TORSION POINTS ON CM ELLIPTIC CURVES OVER REAL NUMBER FIELDS

ABBEY BOURDON, PETE L. CLARK, AND JAMES STANKEWICZ

Abstract. We study torsion subgroups of elliptic curves with complex multiplication (CM) defined over number fields which admit a real embedding. We give a complete classification of the groups which arise up to isomorphism as the torsion subgroup of a CM elliptic curve defined over a number field of odd degree: there are infinitely many. Restricting to the case of prime degree, we show that there are only finitely many isomorphism classes. More precisely, there are six “Olson groups” which arise as torsion subgroups of CM elliptic curves over number fields of every degree, and there are precisely 17 “non-Olson” CM elliptic curves defined over a prime degree number field.

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We denote by $\mathcal{P}$ the set of all prime numbers. For $n \in \mathbb{Z}^+$ let $\zeta_n = e^{\frac{2\pi i}{n}} \in \mathbb{C}$, and put $\mathbb{Q}(\zeta_n)^+ = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$. For a field $F$, let $\overline{F}$ be an algebraic closure, let $F^{\text{sep}}$ be the maximal separable subextension of $\overline{F}/F$, and let $g_F = \text{Aut}(F^{\text{sep}}/F) = \text{Aut}(\overline{F}/F)$ be the absolute Galois group of $F$. A real number field is a number field which admits an embedding into $\mathbb{R}$. Thus every odd degree number field is real.

1. Introduction

This paper continues an exploration of torsion points on elliptic curves with complex multiplication (CM) initiated by the last two authors in collaboration with B. Cook, P. Corn and A. Rice [CCRS13], [CCRS14].

The subject of torsion points on CM elliptic curves begins with the following result.

**Theorem 1.1.** (Olson [Ol74]) Let $E/\mathbb{Q}$ be a CM elliptic curve. Then $E(\mathbb{Q})\,\text{[tors]}$ is isomorphic to one of: the trivial group $\{\bullet\}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Conversely, each such group occurs for at least one CM elliptic curve $E/\mathbb{Q}$.

We say a finite commutative group $G$ is an Olson group if it is isomorphic to one of the six groups given in the conclusion of Theorem 1.1. An elliptic curve $E/F$ is Olson if $E(F)\,\text{[tors]}$ is an Olson group.

The tables of [CCRS14, §4] show that for all $d \leq 13$, every Olson group arises as the torsion subgroup of a CM elliptic curve over some degree $d$ number field. In fact, it follows from Theorem 2.1a) below that all six Olson groups occur in the list of torsion subgroups of CM elliptic curves over infinitely many number fields of every degree $d \geq 2$. Similarly, whenever $d_1 | d_2$, the list of torsion subgroups of CM elliptic curves in degree $d_2$ will contain the corresponding list in degree $d_1$. Therefore it is more penetrating to ask which new groups arise in degree $d$. More precisely, for $d \in \mathbb{Z}^+$, let $\mathcal{T}_{\text{CM}}(d)$ be the set of isomorphism classes of torsion subgroups of CM elliptic curves defined over number fields of degree $d$, and for $d \geq 2$ we put

$$\mathcal{T}^\text{new}_{\text{CM}}(d) = \mathcal{T}_{\text{CM}}(d) \setminus \bigcup_{d' | d, d' \neq d} \mathcal{T}_{\text{CM}}(d').$$

From [CCRS14, §4] we compile the following table.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$7$</th>
<th>$8$</th>
<th>$9$</th>
<th>$10$</th>
<th>$11$</th>
<th>$12$</th>
<th>$13$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$# \mathcal{T}^\text{new}_{\text{CM}}(d)$</td>
<td>5</td>
<td>2</td>
<td>9</td>
<td>1</td>
<td>7</td>
<td>0</td>
<td>14</td>
<td>3</td>
<td>0</td>
<td>13</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

**Remark 1.2.** a) The size of $\mathcal{T}^\text{new}_{\text{CM}}(d)$ is strongly influenced by the 2-adic valuation $v_2(d)$: for all $2 \leq d_1, d_2 \leq 13$, $v_2(d_1) < v_2(d_2) \implies \# \mathcal{T}^\text{new}_{\text{CM}}(d_1) < \# \mathcal{T}^\text{new}_{\text{CM}}(d_2)$.

b) There is very little new torsion when $d$ is odd.

c) When we restrict to prime values of $d$, the sequence of values is $5, 2, 1, 0, 0, 0$.

Remark 1.2c) was made to us by M. Schütt. He also asked the following question.

**Question 1.3.** (Schütt) Is $\# \mathcal{T}^\text{new}_{\text{CM}}(p) = 0$ for all sufficiently large primes $p$?

Schütt’s question is equivalent to asking whether there is an absolute bound on the size of the torsion subgroup of all CM elliptic curves defined over all number fields of prime degree. The first main result of this paper is an affirmative answer.
For $b, c \in F$ we define the **Kubert-Tate curve**

$$E(b, c) : y^2 + (1 - c)xy - by = x^3 - bx^2.$$ 

For $\lambda \in F$ we define the **Hesse curve**

$$E_\lambda : X^3 + Y^3 + Z^3 + \lambda XYZ = 0.$$ 

**Theorem 1.4.** Let $E$ be a non-Olson CM elliptic curve defined over a prime degree number field $F$. Then $F$ is isomorphic to one of the fields listed below, and over that field $E$ is isomorphic to exactly one of the 17 listed elliptic curves.

<table>
<thead>
<tr>
<th>Number Field $F$</th>
<th>Elliptic Curve</th>
<th>$E(F)[\text{tors}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Q}(\sqrt{-3})$</td>
<td>$E_0$</td>
<td>$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$</td>
</tr>
<tr>
<td>$\mathbb{Q}(i)$</td>
<td>$E(-\frac{1}{2}, 0)$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{2})$</td>
<td>$E(1 + \frac{1}{2}\sqrt{2}, 0)$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{2})$</td>
<td>$E(-\frac{1}{2}, 0)$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{-7})$</td>
<td>$E(-\frac{1}{3} + \frac{1}{3}\sqrt{-7}, 0)$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{-7})$</td>
<td>$E(-\frac{1}{3} + \frac{1}{3}\sqrt{-7}, 0)$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{3})$</td>
<td>$E(-\frac{1}{3}, -\frac{1}{3})$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{3})$</td>
<td>$E(\frac{1}{3}, \frac{1}{3})$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{3})$</td>
<td>$E(-\frac{1}{3} + \frac{1}{3}\sqrt{-3}, -1)$</td>
<td>$\mathbb{Z}/7\mathbb{Z}$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{2})/(b^3 - 15b^2 + 9b - 1)$</td>
<td>$E(\frac{1}{3}b^2 + \frac{2}{3}b + \frac{1}{3}, b)$</td>
<td>$\mathbb{Z}/9\mathbb{Z}$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{2})/(b^3 + 105b^2 - 336 - 1)$</td>
<td>$E(-\frac{1}{15}b^2 + \frac{22}{15}b + \frac{1}{3}, b)$</td>
<td>$\mathbb{Z}/9\mathbb{Z}$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{2})/(b^3 - 4b^2 - 3b + 1)$</td>
<td>$E(-2b^2 + 4b + 1, b)$</td>
<td>$\mathbb{Z}/14\mathbb{Z}$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{2})/(b^3 - 180b^2 + 3b + 1)$</td>
<td>$E(\frac{1}{10}b^2 + \frac{10}{9}b - \frac{1}{3}, b)$</td>
<td>$\mathbb{Z}/14\mathbb{Z}$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{2})/(b^3 - 9b^2 + 6b^2 + 42b^2 - 7b - 1)$</td>
<td>$E(-\frac{1}{10}b^4 + \frac{1}{5}b^3 + \frac{2}{5}b^2 + \frac{1}{5}b - \frac{1}{10}, b)$</td>
<td>$\mathbb{Z}/11\mathbb{Z}$</td>
</tr>
</tbody>
</table>

In light of Remark 1.2 and Theorem 1.4, it is natural to wonder whether there is an absolute bound on $\#E(F)[\text{tors}]$ as $E$ ranges over all CM elliptic curves defined over a number field $F$ of odd degree. In the following result, we show this is not the case: although there are limitations on the order of a rational torsion point, infinitely many groups do arise. We classify these groups up to isomorphism.

**Theorem 1.5.** (Odd Degree Theorem) Let $F$ be a number field of odd degree, let $E/F$ be a CM elliptic curve, and let $T = E(F)[\text{tors}]$. Then $T$ is isomorphic to one of the following groups:

1. the trivial group $\{\bullet\}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$;
2. the group $\mathbb{Z}/\ell^n\mathbb{Z}$ for a prime number $\ell \equiv 3 \pmod{8}$ and $n \in \mathbb{Z}^+$;
3. the group $\mathbb{Z}/2\ell^n\mathbb{Z}$ for a prime number $\ell \equiv 3 \pmod{4}$ and $n \in \mathbb{Z}^+$.

Conversely, each of the above groups arises up to isomorphism as the torsion subgroup $E(F)$ of a CM elliptic curve $E$ defined over an odd degree number field $F$.

If we further assume that $F/\mathbb{Q}$ has odd degree $d$ and Galois group $S_d$, then every CM elliptic curve $E/F$ is Olson (§6.3). By contrast, we show in §6.2 that any positive integer arises as the order of a point of some (not necessarily CM) elliptic curve defined over some odd degree number field.
Even if we restrict our attention to number fields of degree twice a prime we see fundamentally different behavior from Theorem 1.4.

**Theorem 1.6.** Assume Schinzel’s Hypothesis H. As \( F \) ranges over all number fields of degree twice a prime number and \( E \) ranges over all CM elliptic curves over \( F \), the set of prime numbers which divide the order of some torsion subgroup \( E(F)_{\text{tors}} \) is infinite. In particular:

\[
\limsup_{p \in \mathcal{P}} \#\mathcal{T}_{\text{CM}}(2p) \geq 1.
\]

We prove this result in a more general form in §6.1.

To prove Theorems 1.4 and 1.5 we need results of the form: “If an \( \mathcal{O}\)-CM elliptic curve over a number field \( F \) has an \( F \)-rational point of order \( N \), then \([F : \mathbb{Q}]\) is divisible by some function of \( N \) and \( \mathcal{O} \).” Prototypical results of this type were given by Silverberg and later by Prasad-Yogananda [Si88], [Si92], [PY01]: the SPY-bounds. They were refined by Clark-Cook-Stankewicz [CCRS13, Theorem 3].

While pursuing further refinements of the SPY-bounds, we noticed a real cyclotomy phenomenon in the tables of [CCRS14]: for every CM elliptic curve \( E/F \) in our tables containing an \( F \)-rational point of order \( N \geq 3 \), \( F \) contains either the CM field \( K \) or \( \mathbb{Q}(\zeta_N)^+ \). In particular, if \( F \) has odd degree then \( F \supset \mathbb{Q}(\zeta_N)^+ \).

Lying at the heart of this work are two results confirming this phenomenon under certain mild additional hypotheses. If we assume that \( N \) is prime to the discriminant \( \Delta \) of the CM order then we show that \( F \) contains an index 2 subfield of \( \mathbb{Q}(\zeta_N) \) if it does not contain the CM field. In general there may be more than one such subfield of \( \mathbb{Q}(\zeta_N) \), but when \( N \) is an odd prime power the unique one is \( \mathbb{Q}(\zeta_N)^+ \).

When \( N \) is an even prime power we get the slightly weaker result that \( F \) contains \( \mathbb{Q}(\zeta_{N/2})^+ \) if it does not contain the CM field. Since \( \mathbb{Q}(\zeta_N)^+ \) is characterized among index 2 subfields of \( \mathbb{Q}(\zeta_N) \) by being a real number field, real cyclotomy is also confirmed when \( F \) has a real embedding.

In fact, when \( F \) is real we can show that it contains \( \mathbb{Q}(\zeta_N)^+ \) without the assumption \( \gcd(N, \Delta) = 1 \). This requires a more detailed argument involving an explicit matrix representation of the \( \mathcal{O} \)-module structure and the complex conjugation action on \( E[N] \) (Theorem 4.9). Here we combine the theory of uniformization of real elliptic curves by real lattices \( \Lambda = \mathcal{X} \subset \mathbb{C} \) with the ideal theory of quadratic orders. We did not find this pleasant, classical-looking material in the literature in the form that we need it, so we develop it in detail in §3.

We prove Theorem 1.4 in §5, using the results of §4 and a Theorem of J.L. Parish which is recalled in §5.1. The proof highlights the relevance of Sophie Germain primes, which provided motivation for Theorem 1.6. We prove Theorem 1.5 in §7.

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2. Torsion Points of Abelian Varieties under Base Extension

**Theorem 2.1.** Let \( A/F \) be an abelian variety over a number field, and let \( d \geq 2 \).

a) There are infinitely many \( L/F \) such that \([L : F] = d\) and \( A(L)_{\text{tors}} = A(F)_{\text{tors}}\).
b) If $d$ is prime, then for all but finitely many $L/F$ with $[L : F] = d$, we have
$A(L)[\text{tors}] = A(F)[\text{tors}]$.

c) For all but at most finitely many quadratic twists $A'$ of $A/F$ we have $A'(F)[\text{tors}] = A'(F)[2] = A(F)[2]$.

Proof. a) By work of Masser [Ma87, Corollary 2], there is $N \in \mathbb{Z}^+$ depending only on $[L : \mathbb{Q}]$ such that $A(L)[\text{tors}] = A(L)[N]$. Let $M = F(A[N])$. Then $A(L)[\text{tors}] \supseteq A(F)[\text{tors}]$ implies $A(L)[N] \supseteq A(F)[N]$ and thus $M \cap L \supseteq F$. For each $d \geq 2$ there are infinitely many degree $d$ $L/F$ with $M \cap L = F$: let $v$ be a finite place of $F$ which is unramified in $M$, and choose $L/F$ to be totally ramified at $v$. This gives one extension $L_1/F$; replacing $M$ with $L_1M$ gives another extension $L_2/F$; and so forth.

b) If $d = [L : F]$ is prime, then $M \cap L \supseteq F$ implies $L \subset M$.

c) For $t \in F^\times/F^\times 2$, we denote by $A^t/F$ the quadratic twist by $t$ and the involution $[-1]$ on $A$. We have monomorphisms $A(F) \hookrightarrow A(F(\sqrt{t}))$, $A'(F) \hookrightarrow A(F(\sqrt{t}))$, and

$$A(F) \cap A'(F) = A(F)[2] = A'(F)[2].$$

By part b), for all but finitely many $t$ we have $A(F)[\text{tors}] = A(F(\sqrt{t}))[\text{tors}]$ and thus $A'(F)[\text{tors}] = A'(F)[2] = A(F)[2]$.

\textbf{Remark 2.2.} In 2001, Qiu and Zhang used Merel’s theorem to prove Theorem 2.1b) when $A$ is an elliptic curve [QZ01, Theorem 1]. When $A$ is an elliptic curve and $F = \mathbb{Q}$, Theorem 2.1c) was proved by Gouvêa and Mazur [GM91, Proposition 1]. This extends to a number field $F$ unless $A$ has complex multiplication by an imaginary quadratic field $K \subset F$. Results of Silverberg [Si88] handle the case of all abelian varieties with complex multiplication. Alternately, Mazur and Rubin use Merel’s theorem to establish Theorem 2.1c) for elliptic curves; however, as in their application $A(F)$ has no points of order 2, they record the result (only) in the form that all but finitely many quadratic twists of $A/F$ have no odd order torsion [MR10, Lemma 5.5]. The full statement of Theorem 2.1c) for elliptic curves first appears in a recent work of F. Najman [Na13, Theorem 12].

3. R-Structures, Complex Conjugation and Cartan Subgroups

3.1. Orders and Ideals in Imaginary Quadratic Fields.

Let $K$ be a number field. A \textbf{lattice} in $K$ is a $\mathbb{Z}$-module $\Lambda \subset K$ obtained as the $\mathbb{Z}$-span of a $\mathbb{Q}$-basis for $K$. An \textbf{order} $\mathcal{O}$ in $K$ is a lattice which is also a sub-ring. The ring of integers $\mathcal{O}_K$ is an order of $K$; conversely, since every element of an order $\mathcal{O}$ is integral over $\mathbb{Z}$, $\mathcal{O} \subset \mathcal{O}_K$ with finite index. For any lattice $\Lambda$,

$$\mathcal{O}(\Lambda) = \{ x \in K \mid x\Lambda \subset \Lambda \}$$

is an order of $K$, and $\Lambda$ is a fractional $\mathcal{O}(\Lambda)$-ideal of $K$. For all $\alpha \in K^\times$ we have $\mathcal{O}(\alpha\Lambda) = \mathcal{O}(\Lambda)$. For any order $\mathcal{O}$, a fractional $\mathcal{O}$-ideal $\Lambda$ is \textbf{proper} if $\mathcal{O} = \mathcal{O}(\Lambda)$. A fractional $\mathcal{O}_K$-ideal is necessarily proper, whereas for any nonmaximal order $\mathcal{O}$, $[\mathcal{O}_K : \mathcal{O}]\mathcal{O}_K$ is an $\mathcal{O}$-ideal which is not proper.

\textbf{Lemma 3.1.} Let $\mathcal{O}$ be an order in a quadratic field $K$, and let $\Lambda$ be a fractional $\mathcal{O}$-ideal. The following are equivalent:

(i) $\Lambda$ is a projective $\mathcal{O}$-module.
(ii) For every prime number \( p \), \( \Lambda \otimes \mathbb{Z}_p \) is a principal fractional \( \mathcal{O} \otimes \mathbb{Z}_p \)-ideal. (iii) \( \Lambda \) is a proper \( \mathcal{O} \)-ideal.

Proof. To prove (i) \( \iff \) (ii) is an exercise in commutative algebra [Ei95, Exercise 4.11]. The local characterization of lattices in any number field gives (ii) \( \implies \) (iii) [La87, p. 97] and the converse is standard [La87, Theorem 9, p. 98]. \( \square \)

From now on we assume that \( K \) is an imaginary quadratic field. If \( \mathcal{O}' \subset \mathcal{O} \) are quadratic orders in \( K \), their discriminants are related as follows:

\[
\Delta(\mathcal{O}') = [\mathcal{O}' : \mathcal{O}]^2 \Delta(\mathcal{O}).
\]

For an order \( \mathcal{O} \) in \( K \), we define the conductor \( f = [\mathcal{O}_K : \mathcal{O}] \). Let us write \( \Delta_K \) for \( \Delta(\mathcal{O}_K) \). Then if \( \mathcal{O} \) has conductor \( f \), we have

\[
\Delta(\mathcal{O}) = f^2 \Delta_K.
\]

Observe that \( \Delta(\mathcal{O}) \) is negative and congruent to 0 or 1 modulo 4; we call such integers **imaginary quadratic discriminants**. For any \( K \) and \( f \in \mathbb{Z}^+, \mathbb{Z}[\sqrt{\Delta_K + \sqrt{\Delta_K}}] \) is the unique order \( \mathcal{O} \) in \( K \) of conductor \( f \) [La87, p. 90]. It follows that for every imaginary quadratic discriminant \( \Delta \), there is a unique imaginary quadratic order \( \mathcal{O}(\Delta) \) of discriminant \( \Delta \).

### 3.2. Basics on CM Elliptic Curves.

Let \( A/F \) be an abelian variety over a field \( F \). By \( \text{End} A \) we mean the ring of endomorphisms of \( A_{F_{\text{sep}}} \), endowed with the structure of a \( \mathfrak{g}_F \)-module. It is known that \( \text{End}^0 A = \text{End} A \otimes \mathbb{Q} \) is a semisimple \( \mathbb{Q} \)-algebra and \( \text{End} A \) is an order in \( \text{End}^0 A \). When \( F \) has characteristic 0 and \( A = E \) is an elliptic curve, \( \text{End}^0 E \) is either \( \mathbb{Q} \) or an imaginary quadratic field \( K \); in the latter case we say that \( E \) has **complex multiplication** (CM). Thus \( \text{End} E \) is\(^1\) an imaginary quadratic order \( \mathcal{O} \), and we say that \( E \) has \( \mathcal{O}\)-CM. We summarize some basic facts of CM theory [CCRS13, Fact1]. Proofs are found throughout the literature [Co89, La87, Si94].

**Fact 1.**

1. **There exists at least one complex elliptic curve with \( \mathcal{O}\)-CM.**
2. **Let \( E, E' \) be any two complex elliptic curves with \( \mathcal{O}(\Delta) \)-CM. The\( \ j \)-invariants \( j(E) \) and \( j(E') \) are Galois conjugate algebraic integers. In other words, \( j(E) \) is a root of some monic polynomial with \( \mathbb{Z} \)-coefficients, and if \( P(t) \) is the minimal such polynomial, \( P(j(E')) = 0 \) also.**
3. **Thus there is a unique irreducible, monic polynomial \( H_\Delta(t) \in \mathbb{Z}[t] \) whose roots are the \( \ j \)-invariants of all \( \mathcal{O}(\Delta) \)-CM elliptic curves.**
4. **The degree of \( H_\Delta(t) \) is the class number \( h(\Delta) = \# \text{Pic} \mathcal{O}(\Delta) \). In particular, when \( \mathcal{O} = \mathcal{O}_K \) we have \( \deg(H_\Delta(t)) = h(K) \), the class number of \( K \).**
5. **Let \( F_\Delta := \mathbb{Q}(t)/H_\Delta(t) \). Then \( F_\Delta \) can be embedded in the real numbers, so in particular is linearly disjoint from the imaginary quadratic field \( K \). Let \( K_\Delta \) denote the compositum of \( F_\Delta \) and \( K \). Then \( K_\Delta/K \) is abelian, with Galois group canonically isomorphic to \( \text{Pic} \mathcal{O} \).**

\(^1\)An imaginary quadratic order \( \mathcal{O} \) has a unique nontrivial ring automorphism (complex conjugation), so there are two different ways to identify \( \mathcal{O} \) with \( \text{End} E \). As is standard, we take the identification which is compatible with the action of \( \mathcal{O} \) on the tangent space at the origin.
Let $E/C$ be an elliptic curve with $\mathcal{O}$-CM; by the Uniformization Theorem there is a lattice $\Lambda \subset \mathbb{C}$ such that $1 \in \Lambda$ and $E \cong \mathbb{C}/\Lambda$. Then $\Lambda$ is a fractional $\mathcal{O}$-ideal of $K$ and $\mathcal{O}(\Lambda) = \mathcal{O}$, so by Lemma 3.1 $\Lambda$ is a projective $\mathcal{O}$-module. Conversely, if $\Lambda$ is a rank one projective $\mathcal{O}$-module, then $E_\Lambda = (\Lambda \otimes \mathcal{O} \mathbb{C})/\Lambda$ is an elliptic curve, and the $\mathbb{C}$-isomorphism class of $E_\Lambda$ depends only on the isomorphism class of $\Lambda$ as an $\mathcal{O}$-module. The map $\Lambda \mapsto E_\Lambda$ induces a bijection from Pic$\mathcal{O}$ to the set of isomorphism classes of elliptic curves $E/C$ with $\text{End} \ E \cong \mathcal{O}$.

3.3. $\mathbb{R}$-structures on Elliptic Curves.

For a subset $S \subset \mathbb{C}$, we put $\overline{S} = \{ \overline{z} \mid z \in S \}$. We say a lattice $\Lambda \subset \mathbb{C}$ is real if $\overline{\Lambda} = \Lambda$. For a lattice $\Lambda \subset \mathbb{C}$, we associate the complex torus $\mathbb{C}/\Lambda$ to the Weierstrass equation

$$\Lambda : y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda),$$

via the Eisenstein series $g_2, g_3$ [Si94, Proposition VI.3.6].

**Lemma 3.2.** a) Let $E_{/\mathbb{C}}$ be an elliptic curve. The following are equivalent:

(i) There is an elliptic curve $(E_0)_{/\mathbb{R}}$ such that $(E_0)_{/\mathbb{C}} \cong E$.
(ii) $j(E) \in \mathbb{R}$.
(iii) $E \cong E_\Lambda$ for a real lattice $\Lambda$.

b) Let $\Lambda_1, \Lambda_2$ be real lattices. The following are equivalent:

(i) There is $\alpha \in \mathbb{R}^\times$ such that $\Lambda_2 = \alpha \Lambda_1$.
(ii) $E_{\Lambda_1}$ and $E_{\Lambda_2}$ are isomorphic as elliptic curves over $\mathbb{R}$.

**Proof.** To prove a), it is immediate that (i) $\implies$ (ii). For (ii) $\implies$ (iii), take $g_2, g_3 \in \mathbb{R}$ such that $E' : y^2 = 4x^3 - g_2x - g_3$ is an elliptic curve with $j$-invariant $j(E)$ [Si86, Proposition III.1.4]. There is a unique lattice $\Lambda \subset \mathbb{C}$ such that $g_2(\Lambda) = g_2$ and $g_3(\Lambda) = g_3$ and thus $E' \cong \mathbb{C}/\Lambda$ [Si86, Theorem VI.5.1]. Since $j(E) = j(E')$ and $\mathbb{C}$ is algebraically closed, we have $E \cong \mathbb{C}/\Lambda$. Since $g_2(\Lambda) = g_2(\overline{\Lambda}), g_3(\Lambda) = g_3(\overline{\Lambda})$, we have $\Lambda = \overline{\Lambda}$. Finally, if $\Lambda$ is a real lattice then $g_2(\Lambda), g_3(\Lambda) \in \mathbb{R}$ and (iii) $\implies$ (i) is immediate. (Alternately, since $\overline{\Lambda} = \Lambda$, complex conjugation on $\mathbb{C}$ descends to an antiholomorphic involution on $\mathbb{C}/\Lambda$ and thus gives descent data for an $\mathbb{R}$-structure on $E$.)

To prove b), if $\Lambda_1, \Lambda_2 \subset \mathbb{C}$, we have $\mathbb{C}/\Lambda_1 \cong \mathbb{C}/\Lambda_2$ if $\Lambda_2 = \alpha \Lambda_1$ for some $\alpha \in \mathbb{C}^\times$. In terms of Weierstrass equations, $E_{\alpha \Lambda}$ is the quadratic twist of $E_\Lambda$ by $\alpha^2$. Thus $E_{\alpha \Lambda} \cong E_{\alpha \Lambda_1} = E_{\Lambda_2}$ if $\alpha \in \mathbb{R}$. Conversely, if $E_{\alpha \Lambda_1} \cong E_{\Lambda_2}$, then the standard theory of Weierstrass equations [Si86, § III.1] shows: there is $\alpha \in \mathbb{R}^\times$ with $g_2(\Lambda_2) = \alpha^4g_2(\Lambda_1) = g_2(\alpha^{-1}\Lambda_1), g_3(\Lambda_2) = \alpha^6g_3(\Lambda_1) = g_3(\alpha^{-1}\Lambda_1)$ and thus $\Lambda_2 = \alpha^{-1}\Lambda_1$. □

**Lemma 3.3.** a) Let $\Lambda$ be a real lattice. If $j(E_\Lambda) \neq 1728$, then $E_{i\Lambda}$ and $E_{\Lambda}$ are isomorphic as $\mathbb{C}$-elliptic curves but not as $\mathbb{R}$-elliptic curves. If $j(E_\Lambda) = 1728$, then $E_{\mathbb{R}}$ and $E_{\mathbb{C}}$ are isomorphic as $\mathbb{C}$-elliptic curves but not as $\mathbb{R}$-elliptic curves.

b) Let $j \in \mathbb{R}$. Then there are precisely two $\mathbb{R}$-isomorphism classes of elliptic curves $E_{/\mathbb{R}}$ with $j(E) = j$.

**Proof.** If $j \neq 1728$ then $g_3(\Lambda) \neq 0$, so $g_3(i\Lambda) = -g_3(\Lambda)$, whereas as above any real change of variables takes $g_3(\Lambda) \mapsto \alpha^{-5}g_3(\Lambda)$ for some $\alpha \in \mathbb{R}^\times$. If $j = 1728$ then $g_3(\Lambda) = 0$, so the above argument shows instead that $i\Lambda = \Lambda$ (as it should, since $\Lambda$ is homothetic to $\mathbb{Z}[i]$). In this case $g_2(\Lambda) \neq 0$ and $g_2(\mathbb{R}) = -g_2(\Lambda)$, whereas any real change of variables takes $g_3(\Lambda) \mapsto \alpha^{-4}g_3(\Lambda)$ for some $\alpha \in \mathbb{R}^\times$. The standard theory of real elliptic curves gives b) [Si94, Prop. V.2.2]. □
Lemma 3.4. Let $\mathcal{O}$ be an order in the imaginary quadratic field $K$, and let $I$ be a proper fractional $\mathcal{O}$-ideal. The following are equivalent:

(i) $[I] = [\mathcal{O}] \in \text{Pic} \mathcal{O}$.
(ii) $I^2$ is principal.
(iii) $j(I) \in \mathbb{R}$.

Proof. a) Since $I^2 = N_{K/\mathbb{Q}}(I)\mathcal{O}$, we have $[I] = [\mathcal{O}]^{-1} \in \text{Pic} \mathcal{O}$, so (i) $\iff$ (ii). Work of Shimura gives (ii) $\iff$ (iii) [Sh94, (5.4.3)].

For an imaginary quadratic discriminant $\Delta$, let $\tau_\Delta = \frac{\Delta + \sqrt{\Delta}}{2}$, so $\mathcal{O}(\Delta) = \mathbb{Z}[\frac{\Delta + \sqrt{\Delta}}{2}]$ is the imaginary quadratic order of discriminant $\Delta$. Then $j(C/\mathcal{O}) = j(\tau_\Delta)$. Let $\sigma_1, \ldots, \sigma_h(\Delta) : \mathbb{Q}(j(\tau_\Delta))/\mathbb{Q} \hookrightarrow \mathbb{C}$ be the $\# \text{Pic} \mathcal{O}(\Delta)$ field embeddings, with $\sigma_1$ taken to be inclusion. By Lemma 3.4, $j(\tau_\Delta) \in \mathbb{R}$. The other embeddings $\sigma_2, \ldots, \sigma_h$ may in general be either real or complex: Lemma 3.4 implies that the number of real embeddings is $\#(\text{Pic} \mathcal{O})[2]$.

Lemma 3.5. For an imaginary quadratic discriminant $\Delta$, let $r$ be the number of distinct odd prime divisors of $\Delta$. We define $\mu$ as follows:

$$
\mu = \begin{cases} 
  r, \Delta \equiv 1 \pmod{4} \text{ or } \Delta \equiv 4 \pmod{16} \\
  r + 1, \Delta \equiv 8, 12 \pmod{16} \text{ or } \Delta \equiv 16 \pmod{32} \\
  r + 2, \Delta \equiv 0 \pmod{32}.
\end{cases}
$$

a) We have $(\text{Pic} \mathcal{O}(\Delta))[2] \cong (\mathbb{Z}/2\mathbb{Z})^{\mu}$. 

b) There are precisely $2^\mu$ $\mathbb{R}$-homothety classes of $\mathcal{O}$-CM real lattices.

Proof. Part a) is essentially due to Gauss [Co89, Proposition 3.11],[HK13, Theorem 5.6.11]. Part b) is obtained by combining part a) with Lemmas 3.3 and 3.4. 

A fractional $\mathcal{O}$-ideal $I$ is primitive if $I \subset \mathcal{O}$ and for all $e \geq 2$, $I \not\subset e\mathcal{O}$.

Lemma 3.6. a) Let $E \cong \mathbb{R} E_\Lambda$ be a real $\mathcal{O}$-CM elliptic curve. The $\mathbb{R}$-homothety class of $\Lambda$ contains a unique primitive $\mathcal{O}$-ideal $I$. The ideal $I$ is proper and real.

b) [HK13, Theorem 5.6.4] There are precisely $2^\mu$ primitive proper real $\mathcal{O}$-ideals.

Proof. The lattice $\Lambda$ contains an element $a + bi$ with $a \neq 0$. Since $\Lambda$ is real by Lemma 3.2, we have $a - bi \in \Lambda$ and thus also $2a = (a + bi) + (a - bi) \in \Lambda$. Then $\frac{\Lambda}{\mathcal{O}}$ is a proper $\mathcal{O}$-ideal which is $\mathbb{R}$-homothetic to $\Lambda$. If two fractional $\mathcal{O}$-ideals are $\mathbb{R}$-homothetic, then one is real iff the other is real, and the $\mathbb{R}$-homothety class of any fractional $\mathcal{O}$-ideal contains a unique primitive $\mathcal{O}$-ideal. To prove part b), combine part a) with Lemma 3.5b).

Ideals of this type are completely classified. We use the following notation: for $\alpha, \beta \in \mathbb{C}$ which are linearly independent over $\mathbb{R}$, we define the lattice

$$
[\alpha, \beta] = \{a\alpha + b\beta \mid a, b \in \mathbb{Z}\}.
$$

Theorem 3.7. [HK13, Theorem 5.6.4] Let $\mathcal{O}$ be an order in $K$ of discriminant $\Delta$. A primitive proper $\mathcal{O}$-ideal $I$ is real iff it is one of the following two types:

1. $I = \left[a, \frac{\sqrt{\Delta}}{2}\right]$, where $a \in \mathbb{Z}^+$, $4a|\Delta$ and $\gcd\left(a, \frac{\Delta}{4a}\right) = 1$

2. $I = \left[a, \frac{a + \sqrt{\Delta}}{2}\right]$, where $a \in \mathbb{Z}^+$, $4a|a^2 - \Delta$ and $\gcd\left(a, \frac{a^2 - \Delta}{4a}\right) = 1$. 

Moreover, if \( \Delta \equiv 1 \) (mod 4), there are no such ideals of type (1).

**Corollary 3.8.** Let \( I \) be a primitive proper real \( \mathcal{O} \)-ideal. Then \( |\mathcal{O} : I| \mid \Delta \).

**Proof.** For all ideals \( I \) of the form (1) and (2) above, we have that \( |\mathcal{O} : I| = a \mid \Delta \) [HK13, Theorem 5.4.2]. □

Let \( R \) be a domain with fraction field \( K \). For fractional \( R \)-ideals \( I \) and \( J \) we define the colon ideal

\[
(J : I) = \{ x \in K \mid xI \subseteq J \};
\]

it is also a fractional \( R \)-ideal. If \( I \) is invertible, then \( (J : I) = I^{-1}J \).

Let \( E/F \) be an \( \mathcal{O} \)-CM elliptic curve defined over a number field, and let \( I \) be a nonzero ideal of \( \mathcal{O} \). We then have an \( I \)-torsion kernel

\[
E[I] = \{ x \in E(\mathcal{F}) \mid \forall \alpha \in I, \ ax = 0 \}.
\]

If \( I \subset J \) then \( E[J] \subset E[I] \). Since \( I \) contains the positive integer \( |\mathcal{O} : I| \), \( E[I] \subset E[|\mathcal{O} : I|] \) and thus \( E[I] \) is a finite \( \mathcal{O} \)-submodule of \( E(\mathcal{F}) \). Evidently \( E[I] \) is stabilized by the action of \( g_{FK} \), so it corresponds to a finite étale \( FK \)-subgroup scheme of \( E \). If \( FK \supseteq F \), then the nontrivial element \( c \in \text{Aut}(FK/F) \) acts as complex conjugation on \( \text{End}^0 E \) and thus \( c(E[I]) = E[\overline{I}] \). It follows that \( E[I] \) is defined over \( F \) iff \( I \) is real. For any nonzero ideal \( I \) of \( \mathcal{O} \) we have an isogeny \( E \to E/E[I] \), defined over \( FK \) in general and over \( F \) if \( I \) is real.

Let \( \iota : F \hookrightarrow \mathbb{C} \) be a field embedding. Then \( E \cong_{\mathbb{C}} E_{\lambda} \) for some proper \( \mathcal{O} \)-ideal \( \lambda \). Observe that the kernel of the natural map \( E_{\lambda} \to E_{(\lambda, I)} \) is \( (\lambda : I)/\lambda = (\mathbb{C}/\lambda)[I] \), so \( E/E[I] \cong_{\mathbb{C}} E_{(\lambda, I)} \). Furthermore, if \( I \) is invertible, then \( E_{(\lambda, I)} = E_{I^{-1}\lambda} \). Thus we get an explicit description of the \( I \)-torsion kernel and the associated isogeny in terms of uniformizing lattices.

Now let \( \iota : F \hookrightarrow \mathbb{R} \) be a field embedding. By Lemma 3.2 we may choose \( \lambda \) to be a real lattice. Suppose that \( I \) is moreover real. Then so is \( (\lambda : I) \) and thus \( E_{\lambda} \to E_{(\lambda, I)} \) is an explicit description of the \( I \)-torsion kernel and the associated isogeny in terms of uniformizing real lattices.

**Theorem 3.9.** Let \( \Delta \) be an imaginary quadratic discriminant.

a) Let \( F \subset \mathbb{R} \), and let \( E_{/F} \) be an \( \mathcal{O} \)-CM elliptic curve. Then there is an \( \mathcal{O} \)-CM elliptic curve \( E_{/F} \) such that \( E \cong_{\mathbb{R}} E_{\mathcal{O}} \) and an \( F \)-rational isogeny \( \varphi : E \to E' \).

b) Let \( N \) be a positive integer which is prime to \( \Delta \). Then the isogeny \( \varphi \) of part a) induces a \( g_F \)-module isomorphism \( E[N] \cong E'[N] \).

**Proof.** There is a primitive, proper, real \( \mathcal{O} \)-ideal \( \lambda \) such that \( E \cong_{\mathbb{R}} E_{\lambda} \). Let \( I \) be any proper, real \( \mathcal{O} \)-ideal. As above there is an \( F \)-rational isogeny \( \varphi_I : E \to E/E[I] \) which over \( \mathbb{R} \) is given as \( E_{\lambda} \to E_{(\lambda, I)} = E_{I^{-1}\lambda} \). Taking \( \varphi = \varphi_{\lambda} \) establishes part a). The degree of \( \varphi_{\lambda} \) is \( |\mathcal{O} : \lambda| \), which by Corollary 3.8 divides \( \Delta \). It follows that \( \deg \varphi_{\lambda} \) is prime to \( N \) and thus \( \varphi_{\lambda} : E[N] \cong E'[N] \). □

**Remark 3.10.** Recently we found a paper of S. Kwon [Kw99] which contains related results. Especially, the discussion preceding Theorem 3.9 is very closely related to (but slightly more general than) [Kw99, Prop. 2.3.1], and the classification of primitive, proper real ideals in an imaginary quadratic order is given in [Kw99, §3] and is used to classify cyclic isogenies on CM elliptic curves rational over \( \mathbb{Q}(j(E)) \).
3.4. Cartan Subgroups.

Let \( \Lambda \subset \mathbb{C} \) be a lattice, and let \( E_\Lambda = \mathbb{C}/\Lambda \). For \( N \in \mathbb{Z}^+ \) and \( \ell \in \mathcal{P} \), put

\[
\Lambda_N = \left( \frac{1}{N} \right) \Lambda / \Lambda = E_\Lambda[N]
\]

\[
T_\ell \Lambda = \lim_{\leftarrow} \Lambda_{\ell^n} = T_\ell(E_\Lambda).
\]

If \( F \subset \mathbb{C} \) and \( E/F \) is an elliptic curve, then \( E/C \cong E_\Lambda \) for some lattice \( \Lambda \), uniquely determined up to homothety. If \( F \subset \mathbb{R} \), then \( E/R \cong E_\Lambda \) for some real lattice \( \Lambda \), uniquely determined up to real homothety.

We have \( E(\mathbb{C})[\text{tors}] = E(\overline{\mathbb{F}})[\text{tors}] \), so \( \text{Aut}(\mathbb{C}/F) \) acts on \( \Lambda_N, T_\ell \Lambda \) and \( \hat{\Lambda} \). We assume that \( \text{Aut}(\mathbb{C}/F) \) is a normal subgroup of \( \text{Aut}(\mathbb{C}/F) \): this holds if \( F \) is a number field or if \( F = \mathbb{R} \). Then we get an induced action of \( g_F \) on \( \Lambda_N \). If moreover \( F \subset \mathbb{R} \), then complex conjugation \( c \in \text{Aut}(\mathbb{C}/\mathbb{R}) \subset \text{Aut}(\mathbb{C}/F) \rightarrow g_F \) acts on \( \Lambda_N \).

Let \( O \) be an imaginary quadratic order. For \( N, \ell \) as above, consider the \( O \)-algebras

\[
O_N = O \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z},
\]

\[
T_\ell(O) = O \otimes_{\mathbb{Z}} \mathbb{Z}_\ell = \lim_{\leftarrow} O_{\ell^n}.
\]

Let \( E/F \) be an \( O \)-CM elliptic curve. As above, there is a proper integral \( O \)-ideal \( \Lambda \), with uniquely determined class in \( \text{Pic} O \), such that \( E/C \cong E_\Lambda \). Then \( \Lambda_N \) (resp. \( T_\ell \Lambda \)) has a natural \( O_N \)-module (resp. \( T_\ell(O) \)-module) structure. If \( F \subset \mathbb{R} \) we may take \( \Lambda \) to be a real ideal: \( \overline{\Lambda} = \Lambda \).

**Lemma 3.11.**

a) For every \( N \in \mathbb{Z}^+ \), \( \Lambda_N = E[N] \) is free of rank 1 as an \( O_N \)-module.

b) \( T_\ell \Lambda = T_\ell(E) \) is free of rank 1 as \( T_\ell(O) \)-module.

**Proof.** Part b) is known by work of Serre and Tate [ST68, p. 502] while part a) can be deduced from work of Parish [Pa89, Lemma 1]. Either part can be used to deduce the other. \( \square \)

In particular, for all primes \( \ell \) we have a homomorphism of \( \mathbb{Z}_\ell \)-algebras

\[
\iota_\ell : T_\ell(O) \rightarrow \text{End} T_\ell(E).
\]

The map \( \iota_\ell \) is \( g_F \)-equivariant; further, it is injective with torsion-free cokernel [Mi86, Lemma 12.2]. Tensoring with \( \mathbb{Z}/\ell^n\mathbb{Z} \) and applying primary decomposition, we get for each \( N \in \mathbb{Z}^+ \) an injective \( g_F \)-equivariant ring homomorphism

\[
O_N \hookrightarrow \text{End} E[N].
\]

Tensoring to \( \mathbb{Q}_\ell \) gives

\[
\iota_\ell^0 : V_\ell(O) \hookrightarrow \text{End} V_\ell(E).
\]

We define the **Cartan subalgebras**

\[
C_\ell = \iota_\ell(T_\ell(O)) \subset \text{End} T_\ell(E),
\]

\[
C_\ell^0 = \iota_\ell^0(V_\ell(O)) \subset \text{End} V_\ell(E)
\]

and the **Cartan subgroups**

\[
C_\ell^\times \subset \text{Aut} T_\ell(E),
\]
\((C_{\ell}^0)^{\times} \subset \text{Aut} \ V_\ell(E)\).

Then \(C_{\ell}^0 \cong K \otimes \mathbb{Q}_\ell\) is a maximal étale subalgebra of \(\text{End} \ V_\ell(E) \cong M_2(\mathbb{Q}_\ell)\). We may view \(C_{\ell}^0 \to M_2(\mathbb{Q}_\ell)\) as the regular representation. We write \(C(C_{\ell}^0)\) for the commutant and \(N(C_{\ell}^0)\) for the normalizer of \(C_{\ell}^0\) inside \(\text{End} \ V_\ell(E)\). By the Double Centralizer Theorem [Pi82, §12.7], we have

\[ C(C_{\ell}^0) = C_{\ell}^0. \]

Using the Skolem-Noether Theorem [Pi82, §12.6], we find that

\[ NC_{\ell}^0/(C_{\ell}^0)^{\times} \cong \text{Aut}_{\mathbb{Q}_\ell} C_{\ell}^0 \]

has order 2.

The fixed field of the kernel of the representation \(g_F \to V_\ell(O)\) is \(FK\), so

\[ \rho_{\ell^\infty}(g_{FK}) \subseteq C_{\ell}^0, \]

and if \(FK \supseteq F\) then

\[ \rho_{\ell^\infty}(g_F) \not\subseteq C_{\ell}^0. \]

In fact [ST68, §4, Corollary 2] we have

\[ \rho_{\ell^\infty}(g_{FK}) \subseteq \iota_{\ell}(T_\ell(O)^{\times}). \]

Moreover, \(g_F\)-equivariance of \(i_{\ell}^0\) gives

\[ \rho_{\ell^\infty}(g_F) \subseteq NC_{\ell}^0. \]

This recovers a standard result of Serre [Se66, Theorem 5].

**Lemma 3.12.** Let \(G_{\ell^\infty} = \rho_{\ell^\infty}(g_F)\) be the image of the \(\ell\)-adic Galois representation. The following are equivalent:

(i) \(G_{\ell^\infty}\) lies in the Cartan subgroup.

(ii) \(G_{\ell^\infty}\) is commutative.

(iii) \(K \subseteq F\).

We now deduce a stronger version of a result of Serre [CCRS13, Lemma 15]. Let us first note the following in the case \(\Lambda = \mathcal{O}\).

**Lemma 3.13.** Let \(K = \mathbb{Q}(\sqrt{\Delta_0})\) be an imaginary quadratic field, and let \(\mathcal{O}\) be an order in \(K\) of discriminant \(\Delta = \ell^2 \Delta_0\): thus \(\mathcal{O} = \mathbb{Z}\left[\frac{\Delta+\sqrt{\Delta}}{2}\right]\). Let \(c\) be the nontrivial element of \(\text{Aut}(K/\mathbb{Q})\). Let \(N \geq 2\), put \(\mathcal{O}_N = (1/N)\mathcal{O}/\mathcal{O}\) and \(i\mathcal{O}_N = (\frac{i}{N})\mathcal{O}/(i\mathcal{O})\).

(a) If \(\Delta\) is even or \(N\) is odd, then \(\frac{1}{N}, \frac{\sqrt{\Delta}}{N}\) (resp. \(\frac{i}{N}, \frac{\sqrt{\Delta}}{N}\)) is a \(\mathbb{Z}/N\mathbb{Z}\)-basis for \(\mathcal{O}_N\) (resp. \(i\mathcal{O}_N\)). The corresponding matrix of \(c\) is

\[
\begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix}
\]

(b) In all cases \(\frac{1}{N}, \frac{\Delta+\sqrt{\Delta}}{2N}\) (resp. \(i\left(\frac{\Delta+\sqrt{\Delta}}{2N}\right)\)) is a \(\mathbb{Z}/N\mathbb{Z}\)-basis for \(\mathcal{O}_N\) (resp. \(i\mathcal{O}_N\)). The corresponding matrix of \(c\) is

\[
\begin{bmatrix}
1 & \Delta \\
0 & -1 \\
\end{bmatrix}
\]

**Corollary 3.14.**

(a) If \(N \geq 3\), then \(c\) acts nontrivially on \(\mathcal{O}_N\).

(b) If \(N = 2\), then \(c\) acts nontrivially on \(i\mathcal{O}_N\) iff \(\Delta\) is odd.

**Lemma 3.15.** Let \(K\) be an imaginary quadratic field, and let \(\mathcal{O}\) be an order in \(K\) of discriminant \(\Delta = \ell^2 \Delta_0\). Let \(F\) be a field of characteristic 0, and let \(E_{\ell^\prime}/F\) be an \(\mathcal{O}\)-CM elliptic curve. Let \(N \in \mathbb{Z}^+\), and suppose at least one of the following holds:

* \(N \geq 3\);
\begin{itemize}
  \item $N = 2$ and $\Delta$ is odd.
\end{itemize}

Then $F(E[N]) \supset K$.

\textbf{Proof.} We may certainly assume $K \nsubseteq F$. Let $\sigma \in g_F$ be any element which restricts nontrivially to $KF$. Then $\sigma$ acts on $O_N$ as $c$ acts on $O_N$, and by Corollary 3.14, this action is nontrivial. Since $\iota_N : O_N \hookrightarrow \text{End} E[N]$ is injective and $g_F$-equivariant, it follows that $\sigma$ acts nontrivially on $\text{End} E[N]$. For any $G$-module $M$, if $\sigma \in G$ acts nontrivially on $\text{End}(M)$ then $\sigma$ acts nontrivially on $M$.\hfill $\square$

\section{Ray Class Field Containment.}

\textbf{Theorem 3.16.} Let $O$ be an order in an imaginary quadratic field $K$. Let $F$ be a field of characteristic 0, and let $E/F$ be an $O$-CM elliptic curve. Let $N \in \mathbb{Z}^+$. Let $\mathfrak{h}_F : E \to \mathbb{P}^1$ be a \textbf{Weber function} for $E$: that is, $\mathfrak{h}$ is the composition of the quotient map $E \to E/(\text{Aut} E)$ with an isomorphism $E/(\text{Aut} E) \cong \mathbb{P}^1$. Then the field $F(K(\mathfrak{h}(E[N])))$ contains the $N$-ray class field $K^{(N)}$ of $K$.

\textbf{Remark 3.17.} When $O = O_K$, the equality $K(j(C/O_K), h(E[N])) = K^{(N)}$ is one of the central results of classical CM theory [Si94, Theorem II.5.6].

\textbf{Proof.} We use the results and notation of Lang [La87, §10.3]. Applying Theorem 7 first with $a = O$ and $u = \frac{1}{N}$ and then with $a = O_K$ and $u = \frac{1}{N}$. We observe that for an idele $b$, $bO = O \Rightarrow bO_K = O_K$. This is much as in [La87, Thm. 6, §10.3]. We conclude

$$L \supset K(j(C/O), h(\frac{1}{N} + O)) \supset K(j(C/O_K), h(\frac{1}{N} + O_K)).$$

But as an $O_K$-module, $\frac{1}{N}O_K/O_K$ is generated by $\frac{1}{N} + O_K$ [La87, p. 135], so

$$K(j(C/O_K), h(\frac{1}{N} + O_K)) = K(j(C/O_K), h(E[N])) = K^{(N)}.$$\hfill $\square$

\section{Results on Torsion Points on CM Elliptic Curves}

Throughout this section we will use the following setup: $O$ is an imaginary quadratic order with fraction field $K$. Let $\Delta_K$ be the discriminant of $O_K$, $f$ the conductor of $O$, and $\Delta$ the discriminant of $O$, so $\Delta = f^2 \Delta_K$. Let $F$ be a subfield of $C$, and let $E/F$ be an $O$-CM elliptic curve. Again $h_{/F}$ will denote a Weber function.

\subsection{Points of Order 2.}

Let $O$ be an imaginary quadratic order of discriminant $\Delta < -4$, with fraction field $K$. Let $E_F$ be an $O$-CM elliptic curve. Let $F = \mathbb{Q}(j(E))$ and let $L = F(E[2])$, so $L/F$ is Galois of degree dividing 6. By Fact 1, the isomorphism class of $F$ depends only on $\Delta$. Since $\Delta < -4$, the $x$-coordinate is a Weber function on $E$, and thus $L$ does not depend upon the chosen Weierstrass model (any two $O$-CM elliptic curves with the same $j$-invariant are \textbf{quadratic} twists of each other, and 2-torsion points are invariant under quadratic twist). Thus as an abstract number field and a Galois extension thereof, $F$ and $L$ depend only on $\Delta$.

\textbf{Theorem 4.1.} (Parish) \textit{For all $\Delta < -4$, $F \nsubseteq L$.}

\textbf{Proof.} Equivalently, $E$ does not have full 2-torsion defined over $F = \mathbb{Q}(j(E))$. This follows from more precise results of Parish [Pa89, Table 1].\hfill $\square$
**Theorem 4.2.** Let $O$ be an imaginary quadratic order of discriminant $\Delta < -4$ and with fraction field $K$. Let $E/O$ be an elliptic curve with $O$-CM. Let $F = \mathbb{Q}(j(E))$, and let $L = F(E[2])$.

a) We have $K \subset L$ iff $\Delta$ is odd.

b) If $\Delta \equiv 1 \pmod{8}$, then $L = FK$ and $[L : F] = 2$.

c) If $\Delta \equiv 5 \pmod{8}$ and $K \not= \mathbb{Q}(\sqrt{-3})$ then $L = K^{(2)}F$ and $[L : F] = 6$.

d) If $\Delta$ is even, then $[L : F] = 2$.

**Proof.** a) If $\Delta$ is odd, then Lemma 3.15 gives $K \subset L$. Suppose $\Delta$ is even. Then Lemma 3.13 implies $E[2](\mathbb{R}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since $K \not\subset \mathbb{R}$, the result follows.

b) If $\Delta \equiv 1 \pmod{8}$, then the mod 2 Cartan subgroup is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ \times $(\mathbb{Z}/2\mathbb{Z})^2$, better known as the trivial group. It follows that $FK(E[2]) = FK$.

together with part a) this shows $L = F(E[2]) = FK$.

c) By part a) we have $K \subset L$, so by Theorem 3.16 we have $L \supset K^{(2)}F$. Since $\Delta$ is odd, $K(j)$ and $K^{(2)}$ are linearly disjoint over $K^{(1)}$, so

$$[K^{(2)} : K^{(1)}] = [K(j)K^{(2)} : K(j)K^{(1)}] = [FK^{(2)} : FK^{(1)}] \quad \text{and} \quad [L : FK] = \frac{[L : F]}{2}.$$ 

Since $K \not= \mathbb{Q}(\sqrt{-3})$, we have (c.f. Proposition 4.12) $[K^{(2)} : K^{(1)}] = 3$. Since for any elliptic curve $E/F$ we have $[F(E[2]) : F] \mid 6$, the result follows.

d) Since $\Delta$ is even, the mod 2 Cartan subgroup is cyclic of order $2^2 - 2 = 2$, and thus $[FK(E[2]) : FK] \mid 2$. It follows by using the result of part a) that $K \not\subset F(E[2])$, or just the fact that $[L : F] \mid 6$ so we cannot have $[L : F] = 4$, that $[L : F] \mid 2$.

Combining with Theorem 4.1 we get the result.

**Corollary 4.3.** Suppose $\Delta \neq -4$. Let $F$ be a number field, and let $E/F$ be an $O(\Delta)$-CM elliptic curve. If $\mathbb{Z}/2 \times \mathbb{Z}/2 \subset E(F)$, then $2 \mid [F : \mathbb{Q}]$.

**Proof.** If $\Delta = -3$, then by Theorem 4.2a) we have $F \supset K$ hence $2 \mid [F : \mathbb{Q}]$. Otherwise we have $\Delta < -4$, so $\mathbb{Q}(E[2]) = \mathbb{Q}(h(E[2]))$, and the 2-torsion field is independent of the model of $E$. We may thus assume without loss of generality that $E$ is obtained by base extension from an elliptic curve $E/\mathbb{Q}(j(E))$. Applying Theorem 4.2 we get

$$2 \mid [\mathbb{Q}(j(E), E[2]) : \mathbb{Q}(j(E))] \mid [F : \mathbb{Q}].$$

**Remark 4.4.** The elliptic curve $E/\mathbb{Q}$ : $y^2 = x^3 - x$ shows that the hypothesis $\Delta \neq -4$ in Corollary 4.3 is necessary.

**Corollary 4.5.** If $[F : \mathbb{Q}]$ is odd, $E/F$ is a CM elliptic curve and $(\mathbb{Z}/2\mathbb{Z})^2 \subset E(F)$ then $E(F)[12] = (\mathbb{Z}/2\mathbb{Z})^2$.

**Proof.** By Corollary 4.3 we may assume $\Delta = -4$. It is enough to show $E(F)$ has no subgroup isomorphic to $\mathbb{Z}/6\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. This follows from Table 2. 

This result will be sharpened in Corollary 7.11 below.

### 4.2. A refinement of the SPY bounds.

**Theorem 4.6.** Let $O$ be an imaginary quadratic order of discriminant $\Delta = j^2 \Delta_K$; put $K = \mathbb{Q}(\sqrt{\Delta})$. Let $F$ be a number field, and let $E/F$ an elliptic curve with $O$-CM. Let $w(K) = \#O_K^\times$. Suppose $E(F)[\text{tors}]$ contains a point of prime order $\ell > 2$. 

a) If $(\frac{\Delta}{\ell}) = -1$, then
\[ \left( \frac{2(\ell^2 - 1)}{w(K)} \right) h(K) | [FK : Q]. \]

b) If $(\frac{\Delta}{\ell}) = 1$, then
\[ \left( \frac{2(\ell - 1)}{w(K)} \right) h(K) | [FK : Q]. \]

c) If $(\frac{\Delta}{\ell}) = 0$, then:
1. $(\ell - 1)h(K) | [FK : Q]$ if $\ell$ is ramified in $K$.
2. $\left( \frac{2(\ell - 1)^2}{w(K)} \right) h(K) | [FK : Q]$ if $\ell$ is split in $K$.
3. $\left( \frac{2(\ell^2 - 1)}{w(K)} \right) h(K) | [FK : Q]$ if $\ell$ is inert in $K$.

**Proof.** The cases $\mathcal{O} = \mathcal{O}_K$ and $\ell \nmid \Delta$ of Theorem 4.6 were proved in [CCRS13, Theorem 2]. The hypothesis $\mathcal{O} = \mathcal{O}_K$ comes into the proof only via the statement that $K((j(E)))(h(E[N])) = K^{(N)}$, the $N$-ray class field of $K$. And in fact we used only that the former field contains the latter field, which holds for all $\mathcal{O}$ by Theorem 3.16. So it remains to consider the case in which $\ell \mid \Delta$. If $\Delta = -3$ then $\ell = 3$ and the result is clear; if $\Delta = -4$ there is no such $\ell$. Henceforth we assume $\Delta < -4$.

Since $\ell \mid \Delta$ we have by [CCRS13, §2.3] that
\[ \mathcal{O}_\ell = \mathcal{O} \otimes \mathbb{Z}/\ell \mathbb{Z} \cong \mathbb{F} \ell/((\ell)^2). \]

Thus its image $C_\ell = i(\mathcal{O}_\ell) \subset \text{End } E[\ell] \cong M_2(\mathbb{F}_\ell)$ is generated over the scalar matrices by a single nonzero nilpotent matrix $g$. Since the eigenvalues of $g$ are $\mathbb{F}_\ell$-rational we can put it in Jordan canonical form over $\mathbb{F}_\ell$. We get a choice of basis $e_1, e_2$ of $E[\ell]$ such that
\[ C_\ell \cong \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix} \mid \alpha, \beta \in \mathbb{F}_\ell \right\}. \]

Let $x = ae_1 + be_2 \in E(F)$ have order $\ell$. For all $S = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix} \in \rho_\ell(g_{FK})$ we have

\[ (\alpha a + \beta b)e_1 + (\alpha b)e_2 = Sx = x = ae_1 + be_2, \]

and thus
\[ (\alpha - 1)b = (\alpha - 1)a + \beta b = 0. \]

If $\alpha \neq 1$, then $b = 0$ and thus also $a = 0$ – contradiction – so $\alpha = 1$ and $\rho_\ell(g_{FK})$ consists of elements of the form $\begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}$. Hence $\rho_\ell(g_{FK})$ has size 1 or $\ell$.

Case 1: Suppose $\#\rho_\ell(g_{FK}) = 1$. By Theorem 3.16 we have $FK \supset K^{(\ell)}$, and the expression for $[K^{(\ell)} : K^{(1)}]$ found in [CCRS13, Corollary 9] gives
\[ [K^{(\ell)} : Q] = \frac{2(\ell - 1)h(K)}{w(K)} \left( \ell - \left( \frac{\Delta_K}{\ell} \right) \right) | [FK : Q]. \]

This gives the result, in fact with an extra factor of $\ell$ when $\ell \mid \Delta_K$.

Case 2: If $\#\rho_\ell(g_{FK}) = \ell$, then by Lemma 3.15 and Theorem 3.16 we have the following diagram of fields. Note $K^{(1)} \subset K(j(E))$, the ring class field of $K$ with conductor $\ell$.
Lemma 4.7. Let \( \Delta \) be an imaginary quadratic discriminant, and let \( K = \mathbb{Q}(\sqrt{\Delta}) \). Let \( F \) be a number field, and let \( E_{/F} \) be an elliptic curve. Let \( N \geq 3 \), and suppose \( (\mathbb{Z}/N\mathbb{Z})^2 \subset E(FK) \). Then:

a) We have \([\mathbb{Q}(\zeta_N) : \mathbb{Q}(\zeta_N) \cap F]\) \leq 2.

b) Suppose \( F \) is real. Then \( \mathbb{Q}(\zeta_N) \cap F = \mathbb{Q}(\zeta_N)^+ \).

c) Suppose \( \gcd(N, \Delta_K) = 1 \). Then \( \mathbb{Q}(\zeta_N) \subseteq FK \).

d) If \( K \not\subseteq F(\zeta_N) \), then \( \mathbb{Q}(\zeta_N) \subset F \).

e) Suppose \( N \) is an odd prime power. Then \( \mathbb{Q}(\zeta_N)^+ \subset F \).

f) Suppose \( N = 2^a \) with \( a \geq 3 \). Then \( \mathbb{Q}(\zeta_{N/2})^+ \subseteq F \).

Proof. a) Let \( \chi_N : g_F \to (\mathbb{Z}/N\mathbb{Z})^\times \) be the mod \( N \) cyclotomic character, and let \( H_F = \chi_N(g_F) \). As usual the Weil pairing gives \( \mathbb{Q}(\zeta_N) \subset FK \), so \( \chi_N(g_{FK}) \equiv 1 \), and thus \( \#H_F \leq 2 \). Moreover \( F = F(\zeta_N)^{H_F} \supseteq \mathbb{Q}(\zeta_N)^{H_F} \), and the result follows.

b) Since \( N \geq 3 \), \( \mathbb{Q}(\zeta_N) \not\subseteq F \) and by part a) we have \([\mathbb{Q}(\zeta_N) : \mathbb{Q}(\zeta_N) \cap F] = 2 \). Further, \( \mathbb{Q}(\zeta_N) \cap F \subset \mathbb{Q}(\zeta_N) \cap 

\mathbb{Q}(\zeta_N)^+ \).

c) We have \( \mathbb{Q}(\zeta_N) \subset FK \); if equality held, then \( K \subset \mathbb{Q}(\zeta_N) \). But \( K \) is ramified at some prime \( \ell \nmid N \) and \( \mathbb{Q}(\zeta_N) \) is ramified only at primes dividing \( N \).

d) The hypothesis implies that \( FK \) and \( F(\zeta_N) \) are linearly disjoint over \( F \), \( \chi_N|_{g_{FK}} \equiv 1 \) implies \( \#H_F = \{1\} \) and \( \mathbb{Q}(\zeta_N) \subset F \).

e) If \( N \) is an odd prime power, then \( (\mathbb{Z}/N\mathbb{Z})^\times \) is cyclic, so either \( H_F = 1 \) and \( \mathbb{Q}(\zeta_N) = \mathbb{Q}(\zeta_N)^{H_F} \subset F \) or \( H_F = \{\pm 1\} \) and \( \mathbb{Q}(\zeta_N)^+ = \mathbb{Q}(\zeta_N)^{H_F} \subset F \).
f) Since $N = 2^a$ with $a \geq 3$, $(\mathbb{Z}/N\mathbb{Z})^\times$ has three elements of order 2: $-1$ and $2^{a-1} \pm 1$. So we have $H_F \subset \{\pm 1, 2^{a-1} \pm 1\}$, and thus
\[
F \supset \mathbb{Q}(\zeta_N)^{H_F} \supset \mathbb{Q}(\zeta_N)^{\{\pm 1, 2^{a-1} \pm 1\}} = \mathbb{Q}(\zeta_{N/2})^+.
\]
\[\square\]

**Theorem 4.8.** (Real Cyclotomy I) Let $\Delta$ be an imaginary quadratic discriminant, and let $K = \mathbb{Q}(\sqrt{\Delta})$. Let $N \in \mathbb{Z}^+$ be such that $\gcd(N, \Delta) = 1$. Let $F \not\supset K$ be a number field, and let $E/F$ be an $O(\Delta)$-CM elliptic curve. Suppose that $E(F)$ contains a point of order $N$.

a) We have $(\mathbb{Z}/N\mathbb{Z})^2 \subset E(FK)$.

b) $F$ contains an index 2 subfield of $\mathbb{Q}(\zeta_N)$.

c) If $N$ is an odd prime power, then $\mathbb{Q}(\zeta_N)^+ \subset F$. If $N \geq 8$ is an even prime power, then $\mathbb{Q}(\zeta_N)^+ \subset F$.

d) If $F$ is real and $N \geq 3$, then $\mathbb{Q}(\zeta_N)^+ \subset F$.

**Proof.** a) We immediately reduce to the case that $N = \ell^a$ is a power of a prime number $\ell$. Let $T_{\ell}(O) = O \otimes \mathbb{Z}_{\ell}$, and identify $T_{\ell}(O)$ with its isomorphic image in $\text{End}_F(T_{\ell}(E))$. For $b \in \mathbb{Z}^+$, let $O_{\rho^b} = T_{\ell}(O)/\langle \ell^b \rangle$. The hypothesis $\gcd(N, \Delta) = 1$ implies that $T_{\ell}(O)$ is the maximal $\mathbb{Z}_{\ell}$-order in $K_{\ell} = K \otimes \mathbb{Q}_{\ell}$. We know that $T_{\ell}(E)$ is free of rank one as a $T_{\ell}(O)$-module by Lemma 3.11.

Case 1: Suppose $(\frac{\Delta}{\ell}) = 1$. Then $T_{\ell}(O) \cong \mathbb{Z}_{\ell} \times \mathbb{Z}_{\ell}$. Put $\iota = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $NT_{\ell}(O)^\times = \langle T_{\ell}(O)^\times, \iota \rangle$. Because $F$ does not contain $K$, there is $\sigma \in g_{\ell^a}$ such that $\rho_{\ell^a}(\sigma) \in NT_{\ell}(O)^\times \setminus T_{\ell}(O)^\times$. We may choose a $\mathbb{Z}_{\ell}$-basis $\tilde{e}_1, \tilde{e}_2$ of $T_{\ell}(E)$ and represent the $T_{\ell}(O)$-action on $T_{\ell}(E)$ via $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} | \alpha, \beta \in \mathbb{Z}_{\ell}$. Let $\tilde{V}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\tilde{V}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in T_{\ell}(O)$.

For $i = 1, 2$, let $\tilde{V}_i = \langle \tilde{e}_i \rangle_{\mathbb{Z}_{\ell}}$ and observe that each $\tilde{V}_i$ is an $T_{\ell}(O)$-submodule of $T_{\ell}(E)$. For $i = 1, 2$, put $V_i = \tilde{V}_i \mod \ell \subset E[\ell](\overline{F})$. Because we may write $\rho_{\ell^a}(\sigma)$ as $\iota M$ with $M \in T_{\ell}(O)^\times$, we have
\[
\rho_{\ell^a}(\sigma)(V_1) = V_2, \ \rho_{\ell^a}(\sigma)(V_2) = V_1
\]
and thus also
\[
\rho_{\ell^a}(\sigma)(V_1) = V_2, \ \rho_{\ell^a}(\sigma)(V_2) = V_1.
\]

Lift $P \in E[\ell^a](F)$ to a point $\tilde{P} = a\tilde{e}_1 + b\tilde{e}_2 \in T_{\ell}(E)$. We claim $a, b \in \mathbb{Z}_{\ell}^\times$: if not, $P' = [\ell^a-1]P \in V_1 \cup V_2 \setminus \{0\}$, and $\rho_{\ell^a}(P') = P'$ gives a contradiction. It follows that for $i = 1, 2$, $\langle \tilde{I}_i \tilde{P} \rangle_{\mathbb{Z}_{\ell}} = \tilde{V}_i$, so the $T_{\ell}(O)$-submodule generated by $\tilde{P}$ is $T_{\ell}(E)$.

Going modulo $\ell^a$ we get that the $O_{\ell^a}$-submodule generated by $P$ is $E[\ell^a]$, and thus $(\mathbb{Z}/\ell^a\mathbb{Z})^\times \subset E(FK)$.

Case 2: Suppose $(\frac{\Delta}{\ell}) = -1$. Then $T_{\ell}(O)$ is a discrete valuation ring with uniformizing element $\ell$ and fraction field $K_{\ell}$ and thus $O_{\ell^a}$ is a finite principal ring with maximal ideal $\langle \ell \rangle$. The elements of $(O_{\ell^a}, +)$ of order $\ell^a$ are precisely the units, so $O_{\ell^a}^\times$ acts transitively on the order $\ell^a$ elements of $E[\ell^a]$ and thus the $O_{\ell^a}$-submodule of $E[\ell^a]$ generated by $P$ is $E[\ell^a]$.

b) This follows from Lemma 4.7a).

c) If $N \geq 3$ is an odd prime power then by Lemma 4.7e) we have $\mathbb{Q}(\zeta_N)^+ \subset F$. 

Applying Lemma 4.7c) we get

\[ 1 < [FK : \mathbb{Q}(\zeta_N)] = [F : \mathbb{Q}(\zeta_N)^+] \]

The case of an even prime power \( N \geq 8 \) is similar but easier, since the strictness in the containment \( \mathbb{Q}(\zeta_N)^+ \subseteq F \) comes from Lemma 4.7f). For part d) we apply Lemma 4.7b) and deduce the strictness of the containment as above. \( \square \)

In the case when \( F \) is real, we can dispense with the hypothesis \( \gcd(\Delta, N) = 1 \).

**Theorem 4.9.** (Real Cyclotomy II) Let \( \mathcal{O} \) be an order of discriminant \( \Delta \) in an imaginary quadratic field \( K \), let \( F \) be a real number field, and let \( E/F \) be an \( \mathcal{O} \)-CM elliptic curve. Let \( N \geq 1 \), and suppose \( E(F) \) contains a point of order \( N \). Then:

a) \( \mathbb{Q}(\zeta_N) \subseteq FK \) and \( \mathbb{Q}(\zeta_N)^+ \subseteq F \).

b) If \( \gcd(N, \Delta_K) = 1 \) and \( N \geq 3 \), then \( \mathbb{Q}(\zeta_N)^+ \subseteq F \).

d) To establish \( \mathbb{Q}(\zeta_N)^+ \subseteq F \) we reduce to the case in which \( N = \ell^n \) is a prime power. It will then follow that \( \mathbb{Q}(\zeta_N)^+ = \mathbb{Q}(\zeta_N)^c \subset (FK)^c = F \).

First suppose \( \Delta = 4D \). Then \( \mathcal{O} = [1, \sqrt{D}] \), and \( \Lambda' \) is of type I or II as in Theorem 3.7 above. If it is of type I, it follows that

\[ \Lambda = \left[ \frac{t}{r}, \frac{\sqrt{D}}{r} \right], \text{ where } t \in \mathbb{N}, t|D \text{ and } \left( \frac{t}{\ell} \right) = 1. \]

With respect to this basis the action of complex conjugation is given by

\[ T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \]

and the action of \( \mathcal{O} \) on \( \Lambda \) is given by

\[ \alpha + \beta \sqrt{D} \mapsto \left[ \begin{array}{c} \alpha \\ \beta t \end{array} \right], \quad \left( \frac{D}{\ell} \right) = 1. \]

We may choose a \( \mathbb{Z}_\ell \)-basis \( \tilde{e}_1, \tilde{e}_2 \) for \( T_1(E) \) such that the image of the Cartan subgroup in \( \text{GL}_2(\mathbb{Z}_\ell) \) is

\[ \mathcal{C}_\ell^\times = \left\{ \left[ \begin{array}{c} \alpha \\ \beta t \end{array} \right], \alpha^2 - \beta^2 D \in \mathbb{Z}_\ell^\times \right\} \]

and the element \( c \in \mathfrak{g}_F \) induced by complex conjugation corresponds to \( \rho_\ell(c) = T \).

Let \( e_i = \tilde{e}_i \pmod{\ell^n} \). For \( x \in E(F)[\ell^n] \setminus E(F)[\ell^{n-1}] \), we may choose \( a, b \in \mathbb{Z}/\ell^n \mathbb{Z} \) such that \( x = ae_1 + be_2 \). Then

\[ ae_1 + be_2 = x = Tx = ae_1 - be_2, \]

so \( 2b \equiv 0 \pmod{\ell^n} \). We assume for the moment that \( \ell \) is odd, so it follows that \( b \equiv 0 \pmod{\ell^n} \). Thus \( a \in (\mathbb{Z}/\ell^n \mathbb{Z})^\times \).

Let \( G_{\ell^n} = \rho_\ell(\mathfrak{g}_F) \) be the image of the \( \ell \)-adic Galois representation. For \( S = \left[ \begin{array}{c} \alpha \\ \beta t \end{array} \right] \in \mathcal{C}_\ell^\times \cap G_{\ell^n} \), we have

\[ \alpha e_1 + \beta e_2 = x = Sx = \left( \alpha a + \beta b \left( \frac{D}{\ell} \right) \right) e_1 + (\beta at + ab)e_2. \]

Modulo \( \ell^n \) this becomes

\[ ae_1 = x = Sx = aae_1 + \beta ate_2. \]
It follows that \( \alpha \equiv 1 \pmod{\ell^n} \) and \( \beta t \equiv 0 \pmod{\ell^n} \), and thus
\[
S \equiv \begin{bmatrix} 1 & \beta \left( \frac{D}{T} \right) \\ 0 & 1 \end{bmatrix} \quad (\text{mod } \ell^n).
\]
Let \( \sigma \in g_{FK} \). Then there exists \( \beta_0 = \beta_0(\sigma) \) such that
\[
\rho_{\ell^n}(\sigma) = \begin{bmatrix} 1 & \beta_0 \left( \frac{D}{T} \right) \\ 0 & 1 \end{bmatrix}.
\]
By Galois equivariance of the Weil pairing, \( \sigma \zeta_{\ell^n} = \zeta_{\ell^n}^{\det \rho_{\ell^n}(\sigma)} = \zeta_{\ell^n} \), so \( \zeta_{\ell^n} \in FK \).

If \( \ell = 2 \), we must adjust our approach by working mod \( 2^{n-1} \). Indeed, \( Tx = x \) will only imply \( b \equiv 0 \pmod{2^{n-1}} \) and \( a \in (\mathbb{Z}/2^n\mathbb{Z})^\times \). For \( S \in C_{\ell^n}^x \cap G_{\ell^n} \), \( Sx = x \) gives \( \alpha \equiv 1 \pmod{2^{n-1}} \), \( \beta t \equiv 0 \pmod{2^{n-1}} \). In fact, \( \beta t \equiv 0 \pmod{2^n} \) as well. Indeed, by (3), \( b = \beta at + ab \), which means
\[
a^{-1}b(1-\alpha) \equiv \beta t \quad (\text{mod } 2^n).
\]
As \( 2^{n-1} \mid b \) and \( 2^{n-1} \mid (1-\alpha) \), the claim follows since \( n \geq 2 \). Thus
\[
\det S = \alpha^2 - \beta^2 D = \alpha^2 - \beta^2 t \frac{D}{t} \equiv \alpha^2 \equiv 1 \quad (\text{mod } 2^n),
\]
and \( \det \rho_{2^n} |_{g_{FK}} \) is trivial. We conclude \( \zeta_{2^n} \in FK \).

If \( \Lambda' \) is of type II, then
\[
\Lambda = \left[ \frac{t, t + \sqrt{\Delta}}{2r} \right], \quad \text{where } t \in \mathbb{N}, 4t|t^2 - \Delta \text{ and } \left( t, \frac{t^2 - \Delta}{4t} \right) = 1.
\]
With respect to this basis the action of complex conjugation is given by
\[
T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},
\]
and the action of \( \mathcal{O} \) on \( \Lambda \) is given by
\[
\alpha + \beta \sqrt{D} \mapsto \begin{bmatrix} \alpha - \beta \left( \frac{t}{2} \right) \\ \beta t \end{bmatrix} - \beta \left( \frac{t^2 - \Delta}{4t} \right) \alpha + \beta \left( \frac{t}{2} \right).
\]
This gives rise to the Cartan subgroup
\[
C_{\ell^n}^x = \left\{ \begin{bmatrix} \alpha - \beta \left( \frac{t}{2} \right) \\ \beta t \end{bmatrix} - \beta \left( \frac{t^2 - \Delta}{4t} \right) \alpha + \beta \left( \frac{t}{2} \right) \mid \alpha^2 - \beta^2 t^2 + \beta^2 \left( \frac{t^2 - \Delta}{4} \right) \in \mathbb{Z}_\ell^\times \right\}
\]
As before, we may choose a \( \mathbb{Z}_\ell \)-basis \( \vec{e}_1, \vec{e}_2 \) for \( T \ell(E) \) such that \( \rho_{\ell^n}(g_{FK}) \subset C_{\ell^n}^x \) and the element \( c \in g_{FK} \) induced by complex conjugation corresponds to \( \rho_{\ell^n}(c) = T \).

Let \( e_i = \vec{e}_i \pmod{\ell^n} \). For \( x \in E(F)[\ell^n] \setminus E(F)[\ell^{n-1}] \), we may choose \( a, b \in \mathbb{Z}/\ell^n\mathbb{Z} \) such that \( x = ae_1 + be_2 \). Then
\[
ae_1 + be_2 = x = Tx = (a + b)e_1 - be_2,
\]
so \( b \equiv 0 \pmod{\ell^n} \) and \( a \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times \). For \( S = \begin{bmatrix} \alpha - \beta \left( \frac{t}{2} \right) \\ \beta t \end{bmatrix} - \beta \left( \frac{t^2 - \Delta}{4t} \right) \alpha + \beta \left( \frac{t}{2} \right) \) \in \( C_{\ell^n}^x \cap G_{\ell^n} \), we have
\[
ae_1 = x = Sx = \left( \alpha - \beta \left( \frac{t}{2} \right) \right) ae_1 + \beta at e_2 \quad (\text{mod } \ell^n).
\]
Thus \( \alpha - \beta \left( \frac{1}{2} \right) \equiv 1 \pmod{\ell^n} \) and \( \beta t \equiv 0 \pmod{\ell^n} \). It follows that
\[
\alpha + \beta \left( \frac{t}{2} \right) \equiv \alpha + \beta \left( \frac{1}{2} \right) - \beta t = \alpha - \beta \left( \frac{t}{2} \right) \equiv 1 \pmod{\ell^n}.
\]
Hence
\[
S \equiv \begin{bmatrix} 1 & -\beta \left( \frac{t^2 - \Delta}{4t} \right) \\ 0 & 1 \end{bmatrix} \pmod{\ell^n}.
\]
We conclude \( \zeta_{\ell^n} \in FK \) as before.

Finally, we consider the case when \( \Delta \equiv 1 \pmod{4} \). Then \( \mathcal{O} = [1, \frac{1 + \sqrt{\Delta}}{2}] \) and \( \Lambda' \) is of type II as in Theorem 3.7. Thus
\[
\Lambda = \left[ t + \frac{\sqrt{\Delta}}{2r} \right], \text{ where } t \in \mathbb{N}, 4t|t^2 - \Delta \text{ and } \left( t, \frac{t^2 - \Delta}{4t} \right) = 1.
\]
Following the method used above, we may choose a \( \mathbb{Z}/\ell^n \)-basis \( \tilde{e}_1, \tilde{e}_2 \) for \( T_{\ell}(E) \) such that the image of the Cartan subgroup in \( \text{GL}_2(\mathbb{Z}_\ell) \) consists of matrices of the form
\[
\begin{bmatrix}
\alpha - \beta \left( \frac{t - 1}{2} \right) \\
\beta t \\
\beta \left( \frac{\Delta - t^2}{4t} \right) \\
\alpha + \beta \left( \frac{t + 1}{2} \right)
\end{bmatrix}
\]
and the element \( c \in g_F \) induced by complex conjugation corresponds to
\[
T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.
\]
Let \( e_i = \tilde{e}_i \pmod{\ell^n} \). For \( x \in E(F)[\ell^n] \setminus E(F)[\ell^{n-1}] \), we again choose \( a, b \in \mathbb{Z}/\ell^n \mathbb{Z} \) such that \( x = ae_1 + be_2 \). Then \( Tx = x \) gives \( b \equiv 0 \pmod{\ell^n} \) and hence \( a \in (\mathbb{Z}/\ell^n \mathbb{Z})^\times \). For
\[
S = \begin{bmatrix}
\alpha - \beta \left( \frac{t - 1}{2} \right) \\
\beta t \\
\beta \left( \frac{\Delta - t^2}{4t} \right) \\
\alpha + \beta \left( \frac{t + 1}{2} \right)
\end{bmatrix}
\in \mathcal{C}_F^\times \cap G_{\ell^n}, \text{ we have}
\]
\[
ae_1 = x = Sx = \left( \alpha - \beta \left( \frac{t - 1}{2} \right) \right) ae_1 + \beta ate_2 \pmod{\ell^n}.
\]
As before, this implies \( \alpha - \beta \left( \frac{t - 1}{2} \right) \equiv 1 \pmod{\ell^n} \) and \( \beta t \equiv 0 \pmod{\ell^n} \). Since
\[
\alpha + \beta \left( \frac{t + 1}{2} \right) \equiv \alpha + \beta \left( \frac{t + 1}{2} \right) - \beta t = \alpha - \beta \left( \frac{t - 1}{2} \right) \equiv 1 \pmod{\ell^n},
\]
we have
\[
S \equiv \begin{bmatrix} 1 & \beta \left( \frac{\Delta - t^2}{4t} \right) \\ 0 & 1 \end{bmatrix} \pmod{\ell^n}.
\]
It follows that \( \zeta_{\ell^n} \in FK \).

b) Suppose \( \gcd(N, \Delta_K) = 1 \) and \( N \geq 3 \). Seeking a contradiction, we suppose that \( F = \mathbb{Q}(\zeta_N)^+ \). Then \( K \subset FK = \mathbb{Q}(\zeta_N) \), so \( K \) is ramified at some prime divisor of \( N \) and thus \( \gcd(N, \Delta_K) > 1 \).

\[\square\]

Remark 4.10. When \( \ell^n = 2 \), Theorem 4.9a) is vacuous. Part b) holds in this case unless \( F = \mathbb{Q} \) and \( j \in \{0, 54000, -153, 2553\} \). These curves have CM by \( \mathcal{O}(\Delta) \) for \( \Delta \in \{-3, -12, -7, -28\} \) and a point of order 2.

Corollary 4.11. a) Let \( N \geq 5 \), let \( F \) be an odd degree number field, and let \( E_{/F} \) be a CM elliptic curve such that \( E(F) \) contains a point of order \( N \). Then there is a prime \( p \equiv 3 \pmod{4} \) and a positive integer \( a \) such that \( N \in \{p^a, 2p^a\} \).

b) Let \( S \) be the set of positive integers \( N \) such that there is an odd degree number
field $F$ and a CM elliptic curve $E/F$ such that $E(F)$ contains a point of order $N$. Then $S$ has density 0.

Proof. a) By Real Cyclotomy II (Theorem 4.9) we have $\mathbb{Q}(\zeta_N)^+ \subset F$, so $\frac{\varphi(N)}{2} \mid [F : \mathbb{Q}]$ and thus $4 \nmid \varphi(N)$. Since $\varphi(N)$ is divisible by 4 if $N > 4$ is divisible by 4, by a prime $p \equiv 1 \pmod{4}$, or by two odd primes, the result follows.

b) This follows easily from part a). In fact the Prime Number Theorem gives the more precise bound $S \cap [1, X] = O\left(\frac{X}{\log X}\right)$. □

4.4. Square-Root SPY Bounds.

Proposition 4.12. ([Co00, Corollary 3.2.4]) Let $K$ be a number field, and let $m$ a nonzero ideal of $O_K$, hence also a modulus in the sense of class field theory. Let $U = O_K^\times$ and $U_m = \{ \alpha \in U : \operatorname{ord}_p(\alpha - 1) \geq \operatorname{ord}_p m \text{ for all } p \mid m \}$. Then

$$[K_m : K] = \frac{\left[K(U) : m \cdot \prod_{p|m} (1 - [O_K : p]^{-1})\right]}{\prod_{p}^{h(K)}}$$

where $K_m$ is the ray class field of $K$ with modulus $m$.

Theorem 4.13. (Square-Root SPY Bounds) Let $O$ be an imaginary quadratic order of discriminant $\Delta$ and fraction field $K$, let $F$ be a number field, and let $E/F$ be an $O$-CM elliptic curve. Let $N \geq 3$, and suppose $E(F)$ has a point of order $N$.

a) If $(\mathbb{Z}/N\mathbb{Z})^2 \subset E(FK)$, then

$$\varphi(N) \leq \sqrt{\frac{[F : \mathbb{Q}](K)}{h(K)}}.$$

b) The hypothesis of part a) is satisfied when $K \not\subset F$ and $\gcd(\Delta, N) = 1$.

Proof. a) We have

$$\prod_{p \mid N O_K} (1 - [O_K : p]^{-1}) = \prod_{p \mid N} \prod_{p \mid O_K} (1 - [O_K : p]^{-1}).$$

Further, we have

$$\prod_{p \mid O_K} (1 - [O_K : p]^{-1}) = \left\{ \begin{array}{ll} (1 - \frac{1}{p})^2, & \left(\frac{\Delta_K}{p}\right) = 1 \\ (1 - \frac{1}{p}), & \left(\frac{\Delta_K}{p}\right) = 0 \\ (1 - \frac{1}{p}), & \left(\frac{\Delta_K}{p}\right) = -1 \end{array} \right.$$

so

$$\prod_{p \mid N O_K} (1 - [O_K : p]^{-1}) \geq \prod_{p \mid N} \left(1 - \frac{1}{p}\right)^2.$$

Applying Lemma 3.15 and Theorem 3.16, we get

$$FK = F(E[N]) \supseteq K^{(N)}$$

and thus

$$[F : \mathbb{Q}] = \frac{[FK : \mathbb{Q}]}{2} \geq \frac{[K^{(N)} : \mathbb{Q}]}{2} = \frac{[K^{(N)} : K]}{2} = \left[\frac{h_K}{\prod_{p \mid N O_K} (1 - [O_K : p]^{-1})}\right] \geq \frac{h_K N^2}{\prod_{p \mid N} \left(1 - \frac{1}{p}\right)^2} = \frac{h_K}{w_K} \varphi(N)^2.$$

b) This is Theorem 4.8a). □
Example 4.14. Let $N \geq 3$, let $F$ be a cubic number field, and let $E_{/F}$ be a CM elliptic curve such that $E(F)$ contains a point of order $N$. The SPY Bounds give $\varphi(N) \leq [F : Q]w_K \leq 18$; in particular the largest prime value of $N$ permitted is 19. The odd order torsion subgroup of $E(F)$ has size at most 13 [CX08, Main Theorem]; in particular, the largest prime value permitted is 13. Real Cyclotomy II (Theorem 4.9) gives $\varphi(N) | 6$ and thus $N | 4$, $N | 14$ or $N | 18$; in particular, the largest prime value permitted is 7. Theorems 2.1 and 1.4 show that for all of these values of $N$ are actually attained except for $N = 18$.

Suppose now that $E_{/F}$ is $O(\Delta)$-CM and $\gcd(\Delta, N) = 1$. Then the Square Root SPY Bounds give

$$\varphi(N) \leq \lfloor \sqrt{18} \rfloor = 4, \ K = Q(\sqrt{-3}),$$
$$\varphi(N) \leq \lfloor \sqrt{12} \rfloor = 3, \ K = Q(\sqrt{-1}),$$
$$\varphi(N) \leq \lfloor \sqrt{6} \rfloor = 2, \ K \notin \{Q(\sqrt{-3}), Q(\sqrt{-1})\}.$$ Combining with Real Cyclotomy II, we get that the only prime values of $N$ permitted in this case are the “Olson primes” 2 and 3. The four elliptic curves in rows 13 through 16 of the table in Theorem 1.4 do not satisfy the Square Root SPY Bounds. It follows from Theorem 4.13 that for $N = 9$ and $N = 14$ we do not have $(Z/NZ)^2 \subset E(FK)$. This shows that the hypothesis $\gcd(\Delta, N) = 1$ in Real Cyclotomy I (Theorem 4.8) is necessary in order for this stronger form of real cyclotomy to hold. On the other hand, we have

$$Q[b]/(b^3 - 15b^2 - 9b - 1) \cong Q[b]/(b^3 + 105b^2 - 33b - 1) \cong Q(\zeta_9)^+,\n$$
$$Q[b]/(b^3 - 4b^2 + 3b + 1) \cong Q[b]/(b^3 - 18b^2 + 3b + 1) \cong Q(\zeta_{14})^+,$$

in accordance with Real Cyclotomy II (Theorem 4.9).

5. Number Fields of Prime Degree

5.1. Parish’s Theorem.

The proof of Theorem 1.4 will make use of the following striking result.

Theorem 5.1. (Parish [Pa89, Theorem 2]) Let $E_{/F}$ be a CM elliptic curve defined over a number field. If $F = Q(j(E))$, then $E(F)[\text{tors}]$ is an Olson group.

Remark 5.2. Parish gives a complete tabulation of torsion subgroups of CM elliptic curves over $Q(j(E))$, but the precise statement of Theorem 5.1 is left implicit. In view of the importance of this result, let us give a more explicit account.

In [Pa89, Thm. 2], Parish shows that for $j \neq 0, 1728$, the only possible orders of torsion points on a CM elliptic curve rational over $K(j(E))$ are 1, 2, 3, 4, 6. Combining with Olson’s Theorem we see that the only possible orders of torsion points over $Q(j(E))$ are 1, 2, 3, 4, 6 in all cases. The field $Q(j(E))$ is real, and the torsion subgroup of any elliptic curve over any real number field is either cyclic or of the form $Z/2nZ \times Z/2Z$. It remains to rule out $Z/4Z \times Z/2Z$ and $Z/6Z \times Z/2Z$. But as we have already recorded in Theorem 4.1, Parish showed that when $\Delta \notin \{-4, -3\}$, a CM elliptic curve $E$ cannot have full 2-torsion over $Q(j(E))$, and the remaining cases have rational $j$-invariant so are covered by Olson’s Theorem.

The following consequence is immediate.

\footnote{The table of Theorem 1.4 shows that $j = 0$ and $j = 1728$ indeed must be excluded in the statement of Parish’s Theorem 2.}
Corollary 5.3. If $F$ is a number field of prime degree and $E/F$ is a CM elliptic curve with $j(E) \notin \mathbb{Q}$, then $E/F$ is Olson.

5.2. Proof of Theorem 1.4.

Let $F$ be a number field with $[F : \mathbb{Q}] = p$ a prime number: so $F$ is real if $p \neq 2$. Let $E/F$ be an elliptic curve with CM by an order $\mathcal{O}$ of discriminant $\Delta = \ell^2 \Delta_K$ in the imaginary quadratic field $K$. Suppose $E/F$ is not Olson.

Step 1: Suppose $p = 2$. Here, the result is in principle a very special case of work of Müller-Strøher-Zimmer [MSZ89] which shows that there are finitely many pairs $(E,F)$ with $F$ a quadratic number field and $E/F$ a non-Olson elliptic curve with integral moduli – i.e., $j(E) \in \mathcal{O}_F$ – and lists all of them. In practice, we used Parish’s Theorem, the SPY Bounds, and Table 2 to rederive the classification.

Step 2: Suppose $p = 3$. Work of Petho-Weis-Zimmer [PWZ97] shows that as $E/F$ ranges over all elliptic curves over cubic number fields with integral moduli, the only non-Olson group which arises infinitely many times is $\mathbb{Z}/5\mathbb{Z}$. By Corollary 4.11, there is no CM elliptic curve over a cubic field with a point of order 5. As above, although these results suffice in principle, in practice we used Parish’s Theorem, the SPY Bounds, work of Clark-Xarles [CX08, Corollary 2], and Table 2 to rederive the classification.

Step 3: Suppose $p \geq 5$. Suppose $E(F)[\text{tors}]$ has a point of prime order $\ell \geq 5$. By Corollary 5.3, $j(E) \in \mathbb{Q}$. Thus $\mathcal{O}$ has class number 1, so

$$\Delta \in \{-3, -4, -7, -8, -11, -12, -16, -19, -27, -28, -43, -67, -163\}.$$  

If $\ell$ is unramified in $K$ then by Real Cyclotomy II (Theorem 4.9), $\mathbb{Q}(\zeta_\ell)^+ \subset F$ and thus $\frac{\ell-1}{2}$ properly divides $p$. Since $\ell \geq 5$, this is a contradiction. If $\ell$ is ramified in $K$ then $\ell \in \{7, 11, 19, 43, 67, 163\}$. Real Cyclotomy II gives $\frac{\ell-1}{2} | p$, so $p = \frac{\ell-1}{2}$.

- If $\ell = 7$ then $p = 3$.
- If $\ell = 11$ then $p = 5$.
- If $\ell \in \{19, 43, 67, 163\}$, then $\frac{\ell-1}{2}$ is not a prime.

Step 4: It follows from Table 2 that no CM elliptic curve $E$ with rational $j$-invariant defined over number field $F$ of prime degree $p \geq 5$ can have any of the following as subgroups of $E(F)$:

$$\mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/9\mathbb{Z}, \mathbb{Z}/12\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}.$$  

Therefore $p \leq 5$, and if $p = 5$ then $\Delta = -11, F = \mathbb{Q}(\zeta_{11})^+$ and $E(F)$ has a point of order 11. It follows that there is such an elliptic curve [CCRS14, §4.5]. The same kind of computation shows that this curve is unique. Alternately, any other such elliptic curve would be a quadratic twist of $E/F$ by $F(\sqrt{d})/F$, say. For any odd $N \geq 3$, if $E/F$ and the quadratic twist $E_{/F}^{(d)}$ each have points of order $N$, then $E$ has full $N$-torsion over $F(\sqrt{d})$. This would force $F(\sqrt{d}) = FK = \mathbb{Q}(\zeta_{11})$ and thus, by Theorem 3.16, $\mathbb{Q}(\zeta_{11}) \supset K^{(11)}$, but $[\mathbb{Q}(\zeta_{11}) : \mathbb{Q}] = 10$ and $[K^{(11)} : \mathbb{Q}] = 110$.

Remark 5.4. Theorem 1.4 is proved by ruling out non-Olson torsion in prime degree $p \geq 7$ and then computing all non-Olson CM elliptic curves over number
fields of degree \( p \in \{2, 3, 5\} \). These calculations were done en route to previous results [CCRS14, §4.2, §4.3, §4.5], but each group structure which arose in a given degree was recorded only once [CCRS14, Algorithm 3.2]. The proof explains how one could extract the needed calculations from tables appearing in previous work [MSZ89, PWZ97, CCRS14], Corollary 4.11 and some modest calculations with genus zero torsion structures.

However, we did not feel that this was a good approach. Rather, the last two authors knew from their prior work that it would not be overly onerous to recalculate all non-Olson torsion in degrees 2, 3 and 5 from scratch. To achieve the most meaningful corroboration, this recalculation was done by the first author. They are fully consistent with (but more detailed than) the results of [CCRS14].

6. BEYOND PRIME DEGREES

6.1. Proof of Theorem 1.6.

We give the proof of Theorem 1.6. For the reader’s convenience we begin by recalling the statement of Schinzel’s Hypothesis H [SS58], [Po09, §1.8.3].

**Conjecture 6.1.** (Schinzel’s Hypothesis H) Let \( f_1, \ldots, f_r \in \mathbb{Q}[t] \) be irreducible and integer-valued. Suppose: for all \( m \geq 2 \) there is \( n \in \mathbb{Z}^+ \) such that \( m \nmid f_1(n) \cdots f_r(n) \). Then \( \{ n \in \mathbb{Z}^+ \mid |f_1(n)|, \ldots, |f_r(n)| \text{ are all prime numbers} \} \text{ is infinite.} \)

**Theorem 6.2.** Assume Schinzel’s Hypothesis H, and let \( d \in \mathbb{Z}^+ \). Then

\[
\limsup_{p \in \mathcal{P}} \# T_{CM}^{new}(2dp) \geq 1.
\]

**Proof.** Applying Schinzel’s Hypothesis H with \( f_1(x) = x, f_2(x) = 6dx + 1 \), we get infinitely many prime numbers \( p \) such that \( N = 6dp + 1 \) is prime. Thus \( \frac{N-1}{3} = 2dp \).

In particular \( N = 1 \mod 3 \), so \( N \) splits in \( K = \mathbb{Q}(\sqrt{-3}) \), and then there is an \( \mathcal{O}_K \)-CM elliptic curve \( E \) defined over a number field \( F \) of degree \( \frac{N-1}{3} = 2dp \) with an \( F \)-rational point of order \( N \) [CCRS13, Theorem 3]. We claim that for sufficiently large \( N \), \( E(F)[tors] \in T_{CM}^{new}(2dp) \): if so, the result follows. Now there is a positive integer \( N_0 \) (as yet inexplicit) such that for all primes \( N \geq N_0 \), if \( E/F \) is a CM elliptic curve over a number field \( F \) with an \( F \)-rational point of order \( N \), then \( [F : \mathbb{Q}] \geq \frac{N+1}{3} \) [CCRS13, Theorem 1]. Thus for all primes \( N \geq N_0 \), \( E(F)[tors] \) is a torsion subgroup that does not occur in any degree smaller than \( [F : \mathbb{Q}] \), which certainly implies \( E(F)[tors] \in T_{CM}^{new}(2dp) \).

§6.2. UNBOUNDEDNESS OF ODD DEGREE TORSION POINTS ON ELLIPTIC CURVES

**Theorem 6.3.** Let \( N \in \mathbb{Z}^+ \) and let \( d \) be a prime number. The set of algebraic numbers \( j \in \overline{\mathbb{Q}} \) such that there is a number field \( F \) with \( d \nmid [F : \mathbb{Q}] \) and an elliptic curve \( E/F \) with \( j(E) = j \) and a point of order \( N \) in \( E(F) \) is infinite.

**Proof.** Let \( S(d, N) \) be the set of \( j \in \overline{\mathbb{Q}} \) such that there is a number field \( K \) with \( d \nmid [F : \mathbb{Q}] \) and an elliptic curve \( E/F \) with \( j(E) = j \) and such that \( E(F) \) has a point of order \( N \). We want to show that \( S(d, N) \) is infinite. It suffices to show that for every finite subset \( S \subset S(d, N) - \) including the empty set - there is \( j \in S(d, N) \setminus S \).

Let \( \pi : X_1(N) \to X(1) \) be the natural map. Let \( S \subset S(d, N) \) be finite. Identifying \( Y(1) \) with \( \mathbb{A}^1 \), we view \( S \) as a finite set of \( \overline{\mathbb{Q}} \)-valued points of \( Y(1) \). Let \( Z_1 \)}
be an effective \( \mathbb{Q} \)-rational divisor on \( X(1) \) whose support\(^3\) contains \( S \) and all the cusps, let \( Z_N = \pi^*(Z_1) \), and let \( U = X_1(N) \setminus \text{supp}(Z_N) \). By weak approximation, the least positive degree of a divisor on \( U \) is the least positive degree of a divisor on \( X_1(N) \); see [Cl07, Lemma 12] for a complete treatment of a stronger result. Since the cusp at \( \infty \) is a \( \mathbb{Q} \)-rational point on \( X_1(N) \), this common quantity is 1, and thus there is a divisor \( \sum_i n_i[P_i] \) supported on \( U \) such that \( \sum_i n_i[Q(P_i)] : \mathbb{Q} = 1 \). For at least one \( i \) we must have \( d \nmid [\mathbb{Q}(P_i) : \mathbb{Q}] \). The point \( P_i \) corresponds to at least one pair \( (E, x) / \mathbb{Q}(P_i) \) where \( E \) is an elliptic curve and \( x \in E(\mathbb{Q}(P_i)) \) has order \( N \) [DR73, Proposition VI.3.2]. By construction, we have \( j(E) \in S(d, N) \setminus S \). \( \square \)

6.3. Number Fields of \( S_d \)-Type.

Let \( G \) be a finite group. A number field \( F \) is of \( G \)-type if the automorphism group of the normal closure of \( F/\mathbb{Q} \) is isomorphic to \( G \).

**Theorem 6.4.** Let \( d \) be an odd positive integer, and let \( F \) be a degree \( d \) number field of \( S_d \)-type. Then every CM elliptic curve \( E/F \) is Olson.

**Proof.** Step 1: Let \( M \) be the normal closure of \( F/\mathbb{Q} \), and choose an isomorphism \( S_d \cong \text{Aut}(M/\mathbb{Q}) \). Let \( A \) (resp. \( B \)) be the maximal abelian subextension of \( F/\mathbb{Q} \) (resp. of \( M/\mathbb{Q} \)). Then

\[
B = M^{[S_d, S_d]} = M^{A_d},
\]

so \( [B : \mathbb{Q}] = 2 \). Since \( \mathbb{Q} \subset A \subset B \cap F \) and \( [F : \mathbb{Q}] = d \) is odd, we have \( A = \mathbb{Q} \).

Step 2: Let \( E/F \) be a CM elliptic curve. Since \( F \) has odd degree, it is real, so if \( E(F) \) contains a point of order \( N \), by Real Cyclotomry II (Theorem 4.9) \( F \) contains the abelian number field \( \mathbb{Q}(\zeta_N)^+ \). Thus \( \mathbb{Q}(\zeta_N)^+ \subset \mathbb{Q} \) and \( N \in \{1, 2, 3, 4, 6\} \). Because \( d \) is odd, we have \( \mathbb{Q}(\zeta_d) \not\subset F \) and thus \( (\mathbb{Z}/3\mathbb{Z})^2 \not\subset E(F) \). Applying Corollary 4.5, we conclude that \( E/F \) is Olson. \( \square \)

7. The Odd Degree Theorem

7.1. Statement of Theorem.

**Theorem 7.1.** (Odd Degree Theorem) Let \( F \) be a number field of odd degree, let \( E/F \) be a CM elliptic curve, and let \( T = E(F)[\text{tors}] \). Then \( T \) is isomorphic to one of the following groups:

1. the trivial group \( \{\bullet\} \), \( \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \text{or} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \);
2. the group \( \mathbb{Z}/\ell^n\mathbb{Z} \) for a prime number \( \ell \equiv 3 \) (mod 8) and \( n \in \mathbb{Z}^+ \);
3. the group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) for a prime number \( \ell \equiv 3 \) (mod 4) and \( n \in \mathbb{Z}^+ \).

Conversely, each of the above groups arises up to isomorphism as the torsion subgroup \( E(F) \) of a CM elliptic curve \( E \) defined over an odd degree number field \( F \).

7.2. Twisting, Top to Bottom.

For \( N \in \mathbb{Z}^+ \), let \( U(N) = (\mathbb{Z}/N\mathbb{Z})^\times \) and let \( U(N) = U(N)/(\pm 1) \).

**Theorem 7.2.** (Twisting at the Top) Let \( N \geq 3 \) be an integer. Let \( F \) be a field, let \( A_F \) be an abelian variety, and let \( C \subset F \) be an étale subgroup scheme which is cyclic of order \( N \). Then there is an abelian extension \( L/F \) with \( [L : F] \mid \frac{\varphi(N)}{2} \) and a quadratic twist \( A' \) of \( A/F \) such that \( A'(L) \) has a point of order \( N \).

---

\(^3\)We use the standard bijection between closed points and \( \mathbb{P}_1 \)-orbits of \( \mathbb{P}_1 \)-valued points.
Proof. The action of $g_F$ on $C$ gives rise to an isogeny character
$$\Phi : g_F \to U(N).$$
Composing with $U(N) \to U(N)^\pm$, we get the \textbf{plus-minus isogeny character}
$$\Phi^\pm : g_F \to U(N)^\pm.$$ 

Let $M = (F^{\text{sep}})^{\ker \Phi}$. Then $M/F$ is abelian, $[M : F] \mid \varphi(N)$ and $\Phi|_{g_{\text{L}}}$ is trivial, so any generator of $C$ gives a point of order $N$ on $A(M)$. We can however gain back the last factor of 2 by \textit{twisting at the top}. Let $L = (F^{\text{sep}})^{\ker \Phi^\pm}$. Then $L/F$ is abelian, $[L : F] \mid \frac{\varphi(N)}{2}$, and
$$\Phi(g_L) \subset \{\pm 1\}.$$ 

If $\Phi|_{g_L} = 1$, we’re done: a generator of $C$ gives an $L$-rational point on $A$. Otherwise $\Phi|_{g_L} = \epsilon$ is a nontrivial quadratic character on $g_L$, with corresponding separable quadratic extension $M/L$. Let $A'/L$ be the quadratic twist of $A$ by $M/L$. The twisted $g_L$-action on $C$ is by $\Phi \cdot \epsilon = \epsilon \cdot \epsilon = 1$. So $A'(L)$ has a point of order $N$. \(\square\)

\textbf{Theorem 7.3.} (Twisting at the Bottom) Let $\ell \equiv 3 \pmod{4}$ be a prime, $n \in \mathbb{Z}^+$, and put $N = \ell^n$. Let $F$ be a field, $A_{/F}$ an abelian variety, and $C \subset F$ an etale subgroup scheme which is cyclic of order $N$. There is a quadratic twist $A'$ of $A_{/F}$ and a cyclic extension $L/F$ of degree dividing $\frac{\varphi(N)}{2}$ such that $A'(L)$ has a point of order $N$.

\textbf{Proof.} Let $\Phi : g_F \to U(N)$ be the isogeny character associated to $C$. Let $q : U(N) \to U(N)/U(N)^2$ be the quotient map. Since $N = \ell^n$ is an odd prime power, $U(N)/U(N)^2$ has order 2 and is thus (uniquely!) isomorphic to \{\pm 1\}. Moreover, since $p \equiv 3 \pmod{4}$, $-1$ is not a square modulo $p$, so $a \text{ a fortiori}$ is not a square modulo $N$. Thus under the canonical isomorphism $U(N)/U(N)^2 \to \{\pm 1\}$, the class of $-1$ maps to $-1$. Let
$$\epsilon_\Phi = q \circ \Phi : g_F \to U(N)/U(N)^2 \to \{\pm 1\}.$$ 

Let $A'$ be the quadratic twist of $A$ by $\epsilon_\Phi$ (so $A' = A$ if $\epsilon_\Phi$ is trivial), and let
$$\Phi' = \epsilon_\Phi \Phi$$ 

be the associated isogeny character. By the above remarks, about $-1 \in U(N)$ mapping to the nontrivial element of $U(N)/U(N)^2$ it follows that $\epsilon_{\Phi'}$ is the trivial quadratic character, and thus $\Phi'(g_F) \subset U(N)^2$. Thus if $L = (F^{\text{sep}})^{\ker \Phi'}$, then $A'$ has an $L$-rational point of order $N$ and $[L : F] \mid \frac{\varphi(N)}{2}$. \(\square\)

\textbf{Proposition 7.4.} Let $E/F$ be an $O$-\textit{CM} elliptic curve with $F = \mathbb{Q}(j(E))$. Let $\ell \mid \Delta(O)$ be a prime.

\textit{a)} There is a unique prime ideal $p_\ell$ of $O$ such that $p_\ell \cap \mathbb{Z} = (\ell)$.

\textit{b)} We have that $E[p_\ell] = \{x \in E(\mathbb{C}) \mid \ell x = 0 \forall x \in p_\ell\}$ is an $F$-rational subgroup of order $\ell$.

\textbf{Proof.} \textit{a)} Since $\ell \mid \Delta$, we have $O/\ell O \cong \mathbb{Z}/(\ell \mathbb{Z}[\ell])/(\ell^2)$, so there is a unique ideal $p_\ell$ of $O$ with $\#O/p_\ell = \ell$. By uniqueness $\overline{p_\ell} = p_\ell$; thus $E[p_\ell]$ is an $F$-rational subgroup.

\textit{b)} When $O = O_K$, we have $\#E[a] = \#O_K/a$ for all nonzero ideals $a$ of $O_K$ [Si94, Prop. II.1.4].\(^4\) In the general case, we may embed $\mathbb{Q}(j) \to \mathbb{C}$ so as to have

\(^4\)As the remainder of the argument is a bit technical, it may be worthwhile to note that only the case $O = O_K$ will be needed in the proof of Theorem 7.1.
\[ j(E) = j(C/O), \] and then we get
\[ E[p_\ell] = \{ x \in C \mid xp_\ell \subseteq O \}/O. \]

We have
\[ \{ x \in C \mid xp_\ell \subseteq O \} = \{ x \in K \mid xp_\ell \subseteq O \} = (O : p_\ell). \]

Observe that \( E[p_\ell] = (O : p_\ell)/O \) is a vector space over the field \( F_\ell = O/p_\ell; \) it remains to compute its dimension. For any domain \( R, \) any two elements of a fractional \( R \)-ideal are \( R \)-linearly dependent, so \( \dim_{F_\ell}(O : p_\ell)/O \in \{0, 1\}. \) Since \( O \) is a one-dimensional Noetherian domain and \( p_\ell \not\subseteq O \) we have \( (O : p_\ell) \nsubseteq O [Ja89, \S 10.2, \text{Lemma 4}] \) and thus \( 1 = \dim_{F_\ell}(O : p_\ell)/O = \dim_{F_\ell} E[p_\ell]. \]

**Corollary 7.5.** Let \( O \) be an imaginary quadratic order of discriminant \( \Delta, \) and let \( \ell > 2 \) be a prime dividing \( \Delta. \)

\( \text{a) There is a number field } L \text{ of degree } h(O) \left( \frac{\Delta}{\ell^2} \right) \text{ and an } O-\text{CM elliptic curve } E_{/L} \text{ with an } L \text{-rational torsion point of order } \ell. \)

\( \text{b) Suppose } \ell \equiv 3 \pmod{4} \text{ and } O \text{ is the quadratic order of discriminant } -\ell, \text{i.e., the ring of integers of } K = \mathbb{Q}(\sqrt{-\ell}). \text{ Let } j = j(O) \text{ and } F = \mathbb{Q}(j). \text{ Then:} \)

\( \text{i) The number field } F(\zeta_\ell + \zeta_\ell^{-1}) \text{ has degree } h(K)(\ell - 1)/2, \text{ an odd number.} \)

\( \text{ii) There is an elliptic curve } E_{/F} \text{ such that } E(F(\zeta_\ell + \zeta_\ell^{-1})) \text{ has a point of order } \ell. \)

**Proof.**

\( \text{a) Combine Proposition 7.4 and Theorem 7.2.} \)

\( \text{b) (i) Since } [F : \mathbb{Q}] = h(K) \text{ is odd by Lemma 3.5, the genus field } FK \cap \mathbb{Q}^{ab} \text{ is } K, \text{ and thus } F \cap \mathbb{Q}(\zeta_{N})^+ = \mathbb{Q}, \text{i.e., } F \text{ and } \mathbb{Q}(\zeta_\ell + \zeta_\ell^{-1}) \text{ are linearly disjoint over } \mathbb{Q}. \text{ It follows that } [F(\zeta_\ell + \zeta_\ell^{-1}) : \mathbb{Q}] = h(K) \left( \frac{\Delta}{\ell^2} \right), \text{ which is odd.} \)

\( \text{(ii) Let } E_{/F} \text{ be an } O-\text{CM elliptic curve. By Proposition 7.4, } E \text{ has an } F \text{-rational subgroup of order } \ell. \text{ By Theorem 7.3, after replacing } E \text{ by a quadratic twist over } F, \text{ there is an extension } L/F \text{ of degree dividing } \frac{\ell - 1}{2} \text{ such that } E(L) \text{ has a point of order } \ell. \text{ By Real Cyclotomy II, } F(\zeta_\ell + \zeta_\ell^{-1}) \subseteq L. \text{ Thus} \)

\[ [L : \mathbb{Q}] \mid h(K) \left( \frac{\ell - 1}{2} \right) = [F(\zeta_\ell + \zeta_\ell^{-1}) : \mathbb{Q}], \]

so \( L = F(\zeta_\ell + \zeta_\ell^{-1}). \) \qed

**Remark 7.6.** When \( F = \mathbb{Q}, \text{ Corollary 7.5b) is a result of Lozano-Robledo [LR13, Cor. 9.8].} \)

### 7.3. Existence of Odd Degree Torsion.

The groups \{\bullet\}, \( \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \) or \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) are Olson groups, so occur already over \( \mathbb{Q}. \) To complete the existence portion of Theorem 7.1, it remains to construct CM elliptic curves over odd degree number fields with torsion subgroups isomorphic to \( \mathbb{Z}/\ell^n\mathbb{Z} \) or \( \mathbb{Z}/2\ell^n\mathbb{Z} \) for \( \ell \) as in Theorem 7.1 above. We do so in this section.

**Lemma 7.7.** Let \( E_{/F} \text{ be an } O_K-\text{CM elliptic curve such that } F(E[\ell]) = K[(\ell)] \text{ for some prime } \ell > 2. \text{ Suppose additionally that } K \neq \mathbb{Q}(i), \text{ and } \ell = 3 \text{ if } K = \mathbb{Q}(\sqrt{-3}). \text{ Then } F(E[\ell^n]) = K[(\ell^n)] \text{ for all } n \in \mathbb{Z}^+. \)

**Proof.** We will handle the two cases separately.

Case 1: Let \( E_{/F} \text{ be an elliptic curve with CM by the maximal order in } K \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3}) \text{ such that } F(E[\ell]) = K[(\ell)] \text{ for a prime } \ell > 2. \text{ Suppose, for the sake of
contradiction, that \( F(E[\ell^n]) \not\in K^{(\ell^n)} \) for some positive integer \( n \). By [Si94, Thm. II.5.6], we have

\[
K^{(\ell^n)} = K(j(E), h(E[\ell^n])) = FK(h(E[\ell^n])) \subset F(E[\ell^n]).
\]

Since \( K \not\in \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3}) \), we may take \( h(E[\ell^n]) = x(E[\ell^n]) \). Thus \( [F(E[\ell^n]) : K^{(\ell^n)}] = 2 \) if \( F(E[\ell^n]) \not\in K^{(\ell^n)} \). Let \( \sigma \in \mathfrak{g}_F \) generate \( \text{Aut}(F(E[\ell^n]))/K^{(\ell^n)} \). Then \( \sigma \) corresponds to a matrix of order 2 in \( \text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \) which is trivial mod \( \ell \) since \( F(E[\ell]) \subset K^{(\ell^n)} \). But the kernel of the reduction map \( \text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \rightarrow \text{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) \) is an \( \ell \)-group.

Case 2: If \( K = \mathbb{Q}(\sqrt{-3}) \) and \( \ell = 3 \), we have \( \mathcal{O}_K \otimes \mathbb{Z}_3 \cong \mathbb{Z}_3[\sqrt{-3}] \). Thus we may choose a basis \( \tilde{e}_1, \tilde{e}_2 \) for \( T_3(E) \) for which \( \rho_{3,\ell}(\mathfrak{g}_K) \) lands in the Cartan subgroup

\[
C_3^\times = \left\{ \begin{bmatrix} \alpha & \beta \\ -3\beta & \alpha \end{bmatrix} \mid \alpha^2 + 3\beta^2 \in \mathbb{Z}_3^\times \right\}.
\]

Elements of \( \text{Aut}(F(E[3^n])/F(E[3])) = \text{Aut}(F(E[3^n])/K) \) correspond to elements of \( C_3^\times \) modulo \( 3^n \) which are congruent to the identity matrix modulo 3. There are precisely \( 3^{2n-2} \) such matrices, giving an upper bound on \( \# \text{Aut}(F(E[3^n])/K) \).

On the other hand, by [Si94, Thm. II.5.6] we have

\[
K^{(3^n)} = K(j(E), h(E[3^n])) \subset F(E[3^n]).
\]

Since \( [K^{(3^n)} : K] = 3^{2n-2} \), we have \( [F(E[3^n]) : K] = 3^{2n-2} \) and \( F(E[3^n]) = K^{(3^n)}. \) \( \square \)

**Proposition 7.8.** Let \( \ell \equiv 3 \pmod{4} \) be prime, and let \( n \in \mathbb{Z}^+ \). There exists an elliptic curve \( E \) defined over a number field \( F \) such that:

(i) \( E \) has CM by the full ring of integers in \( K = \mathbb{Q}(\sqrt{-\ell}) \);

(ii) \( [F : \mathbb{Q}] \) is odd; and

(iii) \( E(F) \) has a point of order \( \ell^n \).

**Proof.** Let \( E \) be an elliptic curve with CM by \( \mathcal{O}_K \). Choose a model of \( E \) defined over \( F = \mathbb{Q}(j(E)) \). For now, suppose \( \ell \not\equiv 3 \pmod{4} \). By Corollary 7.5, there is a quadratic twist \( E'_{\ell} \) of \( E \) such that \( E'_{\ell}(\zeta_{\ell} + \zeta_{\ell}^{-1}) \) has a point of order \( \ell \). Hence we may assume \( E(\zeta_{\ell} + \zeta_{\ell}^{-1}) \) has a point of order \( \ell \). In fact, this implies \( F(E[\ell]) = K^{(\ell)} \), as we will now show. Since \( E(F(\zeta_{\ell})) \) contains a point of order \( \ell \),

\[
\text{Aut}(F(E[\ell])/F(\zeta_{\ell})) \cong \langle [1 \ b] \rangle,
\]

where \( b \in \mathbb{F}_\ell \). Thus \( F(E[\ell])/F(\zeta_{\ell}) = 1 \) or \( \ell \). By Theorem 3.16, we have \( K^{(\ell)} \subset F(E[\ell]) \), so \( [K^{(\ell)} : \mathbb{Q}] = h(K)\ell(\ell - 1) \) divides \( [F(E[\ell]) : \mathbb{Q}] \). This forces \( [F(E[\ell]) : F(\zeta_{\ell})] = \ell \) and \( F(E[\ell]) = K^{(\ell)} \).
By Lemma 7.7, it follows that $F(E[\ell^n]) = K^{(\ell^n)}$. Viewing $F \subset \mathbb{R}$, we have $E(\mathbb{R}) \cong \mathbb{R}/\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus we have an element of order $\ell^n$ fixed by the element $c \in g_F$ induced by complex conjugation. Then $F(E[\ell^n])^c$ contains the coordinates of a point of order $\ell^n$, and

$$[F(E[\ell^n])^c : \mathbb{Q}] = \frac{1}{2}[F(E[\ell^n]) : \mathbb{Q}] = \frac{1}{2}[K^{(\ell^n)} : \mathbb{Q}] = \frac{1}{2}h(K)(\ell - 1)[\ell^{2n}].$$

Lemma 3.5 gives that $h(K)$ is odd, so $[F(E[\ell^n])^c : \mathbb{Q}]$ is odd as desired.

If $\ell = 3$, consider the elliptic curve $y^2 = x^3 + 16$. This curve has CM by the full ring of integers in $K = \mathbb{Q}(\sqrt{-\ell})$ and one finds (e.g. by direct calculation) that $\mathbb{Q}(E[3]) = K^{(3)} = K$. By Lemma 7.7, $\mathbb{Q}(E[3^n]) = K^{(3^n)}$. As before, $\mathbb{Q}(E[3^n])^c$ contains the coordinates of a point of order $3^n$, and

$$[\mathbb{Q}(E[3^n])^c : \mathbb{Q}] = \frac{1}{2}[\mathbb{Q}(E[3^n]) : \mathbb{Q}] = \frac{1}{2}[K^{(3^n)} : \mathbb{Q}] = 3^{2n-2}. \quad \square$$

**Theorem 7.9.** Let $n \in \mathbb{Z}^+$.

1. If $\ell \equiv 3 \pmod{8}$, there exists a CM elliptic curve $E$ defined over a number field $F$ of odd degree such that $E(F)[\text{tors}] \cong \mathbb{Z}/\ell^n\mathbb{Z}$.

2. If $\ell \equiv 3 \pmod{4}$, there exists a CM elliptic curve $E$ defined over a number field $F$ of odd degree such that $E(F)[\text{tors}] \cong \mathbb{Z}/2\ell^n\mathbb{Z}$.

**Proof.** Suppose $\ell \equiv 3 \pmod{4}$, and let $K = \mathbb{Q}(\sqrt{-\ell})$. By Proposition 7.8, there is an $O_K$-CM elliptic curve $E$ defined over $F = \mathbb{Q}(j(E))$ and an extension $\tilde{F} := F(E[\ell^n])^c$ of odd degree over $\mathbb{Q}$ such that $E(\tilde{F})[\text{tors}]$ contains a point of order $\ell^n$. Since $\tilde{F}$ is odd, we know $E(\tilde{F})[\text{tors}]$ is either cyclic or of the form $\mathbb{Z}/2N\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; otherwise full $N$-torsion would force $\mathbb{Q}(\zeta_N) \subset \tilde{F}$ by the Weil pairing. We first establish that $\ell^n$ is the largest power of $\ell$ dividing $\#E(\tilde{F})[\text{tors}]$. 
Suppose $\ell^{n+1}$ divides $\#E(\bar{F})[\text{tors}]$. By Real Cyclotomy II, $\mathbb{Q}(\zeta_{\ell^{n+1}}) \subset \bar{F}K = K(\ell^n)$. Then elements of
\[
\text{Gal}(F(\mathbb{Q}[\ell^{n+1}])/F(\mathbb{Q}[\ell^n])) = \text{Gal}(K(\ell^{n+1})/K(\ell^n))
\]
correspond to matrices of the form
\[
\left\{ \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \mid \beta \equiv 0 \pmod{\ell^n} \right\}.
\]
There are $\ell$ such matrices. However, $[K(\ell^{n+1}) : K(\ell^n)] = \ell^2$. Thus $\ell^n$ is the largest power of $\ell$ dividing $\#E(\bar{F})[\text{tors}]$.

Suppose $\ell \equiv 3 \pmod{8}$. Since $\ell^n$ is the largest power of $\ell$ dividing $\#E(\bar{F})[\text{tors}]$, by Corollary 4.11 it will suffice to show that $E(\bar{F})$ contains no point of order 2. Suppose first that $\Delta \neq -3$. Since $\Delta \equiv 5 \pmod{8}$, there are no points of order 2 rational over $F = \mathbb{Q}(j(E))$ by Theorem 4.2. By construction, $\bar{F} \subset F(E[\ell^n]) = K(\ell^n)$, and $F(E[2]) = K(2)$ by the proof of Theorem 4.2: $F(E[2])$ has degree $6h(K)$ over $\mathbb{Q}$ and contains $K(2)$, which also has degree $6h(K)$. Thus if $\bar{F}$ contains the coordinates of a point of order 2, then 3 divides $[\bar{F} \cap K(2) : F]$ and hence divides the degree of $K(\ell^n) \cap K(2) = K(1)$ over $F$. But $[K(1) : F] = 2$. Thus $E(\bar{F})$ contains no point of order 2.

We now consider the case where $\Delta = -3$. The curve $y^2 = x^3 + 16$ has no $\mathbb{Q}$-rational points of order 2. If $E(\bar{F})$ contains a point of order 2, then $E(\mathbb{Q}[\sqrt{16}]) = K(\sqrt{x^3 + 16})$ contains a root $\alpha$ of $x^3 + 16$. But this cannot be, since 2 ramifies in $\mathbb{Q}(\alpha)/\mathbb{Q}$ and is unramified in $K(\sqrt{16})$.

Thus if $\ell \equiv 3 \pmod{8}$, we have verified there is an $\mathcal{O}_K$-CM elliptic curve $E$ defined over a number field $\bar{F}$ of odd degree such that $E(\bar{F})[\text{tors}] \cong \mathbb{Z}/\ell^n\mathbb{Z}$. If $(\alpha, 0)$ is a point of order 2, then $[\bar{F}(\alpha) : \mathbb{Q}] = 3 : [F : \mathbb{Q}]$ is odd and $E(\bar{F}(\alpha))[\text{tors}] \cong \mathbb{Z}/2\ell^n\mathbb{Z}$. If $\ell \equiv 7 \pmod{8}$, as described above we have an $\mathcal{O}_K$-CM elliptic curve $E$ defined over $F = \mathbb{Q}(j(E))$ and an extension $\bar{F}/F$ of odd degree over $\mathbb{Q}$ such that $E(\bar{F})[\text{tors}]$ contains a point of order $\ell^n$ and no point of order $\ell^{n+1}$. Since $\Delta \equiv 1 \pmod{8}$, by Theorem 4.2 we have a 2-torsion point rational over $F = \mathbb{Q}(j(E))$. Corollary 4.3 ensures we do not have full 2-torsion. Thus we have $E(\bar{F})[\text{tors}] \cong \mathbb{Z}/2\ell^n\mathbb{Z}$. \hfill \Box

### 7.4. Limitation of Odd Degree Torsion.

**Lemma 7.10.** Let $E/F$ be an $K$-CM elliptic curve defined over a number field $F$ of odd degree. Suppose that $E(F)$ contains a point of prime order $\ell > 2$. Then $K = \mathbb{Q}(\sqrt{-\ell})$.

**Proof.** By Real Cyclotomy II we have $\zeta_\ell \in FK$, so $FK$ contains both $K$ and $\mathbb{Q}(\sqrt{-1})^{\ell^n}$. Since $[FK : \mathbb{Q}] \equiv 2 \pmod{4}$, we have $K = \mathbb{Q}(\sqrt{-1})^{\ell^n} = \mathbb{Q}(\sqrt{-\ell})$. \hfill \Box

**Corollary 7.11.** Let $E/F$ be a CM elliptic curve defined over a number field of odd degree. If $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \hookrightarrow E(F)$, then $\Delta = -4$ and $E(F)[\text{tors}] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

**Proof.** Corollary 4.3 gives $\Delta = -4$, so $K = \mathbb{Q}(\sqrt{-1})$. By Lemma 7.10, $E(F)[\text{tors}]$ is a 2-group, and by Corollary 4.5 we have $E(F)[\text{tors}] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. \hfill \Box
Let $E/F$ be a CM elliptic curve defined over an odd degree number field $F$. We will show that $E(F)[\text{tors}]$ is isomorphic to one of the groups listed in Theorem 7.1. If $E(F)[\text{tors}]$ is not cyclic, then since $F$ is real we have $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \hookrightarrow E(F)$ and the result follows from Corollary 7.11. Since all groups of order at most 4 do occur, we may assume that $E(F)[\text{tors}] \cong \mathbb{Z}/n\mathbb{Z}$ for $N \geq 5$. By Corollary 4.11 there is a prime $\ell \equiv 3 \pmod{4}$ and $n \in \mathbb{Z}^+$ such that $N \in \{\ell^n, 2\ell^n\}$. It remains to rule out $N = \ell^n$ with $\ell \equiv 7 \pmod{8}$. Assuming this to be the case, by Lemma 7.10 the CM field is $K = \mathbb{Q}(\sqrt{-\ell})$, so the CM discriminant $\Delta$ is either even or is 1 modulo 8. Then by Theorem 4.2, every $O(\Delta)$-CM elliptic curve defined over $\mathbb{Q}(j(O))$ has an $F$-rational point of order 2. Since $\Delta \notin \{-3, -4\}$ and quadratic twists preserve the 2-torsion subgroup, it follows that every $O(\Delta)$-CM elliptic curve defined over a field $F$ of characteristic 0 has an $F$-rational point of order 2.

**Appendix: Table of Degree Sequences**

Let $m \mid n$ be positive integers; we exclude the pairs $(1,1), (1,2), (1,3), (2,2)$. Then the modular curve $Y(m,n)$ classifying (roughly: the precise description of the moduli problem involves a Cartier-equivariant isomorphism and is omitted here) $(\mu_m \times \mathbb{Z}/n\mathbb{Z})$-structures on elliptic curves is a fine moduli space. For every $j \in \mathbb{Q}$, the fiber of the morphism $Y(m,n) \to Y(1)$ over $j$ is a finite $\mathbb{Q}(j)$-scheme. The reduced subscheme of the fiber is therefore isomorphic to a finite product $\prod_{i=1}^{N} K_i$ of number fields. By the **degree sequence** for $(m,n)$ and $j$ we mean the sequence of degrees of the number fields $K_i(\zeta_m)$, written in non-decreasing order. These are the degrees of the (unique minimal) fields of definition $K$ such that there is an elliptic curve $E_K$ with $O(\Delta)$-CM and an injection $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \hookrightarrow E(K)$.

In the table below we list the degree sequences for the 13 class number one imaginary quadratic discriminants for certain pairs $(m,n)$. The results of this table are used in the proofs of Corollary 4.5 and Theorem 1.4.

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<th>-4</th>
<th>-7</th>
<th>-8</th>
<th>-11</th>
<th>-12</th>
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**Table 2**

**References**


University of Georgia
E-mail address: abourdon@uga.edu

University of Georgia
E-mail address: pclark@gmail.com

University of Bristol/Heilbronn Institute for Mathematical Research
E-mail address: j.stankewicz@bristol.ac.uk