O}$-submodule of $\mathcal{O}/\ell^{a+1}\mathcal{O}$. With P as in (13), let $Q \in \mathcal{O}/\ell^{a+1}\mathcal{O}$ be such that $\ell Q = P$. Put $M_Q = \{xQ \mid x \in \mathcal{O}\}$ and $I_Q = \{x \in \mathcal{O} \mid xQ = 0\}$, and write

$$M_Q = \mathcal{O}/I_Q \cong_{\mathbb{Z}} \mathbb{Z}/\ell^{a+1}\mathbb{Z} \oplus \mathbb{Z}/\ell^{b'}\mathbb{Z} \tag{14}$$

for $0 \le b' \le a+1$. Because $\ell Q = P$, we have $\ell M_Q = M_P$. Thus we find: if $b = 0$, then $b' \in \{0,1\}$, whereas if $b \ge 1$ then necessarily $b' = b+1$. So: if the $C_{\ell^a}(\mathcal{O})$-orbit on P has the smallest possible size $\varphi(\ell^a)$, then the $C_{\ell^{a+1}}$-orbit on Q either has size $\varphi(\ell^{a+1})$ or size $\varphi(\ell^{a+2})$ (as we will see shortly, both possibilities can occur), whereas if the $C_{\ell^a}(\mathcal{O})$-orbit on P has size $\varphi(\ell^{a+b}) > \varphi(\ell^a)$, then the $C_{\ell^{a+1}}(\mathcal{O})$-orbit on Q has size $\varphi(\ell^{a+b+2})$: i.e., upon lifting from P to Q the size grows by a factor of ℓ^2.

Since $H(\mathcal{O}, \ell^{a+1})$ implies $H(\mathcal{O}, \ell^a)$, for each fixed ℓ and $\mathcal{O}$ there are two possibilities.
Type I: $H(\mathcal{O}, \ell^a)$ holds for all $a \in \mathbb{Z}^+$.
In Type I, for all $a \in \mathbb{Z}^+$ the least size of a $C_{\ell^a}(\mathcal{O})$-orbit is $\varphi(\ell^a)$.
Type II: There is some $A \in \mathbb{Z}^+$ such that $H(\mathcal{O}, \ell^a)$ holds iff $a \le A$.
In Type II, for $1 \le a \le A$, the least size of a $C_{\ell^a}(\mathcal{O})$-orbit is $\varphi(\ell^a)$, but for all $a \ge A$, whenever we lift a point of order ℓ^a to a point of order ℓ^{a+1} the size of the Cartan orbit grows by a factor of ℓ^2, so for all $a > A$ the least size of a $C_{\ell^a}(\mathcal{O})$-orbit is $\ell^{a-A}\varphi(\ell^a)$.

We now determine the smallest size of a $C_{\ell^a}(\mathcal{O})$-orbit on an order ℓ^a point of $\mathcal{O}/\ell^a\mathcal{O}$ by using Theorem 5.15 to determine the type and compute the value of A in Type II.

Case 1: Suppose $\left(\frac{\Delta_K}{\ell}\right) = 1$. Then for all $a \in \mathbb{Z}^+$ $H(\mathcal{O}_K, \ell^a)$ holds, so Δ_K is a square modulo $4\ell^a$, hence $\Delta = \mathfrak{f}^2\Delta_K$ is also a square modulo $4\ell^a$, so $H(\mathcal{O}, \ell^a)$ holds, and we are in Type I.
Case 2: Suppose $\left(\frac{\Delta_K}{\ell}\right) = -1$, and put $k = \mathrm{ord}_\ell(\mathfrak{f})$.

• Let $\ell > 2$. If $a \le 2k$, then $\ell^a \mid \Delta$, so Δ is a square mod ℓ^a and hence also mod $4\ell^a$: thus $H(\mathcal{O}, \ell^a)$ holds. However, if $a = 2k+1$ then we claim $H(\mathcal{O}, \ell^a)$ does not hold. Indeed, suppose there is $s \in \mathbb{Z}$ such that $\Delta = \mathfrak{f}^2\Delta_K \equiv s^2 \pmod{\ell^a}$. Then $\ell^k \mid s$; taking $S = \frac{s}{\ell^k}$ we have $\frac{\mathfrak{f}^2}{\ell^{2k}}\Delta_K \equiv S^2 \pmod{\ell^{a-2k}}$, which implies that Δ_K is a square modulo ℓ: contradiction. So we are in Type II with $A = 2k$.

• Let $\ell = 2$, and write $\mathfrak{f} = 2^k F$. Suppose $a \le 2k$. Since $4 \mid \Delta_K - 1$, we have

$$2^{a+2} \mid (2^kF)^2(\Delta_K - 1) = \Delta - (2^kF)^2,$$

so $H(\mathcal{O}, 2^a)$ holds. Suppose $a \ge 2k+1$. If Δ is a square modulo 2^{a+2}, then we find that $\Delta_K \equiv 1 \pmod 8$, so $\left(\frac{\Delta_K}{2}\right) = 1$: contradiction. So we are in Type II with $A = 2k$.
Case 3: Suppose $\left(\frac{\Delta_K}{\ell}\right) = 0$, and put $k = \mathrm{ord}_\ell(\mathfrak{f})$.

• Let $\ell > 2$. If $a \le 2k+1$, then $\ell^a \mid \Delta$, so Δ is a square mod ℓ^a and hence also mod $4\ell^a$: thus $H\mathcal{O}, \ell^a)$ holds. However, if $a = 2k+2$ then we claim $H(\mathcal{O}, \ell^a)$ does not hold. Indeed, $\mathrm{ord}_\ell(\Delta) = 2k+1 < a$, so if $\Delta \equiv s^2 \pmod{\ell^a}$, then $\mathrm{ord}_\ell(s^2) = 2k+2$: contradiction. So we are in Type II with $A = 2k+1$.

• Let $\ell = 2$, and write $\mathfrak{f} = 2^k F$. Suppose $a \le 2k+1$. Since $4 \mid \Delta_K$, there is $s \in \mathbb{Z}$ such that $8 \mid \Delta_K - s^2$, so

$$2^{a+2} \mid 2^{2k+3} \mid (2^kF)^2(\Delta_K - s^2) = \Delta - (2^kFs)^2,$$

so $H(\mathcal{O}, 2^a)$ holds. Suppose $a \ge 2k+2$. If Δ is a square modulo 2^{a+2}, then Δ_K is a square modulo 2^{a+2-2k}, hence modulo 16: contradiction. So we are in Type II with $A = 2k+1$.

6.5. **Torsion over $K(j)$: Part II.** We return to complete the classification of torsion on $\mathcal{O}$-CM elliptic curves $E_{/K(\mathfrak{f})}$ begun in §5.7.

II. Suppose $\Delta = -4$, so $j = 1728$ and $K(\mathfrak{f}) = K = \mathbb{Q}(\sqrt{-1})$.

• By Theorem 5.2, if $E(K)$ has a point of order N, then $\varphi(N) \mid 4$, so

$$N \in \{1, 2, 3, 4, 5, 6, 8, 10\}.$$

• Using Theorem 6.2 we get

$$T(\mathcal{O}, 1) = T(\mathcal{O}, 2) = T(\mathcal{O}, 4) = T(\mathcal{O}, 5) = T(\mathcal{O}, 10) = 1,$$
$$T(\mathcal{O}, 3) = T(\mathcal{O}, 6) = 2,\ T(\mathcal{O}, 8) = 4.$$

• We have $C_2(\mathcal{O}) = \mu_4/\{\pm 1\}$. Thus $\#C_2(\mathcal{O}) = 2$ so every $\mathcal{O}$-CM elliptic curve $E_{/K}$ has a K-rational point of order 2, and some $\mathcal{O}$-CM elliptic curve $E_{/K}$ has $E[2] = E[2](K)$.
• Because $\widetilde{T}(\mathcal{O}, 5) = 4$, if an $\mathcal{O}$-CM elliptic curve $E_{/K}$ has a K-rational point of order 5, the index of the mod 5 Galois representation in $C_5(\mathcal{O})$ is divisible by 4. Because $\#C_2(\mathcal{O}) = 2$, if an $\mathcal{O}$-CM elliptic curve $E_{/K}$ has full 2-torsion then the index of the mod 2 Galois representation in $C_2(\mathcal{O})$ is divisible by 2. Thus if an $\mathcal{O}$-CM elliptic curve $E_{/K}$ had $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z} \hookrightarrow E(K)[\text{tors}]$, the index of the mod 10 Galois representation in $C_{10}(\mathcal{O})$ would be divisible by 8, contradicting Theorem 1.1.
• If $N \geq 3$ then $\#C_N(\mathcal{O}) > \#\mathcal{O}^\times$, so no $\mathcal{O}$-CM elliptic curve $E_{/K}$ has $E[N] = E[N](K)$.
Thus the groups which can occur as $E(K)[\text{tors}]$ are precisely

$$\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/10\mathbb{Z}.$$

III. Suppose $\Delta = -3$, so $j = 0$ and $K(\mathfrak{f}) = K = \mathbb{Q}(\sqrt{-3})$.

• By Theorem 5.2, if $E(K(\mathfrak{f}))$ has a point of order N, then $\varphi(N) \mid 6$, so

$$N \in \{1, 2, 3, 4, 6, 7, 9, 14, 18\}.$$

• Using Theorem 6.2 we get

$$T(\mathcal{O}, 1) = T(\mathcal{O}, 2) = T(\mathcal{O}, 3) = T(\mathcal{O}, 6) = T(\mathcal{O}, 7) = 1,$$
$$T(\mathcal{O}, 4) = 2,\ T(\mathcal{O}, 9) = T(\mathcal{O}, 14) = 3,\ T(\mathcal{O}, 18) = 9.$$

• We have $C_2(\mathcal{O}) = \mu_6/\{\pm 1\}$. Thus as we range over all $\mathcal{O}$-CM elliptic curves $E_{/K}$, the group $E(K)[2]$ can be trivial (using Theorem 4.8) or have size 4, but it cannot have size 2.
• We have $C_3(\mathcal{O}) = \mu_6$. Thus there is an $\mathcal{O}$-CM elliptic curve $E_{/K}$ with $E[3] = E[3](K)$.
• If $N \geq 4$ then $\#C_N(\mathcal{O}) > \#\mathcal{O}^\times$, so no $\mathcal{O}$-CM elliptic curve $E_{/K}$ has $E[N] = E[N](K)$.
Thus the groups which can occur as $E(K)[\text{tors}]$ are precisely

$$\{e\}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/7\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

Remark 6.9. *a) Case I. of the above calculation is a more detailed and explicit version of one of the main results of* [Pa89]. *Parish offers addenda on Cases II. and III., but without proof, and the possibilities* $E(K(\mathfrak{f}))[\text{tors}] \cong \mathbb{Z}/10\mathbb{Z}$ *in Case II. and* $E(K(\mathfrak{f}))[\text{tors}] \cong \mathbb{Z}/7\mathbb{Z}$ *and* $E(K(\mathfrak{f}))[\text{tors}] \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ *in Case III are not mentioned.*
b) In Cases II. and III. a classification of the possibilities for $E(K(\mathfrak{f}))[\text{tors}]$ apart *from the "Olson groups"* $\{e\}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *is done in* [BCS, Thm. 1.4]. *The older method of proof used computer calculations on degrees of preimages of* $j = 0$ *and* $j = 1728$ *on modular curves* [BCS, Table 2].

6.6. **Isogenies over $K(j)$: Part II.** We return to complete the classification of $K(j)$-rational cyclic isogenies for elliptic curves with CM by the orders of discriminants $\Delta = -4$ and $\Delta = -3$. Recall that these cases have additional complexity coming from the fact that μ_K acts nontrivially on the projectivized torsion group $\mathbb{P}E[N]$. In this case, there is an $\mathcal{O}$-CM elliptic curve $(E_0)_{/K}$ for which the projective mod N Galois representation

$$\mathbb{P}\rho_N : \mathfrak{g}_K \to C_N(\mathcal{O})/(\mathbb{Z}/N\mathbb{Z})^\times$$

is surjective, and thus as we vary over all possible K-models of E_0, the representation $\mathbb{P}\rho_N$ twists by a character

$$\mathbb{P}\chi : \mathfrak{g}_K \to \mu_K/\{\pm 1\}.$$

Thus the index of $\mathbb{P}\rho_N(\mathfrak{g}_K)$ in $C_N(\mathcal{O})/(\mathbb{Z}/N\mathbb{Z})^\times$ divides 2 when $w_K = 4$ and divides 3 when $w_K = 6$.

We will rule out the existence of K-rational cyclic N-isogenies for various values of N using the following "$\widetilde{T}$-argument": suppose that $\widetilde{T}(\mathcal{O}, N) > \varphi(N)\frac{w_K}{2}$. Then every $C_N(\mathcal{O})$-orbit on a point of order N in $\mathcal{O}/N\mathcal{O}$ has size a multiple of $\widetilde{T}(\mathcal{O}, N)$, so every $C_N(\mathcal{O})/(\mathbb{Z}/N\mathbb{Z})^\times$-orbit on $\mathbb{P}E[N]$ has size a multiple of $\frac{\widetilde{T}(\mathcal{O},N)}{\varphi(N)}$, which by our hypothesis is greater than $\frac{w_K}{2}$. So after passing to a field extension L of degree $\frac{w_K}{2}$ to trivialize $\mathbb{P}\chi$, we find that $\mathfrak{g}_L$ acts without fixed points on $\mathbb{P}E[N]$, and there is no L-rational cyclic N-isogeny and thus certainly no K-rational cyclic N-isogeny.

Let $\mathcal{O}$ be the order of discriminant $\Delta = -4$, so $K(j) = K = \mathbb{Q}(\sqrt{-1})$ and $w_K = 4$.
• If $\ell \equiv 1 \pmod 4$, then for all $a \in \mathbb{Z}^+$ we have that -4 is a square in $\mathbb{Z}/4\ell^a\mathbb{Z}$ so there is a K-rational cyclic ℓ^a-isogeny. In fact we get that *every* $\mathcal{O}$-CM elliptic curve $E_{/K}$ has a K-rational cyclic ℓ^a-isogeny.
• If $\ell \equiv 3 \pmod 4$, since $\frac{\widetilde{T}(\mathcal{O},\ell)}{\varphi(\ell)\frac{w_K}{2}} = \frac{\ell^2-1}{2(\ell-1)} = \frac{\ell+1}{2} > 1$, by the $\widetilde{T}$-argument there is no K-rational ℓ-isogeny.
• If $\ell = 2$, then since $T(\mathcal{O}, 4) = 1$, we can have a K-rational point of order 4 (as already seen in §6.5), hence a cyclic K-rational 4-isogeny. Since $\frac{\widetilde{T}(\mathcal{O},8)}{\varphi(8)\frac{w_K}{2}} = \frac{16}{4\cdot 2} > 1$, by the $\widetilde{T}$-argument there is no cyclic K-rational 8-isogeny.
Any elliptic curve over a number field admitting a rational cyclic N-isogeny also admits a rational cyclic M-isogeny for all $M \mid N$. Moreover, if an elliptic curve $E_{/F}$ admits F-rational cyclic $N_1, \ldots, N_r$ isogenies for pairwise coprime $N_1, \ldots, N_r$, then the subgroup generated by the kernels of these isogenies is F-rational and cyclic of order $N_1 \cdots N_r$ so E admits an F-rational cyclic $N_1 \cdots N_r$-isogeny. The assertion of Theorem 5.18b) now follows.

Let $\mathcal{O}$ be the order of discriminant $\Delta = -3$, so $K(j) = K = \mathbb{Q}(\sqrt{-3})$ and $w_K = 6$.
• If $\ell \equiv 1 \pmod 3$, then similarly to the $\Delta = -4$ case above we get that every $\mathcal{O}$-CM elliptic curve $E_{/K}$ has a K-rational cyclic ℓ^a-isogeny for all $a \in \mathbb{Z}^+$.
• If $\ell \equiv 2 \pmod 3$ and $\ell > 2$, then since $\frac{\widetilde{T}(\mathcal{O},\ell)}{\varphi(\ell)\frac{w_K}{2}} = \frac{\ell^2-1}{3(\ell-1)} = \frac{\ell+1}{3} > 1$, by the $\widetilde{T}$-argument there is no cyclic K-rational ℓ-isogeny.
• If $\ell = 2$, then since $T(\mathcal{O}, 2) = 1$ there is an $\mathcal{O}$-CM elliptic curve $E_{/K}$ with a K-rational 2-isogeny.
•. Since $\frac{\widetilde{T}(\mathcal{O},4)}{\varphi(4)\frac{w_K}{2}} = \frac{12}{2\cdot 3} > 1$, by the $\widetilde{T}$-argument there is no cyclic K-rational 4-isogeny.
• We claim that there is an $\mathcal{O}$-CM elliptic curve $E_{/K}$ with a K-rational cyclic 9-isogeny. Let $\mathfrak{p}$ be the unique prime ideal of $\mathcal{O}$ lying over 3, and let P be a generator of the cyclic $\mathcal{O}$-module $E[\mathfrak{p}^3] \subset E[9]$, so P has order 9. By Lemma 6.4, the $C_9(\mathcal{O})$-orbit on P can be identified with the unit group $(\mathcal{O}/\mathfrak{p}^3)^\times$, of order 18. The $\mathcal{O}$-module generated by P is also isomorphic to $(\zeta_3 - 1)\mathcal{O}/9\mathcal{O}$, and using this representation it is easy to compute that the group $(\mathcal{O}/\mathfrak{p}^3)^\times$ is generated by the images of the scalar matrices $(\mathbb{Z}/9\mathbb{Z})^\times$ and the cube roots of unity. Thus Galois acts on the image of P in $\mathbb{P}E[9]$ via a character $\mathbb{P}\chi$. After twisting by the inverse of this character, the image of P in $\mathbb{P}E[9]$ becomes fixed by Galois and we get a K-rational cyclic 9-isogeny.
• Since $\frac{\widetilde{T}(\mathcal{O},18)}{\varphi(18)\frac{w_K}{2}} = \frac{54}{3\cdot 6} > 1$, by the $\widetilde{T}$-argument there is no K-rational cyclic 18-isogeny.
• Since $\frac{\widetilde{T}(\mathcal{O},27)}{\varphi(27)\frac{w_K}{2}} = \frac{162}{3\cdot 18} > 1$, by the $\widetilde{T}$-argument there is no K-rational cyclic 27-isogeny.

• From §6.5 (or Theorem 6.2) we know there is an $\mathcal{O}$-CM elliptic curve $E_{/K}$ with a rational point of order 6, hence certainly a cyclic K-rational 6-isogeny.
Using the same considerations as in the $\Delta = -4$ case above we get the assertion of Theorem 5.18c).

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