

TORSION POINTS ON CM ELLIPTIC CURVES OVER REAL NUMBER FIELDS

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ABSTRACT. We show that there are only finitely many isomorphism classes of groups of the form $E(F)[\text{tors}]$, where F is a number field such that $[F : \mathbb{Q}]$ is prime and E/F is an elliptic curve with complex multiplication (CM). There are six “Olson groups” which arise as torsion subgroups of CM elliptic curves over number fields of every degree, and there are precisely 17 “non-Olson” CM elliptic curves defined over a prime degree number field.

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We denote by \mathcal{P} the set of all prime numbers. For $n \in \mathbb{Z}^+$ let $\zeta_n = e^{\frac{2\pi i}{n}} \in \mathbb{C}$, and put $\mathbb{Q}(\zeta_n)^+ = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$. A **real number field** is a number field which admits an embedding into \mathbb{R} . Thus every odd degree number field is real.

Date: October 8, 2014.

1. INTRODUCTION

This paper continues an exploration on torsion points on elliptic curves with complex multiplication (CM) over number fields initiated by the last two authors in collaboration with B. Cook, P. Corn and A. Rice [CCS13], [CCRS14].

The entire subject began with the following result.

Theorem 1.1. (*Olson [Ols74]*) *Let E/\mathbb{Q} be a CM elliptic curve. Then $E(\mathbb{Q})[\text{tors}]$ is isomorphic to one of: the trivial group $\{\bullet\}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Conversely, each such group occurs for at least one CM elliptic curve E/\mathbb{Q} .*

We say a finite commutative group G is an **Olson group** if it is isomorphic to one of the six groups given in the conclusion of Theorem 1.1. An elliptic curve E/F is **Olson** if $E(F)[\text{tors}]$ is an Olson group.

The tables of [CCRS14, §4] show that for all $d \leq 13$, every Olson group arises as the torsion subgroup of a CM elliptic curve over some degree d number field. It follows from Theorem 2.1a) that for all $d \geq 2$, all six Olson groups occur in the list of torsion subgroups of CM elliptic curves in degree d , and moreover for any $d_1 \mid d_2$, the list in degree d_2 contains the list in degree d_1 . So it is more penetrating to ask which *new* groups arise in degree d : for $d \in \mathbb{Z}^+$, let $\mathcal{T}_{\text{CM}}(d)$ be the set of isomorphism classes of torsion subgroups of CM elliptic curves defined over number fields of degree d , and for $d \geq 2$ we put

$$\mathcal{T}_{\text{CM}}^{\text{new}}(d) = \mathcal{T}_{\text{CM}}(d) \setminus \bigcup_{d' \mid d, d' \neq d} \mathcal{T}_{\text{CM}}(d').$$

From [CCRS14, §4] we compile the following table.

d	2	3	4	5	6	7	8	9	10	11	12	13
$\#\mathcal{T}_{\text{CM}}^{\text{new}}(d)$	5	2	9	1	7	0	14	3	4	0	13	0

TABLE 1

Remark 1.2. (a) *The size of $\mathcal{T}_{\text{CM}}^{\text{new}}(d)$ is strongly influenced by the 2-adic valuation $v_2(d)$: for all $2 \leq d_1, d_2 \leq 13$, $v_2(d_1) < v_2(d_2) \implies \#\mathcal{T}_{\text{CM}}^{\text{new}}(d_1) < \#\mathcal{T}_{\text{CM}}^{\text{new}}(d_2)$.*

(b) *There is very little new torsion when d is odd.*

(c) *When we restrict to prime values of d , the sequence of values is 5, 2, 1, 0, 0, 0.*

Remark 1.2c) was made to us by M. Schütt. He also asked the following question.

Question 1.3. (*Schütt*) *Is $\#\mathcal{T}_{\text{CM}}^{\text{new}}(p) = 0$ for all sufficiently large primes p ?*

We will answer Question 1.3 affirmatively. In fact we get a more precise result. To state it requires some notation. For $b, c \in F$ we define the **Kubert-Tate curve**

$$E(b, c) : y^2 + (1 - c)xy - by = x^3 - bx^2.$$

For $\lambda \in F$ we define the **Hesse curve**

$$E_\lambda : X^3 + Y^3 + Z^3 + \lambda XYZ = 0.$$

Theorem 1.4. *Let E be a non-Olson CM elliptic curve defined over a prime degree number field F . Then F is isomorphic to one of the fields listed below, and over that field E is isomorphic to exactly one of the 17 listed elliptic curves.*

Number Field F	Elliptic Curve	$E(F)[\text{tors}]$
$\mathbb{Q}(\sqrt{-3})$	E_0	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$
$\mathbb{Q}(i)$	$E(-\frac{1}{8}, 0)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
$\mathbb{Q}(\sqrt{2})$	$E(1 + \frac{3}{4}\sqrt{2}, 0)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
$\mathbb{Q}(\sqrt{2})$	$E(-\frac{1}{32}, 0)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
$\mathbb{Q}(\sqrt{2})$	$E(\frac{1+\sqrt{2}}{8}, 0)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
$\mathbb{Q}(\sqrt{-7})$	$E(\frac{-31+3\sqrt{-7}}{512}, 0)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
$\mathbb{Q}(\sqrt{-7})$	$E(\frac{-1+3\sqrt{-7}}{32}, 0)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
$\mathbb{Q}(\sqrt{-3})$	$E(-\frac{2}{9}, -\frac{1}{3})$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$
$\mathbb{Q}(\sqrt{3})$	$E(\frac{1-\sqrt{3}}{9}, \frac{-2+\sqrt{3}}{3})$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$
$\mathbb{Q}(\sqrt{3})$	$E(\frac{4}{9}, \frac{1}{3})$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$
$\mathbb{Q}(\sqrt{-3})$	$E(\frac{-1+\sqrt{-3}}{2}, -1)$	$\mathbb{Z}/7\mathbb{Z}$
$\mathbb{Q}(i)$	$E(i, i)$	$\mathbb{Z}/10\mathbb{Z}$
$\mathbb{Q}[b]/(b^3 - 15b^2 - 9b - 1)$	$E(\frac{1}{4}b^2 + \frac{5}{2}b + \frac{1}{4}, b)$	$\mathbb{Z}/9\mathbb{Z}$
$\mathbb{Q}[b]/(b^3 + 105b^2 - 33b - 1)$	$E(-\frac{17}{76}b^2 + \frac{25}{19}b + \frac{1}{76}, b)$	$\mathbb{Z}/9\mathbb{Z}$
$\mathbb{Q}[b]/(b^3 - 4b^2 + 3b + 1)$	$E(-2b^2 + 4b + 1, b)$	$\mathbb{Z}/14\mathbb{Z}$
$\mathbb{Q}[b]/(b^3 - 186b^2 + 3b + 1)$	$E(\frac{2}{27}b^2 + \frac{10}{27}b - \frac{1}{27}, b)$	$\mathbb{Z}/14\mathbb{Z}$
$\mathbb{Q}[b]/(b^5 - 9b^4 + 6b^3 + 42b^2 - 7b - 1)$	$E(-\frac{1}{16}b^4 + \frac{1}{4}b^3 + \frac{5}{8}b^2 + \frac{1}{4}b - \frac{1}{16}, b)$	$\mathbb{Z}/11\mathbb{Z}$

We further probe the even/odd dichotomy with the following conditional result.

Theorem 1.5. *Assume Schinzel's Hypothesis H. As F ranges over all number fields of degree $2p$ for a prime number p and E ranges over all CM elliptic curves over F , the set of prime numbers which divide the order of some torsion subgroup $E(F)[\text{tors}]$ is infinite. In particular:*

$$\limsup_{p \in \mathcal{P}} \#\mathcal{T}_{\text{CM}}^{\text{new}}(2p) \geq 1.$$

The results of this paper lead us to the following question.

Question 1.6. *Is there an absolute bound on $\#E(F)[\text{tors}]$ as E ranges over all CM elliptic curves defined over a number field F of odd degree?*

To prove Theorem 1.4 we need results of the form: “if an \mathcal{O} -CM elliptic curve defined over a number field F has an F -rational point of order N , then $[F : \mathbb{Q}]$ is divisible by some function of N and \mathcal{O} .” Prototypical results of this type were given by Silverberg and later by Prasad-Yogananda [Si88] and [PY01]: the **SPY-bounds**. They were refined by Clark-Cook-Stankewicz [CCS13, Theorem 3].

While pursuing further refinements of the SPY-bounds sufficient to prove Theorem 1.4, we noticed another pattern in the tables of [CCRS14]: for every CM elliptic curve E/F in our tables containing an F -rational point of order $N \geq 3$, F contains either the CM field K or $\mathbb{Q}(\zeta_N)^+$. In particular, if F has odd degree then $F \supset \mathbb{Q}(\zeta_N)^+$. This striking **real cyclotomy** phenomenon has not previously been observed. We prove real cyclotomy in many cases. First, if we assume that N is prime to the discriminant Δ of the CM order then we show that F contains an index 2 subfield of $\mathbb{Q}(\zeta_N)$ if it does not contain the CM field. For general $N \geq 3$ there

is more than one such subfield of $\mathbb{Q}(\zeta_N)$, but when N is an odd prime power the unique one is $\mathbb{Q}(\zeta_N)^+$. When N is an even prime power we get the slightly weaker result that F contains $\mathbb{Q}(\zeta_{N/2})^+$ if it does not contain the CM field. Moreover $\mathbb{Q}(\zeta_N)^+$ is characterized among index 2 subfields of $\mathbb{Q}(\zeta_N)$ by being a real number field, so real cyclotomy is also confirmed when F has a real embedding.

When F is real we can show that it contains $\mathbb{Q}(\zeta_N)^+$ even without the assumption $\gcd(N, \Delta) = 1$. This requires a more detailed argument involving an explicit matrix representation of the \mathcal{O} -module structure and the complex conjugation action on $E[N]$ (Theorem 4.9). Here we combine the theory of uniformizations of real elliptic curves by *real lattices* $\Lambda = \bar{\Lambda} \subset \mathbb{C}$ with the ideal theory of quadratic orders. This material has such a classical feel that we suspect that much of it was known in some form to Weber and Deuring. We develop it in detail in §3.

We prove Theorem 1.4 in §5, using the results of §4 and a Theorem of J.L. Parish which is recalled in §5.1. The proof highlights the relevance of Sophie Germain primes, which provided motivation for Theorem 1.5. We prove a more general form of Theorem 1.5 in §6.1. In §6.2 we show that Question 1.6 with “CM” removed has a negative answer. On the other hand, in §6.3 we show that if F/\mathbb{Q} has odd degree d and Galois group S_d , then every CM elliptic curve E/F is Olson.

Acknowledgments: We thank Jordan Ellenberg, Paul Pollack, Damien Robert, Will Sawin, and Matthias Schütt for useful discussions. J.S. was supported by the Villum Fonden through the network for Experimental Mathematics in Number Theory, Operator Algebras, and Topology.

2. TORSION POINTS ON QUADRATIC TWISTS OF ABELIAN VARIETIES

Theorem 2.1. *Let A/F be an abelian variety over a number field, and let $d \geq 2$.*

- a) *There are infinitely many L/F such that $[L : F] = d$ and $A(L)[\text{tors}] = A(F)[\text{tors}]$.*
- b) *If d is prime, then for all but finitely many L/F with $[L : F] = d$, we have $A(L)[\text{tors}] = A(F)[\text{tors}]$.*
- c) *For all but at most finitely many quadratic twists A^t of A/F we have $A^t(F)[\text{tors}] = A^t(F)[2] = A(F)[2]$.*

Proof. a) By work of Masser [Ma87, Corollary 2], there is $N \in \mathbb{Z}^+$ depending only on $[L : \mathbb{Q}]$ such that $A(L)[\text{tors}] = A(L)[N]$. Let $M = F(A[N])$. Then $A(L)[\text{tors}] \supseteq A(F)[\text{tors}]$ implies $A(L)[N] \supseteq A(F)[N]$ and thus $M \cap L \supseteq F$. For each $d \geq 2$ there are infinitely many degree d L/F with $M \cap L = F$: let v be a finite place of F which is unramified in M , and choose L/F to be totally ramified at v . This gives one extension L_1/F ; replacing M with L_1M gives another extension L_2/F ; and so forth.

b) If $d = [L : F]$ is prime, then $M \cap L \supseteq F$ implies $L \subset M$.

c) For $t \in F^\times/F^{\times 2}$, we denote by A^t/F the quadratic twist by d and the involution $[-1]$ on A . We have monomorphisms $A(F) \hookrightarrow A(F(\sqrt{t}))$, $A^t(F) \hookrightarrow A(F(\sqrt{t}))$, and

$$A(F) \cap A^t(F) = A(F)[2] = A^t(F)[2].$$

By part b), for all but finitely many t we have $A(F)[\text{tors}] = A(F(\sqrt{t}))[\text{tors}]$ and thus $A^t(F)[\text{tors}] = A^t(F)[2] = A(F)[2]$. \square

Remark 2.2. *In 2001, Qiu and Zhang used Merel's theorem to prove Theorem 2.1b) when A is an elliptic curve [QZ01, Theorem 1]. When A is an elliptic curve and $F = \mathbb{Q}$, Theorem 2.1c) was proved by Gouvea and Mazur [GM91, Proposition 1]. This extends to a number field F unless A has complex multiplication by an imaginary quadratic field $K \subset F$. Results of Silverberg [Si88] handle the case of all abelian varieties with complex multiplication. Alternately, Mazur and Rubin use Merel's theorem to establish Theorem 2.1c): however, as in their application $A(F)$ has no points of order 2, they record the result (only) in the form that all but finitely many quadratic twists of A/F have no odd order torsion [MR10, Lemma 5.5]. It seems that the full statement of Theorem 2.1c) for elliptic curves first appears in a preprint of F. Najman [Na13, Theorem 12].*

3. \mathbb{R} -STRUCTURES, COMPLEX CONJUGATION AND CARTAN SUBGROUPS

3.1. Orders and Ideals in Imaginary Quadratic Fields.

Let K be a number field. A **lattice** in K is a \mathbb{Z} -module $\Lambda \subset K$ obtained as the \mathbb{Z} -span of a \mathbb{Q} -basis for K . An **order** \mathcal{O} in K is a lattice which is also a subring. The ring of integers \mathcal{O}_K is an order of K ; conversely, since every element of an order \mathcal{O} is integral over \mathbb{Z} , $\mathcal{O} \subset \mathcal{O}_K$ with finite index. For any lattice Λ ,

$$\mathcal{O}(\Lambda) = \{x \in K \mid x\Lambda \subset \Lambda\}$$

is an order of K , and Λ is a fractional $\mathcal{O}(\Lambda)$ -ideal of K . For all $\alpha \in K^\times$ we have $\mathcal{O}(\alpha\Lambda) = \mathcal{O}(\Lambda)$. For any order \mathcal{O} , a fractional \mathcal{O} -ideal Λ is **proper** if $\mathcal{O} = \mathcal{O}(\Lambda)$. A fractional \mathcal{O}_K -ideal is necessarily proper, whereas for any nonmaximal order \mathcal{O} , $[\mathcal{O}_K : \mathcal{O}]\mathcal{O}_K$ is an \mathcal{O} -ideal which is not proper.

For a field F , let \overline{F} be an algebraic closure, let F^{sep} be the maximal separable subextension of \overline{F}/F , and let $\mathfrak{g}_F = \text{Aut}(F^{\text{sep}}/F) = \text{Aut}(\overline{F}/F)$ be the absolute Galois group of F .

Lemma 3.1. *Let \mathcal{O} be an order in a quadratic field K , and let Λ be a fractional \mathcal{O} -ideal. The following are equivalent:*

- (i) Λ is a projective \mathcal{O} -module.
- (ii) For every prime number p , $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a principal fractional $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ -ideal.
- (iii) Λ is a proper \mathcal{O} -ideal.

Proof. To prove (i) \Leftrightarrow (ii) is an exercise in commutative algebra [E, Exercise 4.11]. The local characterization of lattices in any number field gives (ii) \implies (iii) [Lan87, p. 97] and the converse is standard [Lan87, Theorem 9, p. 98]. \square

From now on we assume that K is an imaginary quadratic field. If $\mathcal{O}' \subset \mathcal{O}$ are quadratic orders in K , their discriminants are related as follows:

$$\Delta(\mathcal{O}') = [\mathcal{O} : \mathcal{O}']^2 \Delta(\mathcal{O}).$$

For an order \mathcal{O} in K , we define the **conductor** $\mathfrak{f} = [\mathcal{O}_K : \mathcal{O}]$. Let us write Δ_K for $\Delta(\mathcal{O}_K)$. Then if \mathcal{O} has conductor \mathfrak{f} , we have

$$\Delta(\mathcal{O}) = \mathfrak{f}^2 \Delta_K.$$

Observe that $\Delta(\mathcal{O})$ is negative and congruent to 0 or 1 modulo 4; we call such integers **imaginary quadratic discriminants**. For any K and $\mathfrak{f} \in \mathbb{Z}^+$, $\mathbb{Z}[\frac{\mathfrak{f}^2 \Delta_K + \mathfrak{f}\sqrt{\Delta_K}}{2}]$ is the unique order \mathcal{O} in K of conductor \mathfrak{f} [Lan87, p. 90]. It follows that for every

imaginary quadratic discriminant Δ , there is a unique imaginary quadratic order $\mathcal{O}(\Delta)$ of discriminant Δ .

3.2. Basics on CM Elliptic Curves.

Let A/F be an abelian variety over a field F . By $\text{End } A$ we mean the ring of endomorphisms of A/F^{sep} , endowed with the structure of a \mathfrak{g}_F -module. It is known that $\text{End}^\circ A = \text{End } A \otimes_{\mathbb{Z}} \mathbb{Q}$ is a semisimple \mathbb{Q} -algebra and $\text{End } A$ is an order in $\text{End}^\circ A$. When F has characteristic 0 and $A = E$ is an elliptic curve, $\text{End}^\circ E$ is either \mathbb{Q} or an imaginary quadratic field K : in the latter case we say that E has **complex multiplication (CM)**. Thus $\text{End } E$ is¹ an imaginary quadratic order \mathcal{O} , and we say that E has \mathcal{O} -CM. We summarize some basic facts of CM theory [CCS13, Fact1]. Proofs are found throughout the literature [Cox89, Lan87, Sil94].

Fact 1. (a) *There exists at least one complex elliptic curve with \mathcal{O} -CM.*

(b) *Let E, E' be any two complex elliptic curves with $\mathcal{O}(\Delta)$ -CM. The j -invariants $j(E)$ and $j(E')$ are Galois conjugate algebraic integers. In other words, $j(E)$ is a root of some monic polynomial with \mathbb{Z} -coefficients, and if $P(t)$ is the minimal such polynomial, $P(j'(E)) = 0$ also.*

(c) *Thus there is a unique irreducible, monic polynomial $H_\Delta(t) \in \mathbb{Z}[t]$ whose roots are the j -invariants of all $\mathcal{O}(\Delta)$ -CM complex elliptic curves.*

(d) *The degree of $H_\Delta(t)$ is the class number $h(\Delta) = \#\text{Pic } \mathcal{O}(\Delta)$. In particular, when $\mathcal{O} = \mathcal{O}_K$ we have $\deg(H_\Delta(t)) = h(K)$, the class number of K .*

(e) *Let $F_\Delta := \mathbb{Q}[t]/H_\Delta(t)$. Then F_Δ can be embedded in the real numbers, so in particular is linearly disjoint from the imaginary quadratic field K . Let K_Δ denote the compositum of F_Δ and K . Then K_Δ/K is abelian, with Galois group canonically isomorphic to $\text{Pic}(\mathcal{O})$.*

Let E/\mathbb{C} be an elliptic curve with \mathcal{O} -CM; by the Uniformization Theorem there is a lattice $\Lambda \subset \mathbb{C}$ such that $1 \in \Lambda$ and $E \cong \mathbb{C}/\Lambda$. Then Λ is a fractional \mathcal{O} -ideal of K and $\mathcal{O}(\Lambda) = \mathcal{O}$, so by Lemma 3.1 Λ is a projective \mathcal{O} -module. Conversely, if Λ is a rank one projective \mathcal{O} -module, then $E_\Lambda = (\Lambda \otimes_{\mathcal{O}} \mathbb{C})/\Lambda$ is an elliptic curve, and the \mathbb{C} -isomorphism class of E_Λ depends only on the isomorphism class of Λ as an \mathcal{O} -module. The map $\Lambda \mapsto E_\Lambda$ induces a bijection from $\text{Pic } \mathcal{O}$ to the set of isomorphism classes of elliptic curves E/\mathbb{C} with $\text{End } E \cong \mathcal{O}$.

3.3. \mathbb{R} -structures on Elliptic Curves.

For a subset $S \subset \mathbb{C}$, we put $\overline{S} = \{\overline{z} \mid z \in S\}$. We say a lattice $\Lambda \subset \mathbb{C}$ is **real** if $\overline{\Lambda} = \Lambda$. For a lattice $\Lambda \subset \mathbb{C}$, we associate the complex torus \mathbb{C}/Λ to the Weierstrass equation

$$E_\Lambda : y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda),$$

via the Eisenstein series g_2, g_3 [Sil94, Proposition VI.3.6].

Lemma 3.2. a) *Let E/\mathbb{C} be an elliptic curve. The following are equivalent:*

(i) *There is an elliptic curve $(E_0)_{/\mathbb{R}}$ such that $(E_0)_{/\mathbb{C}} \cong E$.*

(ii) *$j(E) \in \mathbb{R}$.*

(iii) *$E \cong E_\Lambda$ for a real lattice Λ .*

¹An imaginary quadratic order \mathcal{O} has a unique nontrivial ring automorphism (complex conjugation), so there are two different ways to identify \mathcal{O} with $\text{End } E$. As is standard, we take the identification which is compatible with the action of \mathcal{O} on the tangent space at the origin.

b) Let Λ_1, Λ_2 be real lattices. The following are equivalent:

- (i) There is $\alpha \in \mathbb{R}^\times$ such that $\Lambda_2 = \alpha\Lambda_1$.
- (ii) E_{Λ_1} and E_{Λ_2} are isomorphic as elliptic curves over \mathbb{R} .

Proof. To prove (a), it is immediate that (i) \implies (ii). For (ii) \implies (iii), take $g_2, g_3 \in \mathbb{R}$ such that $E' : y^2 = 4x^3 - g_2x - g_3$ is an elliptic curve with j -invariant $j(E)$ [Si86, Proposition III.1.4]. There is a *unique* lattice $\Lambda \subset \mathbb{C}$ such that $g_2(\Lambda) = g_2$ and $g_3(\Lambda) = g_3$ and thus $E' \cong \mathbb{C}/\Lambda$ [Si86, Theorem VI.5.1]. Since $j(E) = j(E')$ and \mathbb{C} is algebraically closed, we have $E \cong \mathbb{C}/\Lambda$. Since $g_2(\Lambda) = \overline{g_2(\Lambda)} = g_2(\overline{\Lambda})$, $g_3(\Lambda) = \overline{g_3(\Lambda)} = g_3(\overline{\Lambda})$, we have $\overline{\Lambda} = \Lambda$. Finally, if Λ is a real lattice then $g_2(\Lambda), g_3(\Lambda) \in \mathbb{R}$ and (iii) \implies (i) is immediate. Alternately, since $\overline{\Lambda} = \Lambda$, complex conjugation on \mathbb{C} descends to an antiholomorphic involution on \mathbb{C}/Λ and thus gives descent data for an \mathbb{R} -structure on E .

To prove (b), if $\Lambda_1, \Lambda_2 \subset \mathbb{C}$, we have $\mathbb{C}/\Lambda_1 \cong \mathbb{C}/\Lambda_2$ iff $\Lambda_2 = \alpha\Lambda_1$ for some $\alpha \in \mathbb{C}^\times$. In terms of Weierstrass equations, $E_{\alpha\Lambda}$ is the quadratic twist of E_Λ by α^2 . Thus $E_{\Lambda_1} \cong E_{\alpha\Lambda_1} = E_{\Lambda_2}$ if $\alpha \in \mathbb{R}$. Conversely, if $E_{\Lambda_1} \cong E_{\Lambda_2}$, then the standard theory of Weierstrass equations [Si86, § III.1] shows: there is $\alpha \in \mathbb{R}^\times$ with $g_2(\Lambda_2) = \alpha^4 g_2(\Lambda_1) = g_2(\alpha^{-1}\Lambda_1)$, $g_3(\Lambda_2) = \alpha^6 g_3(\Lambda_1) = g_3(\alpha^{-1}\Lambda_2)$ and thus $\Lambda_2 = \alpha^{-1}\Lambda_1$. \square

Lemma 3.3. a) Let Λ be a real lattice. If $j(E_\Lambda) \neq 1728$, then $E_{i\Lambda}$ and E_Λ are isomorphic as \mathbb{C} -elliptic curves but not as \mathbb{R} -elliptic curves. If $j(E_\Lambda) = 1728$, then $E_{\zeta_8\Lambda}$ and E_Λ are isomorphic as \mathbb{C} -elliptic curves but not as \mathbb{R} -elliptic curves.

b) Let $j \in \mathbb{R}$. Then there are precisely two \mathbb{R} -isomorphism classes of elliptic curves E/\mathbb{R} with $j(E) = j$.

Proof. If $j \neq 1728$ then $g_3(\Lambda) \neq 0$, so $g_3(i\Lambda) = -g_3(\Lambda)$, whereas as above any real change of variables takes $g_3(\Lambda) \mapsto \alpha^{-6}g_3(\Lambda)$. If $j = 1728$ then $g_3(\Lambda) = 0$, so the above argument shows instead that $i\Lambda = \Lambda$ (as it should, since Λ is homothetic to $\mathbb{Z}[i]$). In this case $g_2(\Lambda) \neq 0$ and $g_2(\zeta_8\Lambda) = -g_2(\Lambda)$, whereas any real change of variables takes $g_3(\Lambda) \mapsto \alpha^{-4}g_3(\Lambda)$. The standard theory of real elliptic curves gives (b) [Sil94, Prop. V.2.2]. \square

Lemma 3.4. Let \mathcal{O} be an order in the imaginary quadratic field K , and let I be a proper fractional \mathcal{O} -ideal. The following are equivalent:

- (i) $[I] = [\overline{I}] \in \text{Pic } \mathcal{O}$.
- (ii) I^2 is principal.
- (iii) $j(E_I) \in \mathbb{R}$.

Proof. a) Since $I\overline{I} = N_{K/\mathbb{Q}}(I)\mathcal{O}$, we have $[\overline{I}] = [I]^{-1} \in \text{Pic } \mathcal{O}$, so (i) \iff (ii). Work of Shimura gives (ii) \iff (iii) [Sh, (5.4.3)]. \square

For an imaginary quadratic discriminant Δ , let $\tau_\Delta = \frac{\Delta + \sqrt{\Delta}}{2}$, so $\mathcal{O}(\Delta) = \mathbb{Z}[\frac{\Delta + \sqrt{\Delta}}{2}]$ is the imaginary quadratic order of discriminant Δ . Then $j(\mathbb{C}/\mathcal{O}) = j(\tau_\Delta)$. Let $\sigma_1, \dots, \sigma_{h(\Delta)} : \mathbb{Q}(j(\tau_\Delta))/\mathbb{Q} \hookrightarrow \mathbb{C}$ be the $\# \text{Pic } \mathcal{O}(\Delta)$ field embeddings, with σ_1 taken to be inclusion. By Lemma 3.4, $j(\tau_\Delta) \in \mathbb{R}$. The other embeddings $\sigma_2, \dots, \sigma_h$ may in general be either real or complex: Lemma 3.4 implies that the number of real embeddings is $\#(\text{Pic } \mathcal{O})[2]$.

Lemma 3.5. For an imaginary quadratic discriminant Δ , let r be the number of distinct odd prime divisors of Δ . We define μ as follows:

$$\mu = \begin{cases} r, & \Delta \equiv 1 \pmod{4} \text{ or } \Delta \equiv 4 \pmod{16} \\ r+1, & \Delta \equiv 8, 12 \pmod{16} \text{ or } \Delta \equiv 16 \pmod{32} \\ r+2, & \Delta \equiv 0 \pmod{32}. \end{cases}$$

a) We have $(\text{Pic } \mathcal{O}(\Delta))[2] \cong (\mathbb{Z}/2\mathbb{Z})^{\mu-1}$.

b) There are precisely 2^μ \mathbb{R} -homothety classes of \mathcal{O} -CM real lattices.

Proof. Part (a) is due to Gauss [Cox89, Theorem 3.15],[HK, Theorem 5.6.11]. Part (b) is given by combining part (a) with Lemmas 3.3 and 3.4. \square

A fractional \mathcal{O} -ideal I is **primitive** if $I \subset \mathcal{O}$ and for all $e \geq 2$, $I \not\subset e\mathcal{O}$.

Lemma 3.6. a) Let $E \cong_{\mathbb{R}} E_\Lambda$ be a real \mathcal{O} -CM elliptic curve. The \mathbb{R} -homothety class of Λ contains a unique primitive \mathcal{O} -ideal I . The ideal I is proper and real.

b) ([HK, Theorem 5.6.4]) There are precisely 2^μ primitive proper real \mathcal{O} -ideals.

Proof. The lattice Λ contains an element $a + bi$ with $a \neq 0$. Since Λ is real, we have $a - bi \in \Lambda$ and thus also $2a = (a + bi) + (a - bi) \in \Lambda$. Then $\frac{1}{2a}\Lambda$ is a proper \mathcal{O} -ideal which is \mathbb{R} -homothetic to Λ . If two fractional \mathcal{O} -ideals are \mathbb{R} -homothetic, then one is real iff the other is real, and the \mathbb{R} -homothety class of any fractional \mathcal{O} -ideal contains a unique primitive \mathcal{O} -ideal. To prove part (b), combine part (a) with Lemma 3.5b). \square

Ideals of this type are completely classified. We use the following notation: for $\alpha, \beta \in \mathbb{C}$ which are linearly independent over \mathbb{R} , we define the lattice

$$[\alpha, \beta] = \{a\alpha + b\beta \mid a, b \in \mathbb{Z}\}.$$

Theorem 3.7. [HK, Theorem 5.6.4] Let \mathcal{O} be an order in K of discriminant Δ . A primitive proper \mathcal{O} -ideal I is real iff it is one of the following two types:

$$(1) \quad I = \left[a, \frac{\sqrt{\Delta}}{2} \right], \text{ where } a \in \mathbb{Z}^+, 4a \mid \Delta \text{ and } \left(a, \frac{\Delta}{4a} \right) = 1$$

$$(2) \quad I = \left[a, \frac{a + \sqrt{\Delta}}{2} \right], \text{ where } a \in \mathbb{Z}^+, 4a \mid a^2 - \Delta \text{ and } \left(a, \frac{a^2 - \Delta}{4a} \right) = 1.$$

Moreover, if $\Delta \equiv 1 \pmod{4}$, there are no such ideals of type (1).

Corollary 3.8. Let I be a primitive proper real \mathcal{O} -ideal. Then $[\mathcal{O} : I] \mid \Delta$.

Proof. For all ideals I of the form (1) and (2) above, we have that $[\mathcal{O} : I] = a \mid \Delta$ [HK, Theorem 5.4.2]. \square

Theorem 3.9. Let Δ be an imaginary quadratic discriminant.

a) Let $F \subset \mathbb{R}$, and let $E_{/F}$ be an \mathcal{O} -CM elliptic curve. Then there is an \mathcal{O} -CM elliptic curve $E'_{/F}$ such that $E'_{\mathbb{R}} \cong E_{\mathcal{O}}$ and an F -rational isogeny $\varphi : E \rightarrow E'$.

b) Let N be a positive integer which is prime to Δ . Then the isogeny φ of part a) induces a \mathfrak{g}_F -module isomorphism $E[N] \xrightarrow{\sim} E'[N]$.

Proof. a) For a nonzero ideal I of \mathcal{O} , let $E[I] = \{x \in E(\overline{F}) \mid \forall \varphi \in I, \varphi(x) = 0\}$, so $E[I]$ is a finite subgroup which is \mathfrak{g}_{FK} -stable. Put $E' = E/E[I]$. If $E \cong_{\mathbb{R}} E_\Lambda$, then

$$E' \cong_{\mathbb{C}} E_{\frac{1}{[\mathcal{O}:I]}\Lambda}.$$

Suppose now that I is real and proper. Then $E[I]$ is \mathfrak{g}_F -stable and

$$E' \cong_{\mathbb{R}} E_{\frac{1}{[\mathcal{O}, I]}I\Lambda} \cong E_{I^{-1}\Lambda}.$$

Applying this with $I = \Lambda$ gives the result. Part b) follows immediately. \square

Remark 3.10. *After this section was written we found a paper of S. Kwon [Kw99] which contains related results. Especially, an equivalent form of [Kw99, Prop. 2.3a)] appears in the proof of Theorem 2.9a), and the classification of primitive, proper real ideals in an imaginary quadratic order is given in [Kw99, §3] and is used to classify cyclic isogenies on CM elliptic curves rational over $\mathbb{Q}(j(E))$.*

3.4. Cartan Subgroups.

Let $\Lambda \subset \mathbb{C}$ be a lattice, and let $E_\Lambda = \mathbb{C}/\Lambda$. For $N \in \mathbb{Z}^+$ and $\ell \in \mathcal{P}$, put

$$\Lambda_N = \left(\frac{1}{N}\right)\Lambda/\Lambda = E_\Lambda[N]$$

$$T_\ell\Lambda = \varprojlim \Lambda_{\ell^n} = T_\ell(E_\Lambda),$$

$$\widehat{\Lambda} = \prod_{\ell \in \mathcal{P}} T_\ell\Lambda.$$

If $F \subset \mathbb{C}$ and $E_{/F}$ is an elliptic curve, then $E_{/\mathbb{C}} \cong E_\Lambda$ for some lattice Λ , uniquely determined up to homothety. If $F \subset \mathbb{R}$, then $E_{/\mathbb{R}} \cong E_\Lambda$ for some real lattice Λ , uniquely determined up to real homothety.

We have $E(\mathbb{C})[\text{tors}] = E(\overline{F})[\text{tors}]$, so $\text{Aut}(\mathbb{C}/F)$ acts on Λ_N , $T_\ell\Lambda$ and $\widehat{\Lambda}$. We assume that $\text{Aut}(\mathbb{C}/\overline{F})$ is a normal subgroup of $\text{Aut}(\mathbb{C}/F)$: this holds if F is a number field or if $F = \mathbb{R}$. Then we get an induced action of \mathfrak{g}_F on Λ_N . If moreover $F \subset \mathbb{R}$, then complex conjugation $c \in \text{Aut}(\mathbb{C}/\mathbb{R}) \subset \text{Aut}(\mathbb{C}/F) \rightarrow \mathfrak{g}_F$ acts on Λ_N .

Let \mathcal{O} be an imaginary quadratic order. For N, ℓ as above, consider the \mathcal{O} -algebras

$$\mathcal{O}_N = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z},$$

$$T_\ell(\mathcal{O}) = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell = \varprojlim \mathcal{O}_{\ell^n},$$

$$\widehat{\mathcal{O}} = \mathcal{O} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} = \prod_{\ell} T_\ell\mathcal{O}.$$

Let $E_{/F}$ be an \mathcal{O} -CM elliptic curve. As above, there is a proper integral \mathcal{O} -ideal Λ , with uniquely determined class in $\text{Pic } \mathcal{O}$, such that $E_{/\mathbb{C}} \cong E_\Lambda$. Then Λ_N (resp. $T_\ell\Lambda$, resp. $\widehat{\Lambda}$) has a natural \mathcal{O}_N -module (resp. $T_\ell\mathcal{O}$ -module, resp. $\widehat{\mathcal{O}}$ -module) structure. If $F \subset \mathbb{R}$ we may take Λ to be a real ideal: $\overline{\Lambda} = \Lambda$.

Lemma 3.11.

- a) For every $N \in \mathbb{Z}^+$, $\Lambda[N] = E[N]$ is free of rank 1 as an \mathcal{O}_N -module.
b) $T_\ell(\Lambda) = T_\ell(E)$ (resp. $\widehat{\Lambda} = \prod_{\ell \in \mathcal{P}} T_\ell(E)$) is free of rank 1 as $T_\ell(\mathcal{O})$ -module (resp. as a $\widehat{\mathcal{O}}$ -module).

Proof. Part (b) is known by work of Serre and Tate [ST68, p. 502] while part (a) can be deduced from work of Parish [Pa89, Lemma 1]. Either part can be used to deduce the other. \square

In particular, for all primes ℓ we have a homomorphism of \mathbb{Z}_ℓ -algebras

$$\iota_\ell : T_\ell(\mathcal{O}) \rightarrow \text{End } T_\ell E.$$

The map ι_ℓ is \mathfrak{g}_F -equivariant; further, it is injective with torsionfree cokernel [M, Lemma 12.2]. Tensoring with $\mathbb{Z}/\ell^n\mathbb{Z}$ and applying primary decomposition, we get for each $N \in \mathbb{Z}^+$ an injective \mathfrak{g}_F -equivariant ring homomorphism

$$\mathcal{O}_N \hookrightarrow \text{End } E[N].$$

Tensoring to \mathbb{Q}_ℓ gives

$$\iota_\ell^0 : V_\ell(\mathcal{O}) \hookrightarrow \text{End } V_\ell(E).$$

We define the **Cartan subalgebras**

$$\mathcal{C}_\ell = \iota_\ell(T_\ell(\mathcal{O})) \subset \text{End } T_\ell(E),$$

$$\mathcal{C}_\ell^0 = \iota_\ell(V_\ell(\mathcal{O})) \subset \text{End } V_\ell(E)$$

and the **Cartan subgroups**

$$\mathcal{C}_\ell^\times \subset \text{Aut } T_\ell(E),$$

$$(\mathcal{C}_\ell^0)^\times \subset \text{Aut } V_\ell(E).$$

Then $\mathcal{C}_\ell^0 \cong K \otimes \mathbb{Q}_\ell$ is a maximal etale subalgebra of $\text{End } V_\ell(E) \cong M_2(\mathbb{Q}_\ell)$. We may view $\mathcal{C}_\ell^0 \hookrightarrow M_2(\mathbb{Q}_\ell)$ as the regular representation. We write $C(\mathcal{C}_\ell^0)$ for the commutant and $N(\mathcal{C}_\ell^0)$ for the normalizer of \mathcal{C}_ℓ^0 inside $\text{End } V_\ell(E)$. By the Double Centralizer Theorem [Pi, §12.7], we have

$$C(\mathcal{C}_\ell^0) = \mathcal{C}_\ell^0.$$

Using the Skolem-Noether Theorem [Pi, §12.6], we find that

$$N\mathcal{C}_\ell^0/(\mathcal{C}_\ell^0)^\times \cong \text{Aut}_{\mathbb{Q}_\ell} \mathcal{C}_\ell^0$$

has order 2.

The fixed field of the kernel of the representation $\mathfrak{g}_F \rightarrow V_\ell(\mathcal{O})$ is FK , so

$$\rho_{\ell^\infty}(\mathfrak{g}_{FK}) \subset \mathcal{C}_\ell^0,$$

and if $FK \supsetneq F$ then

$$\rho_{\ell^\infty}(\mathfrak{g}_F) \not\subset \mathcal{C}_\ell^0.$$

In fact [ST68, §4, Corollary 2] we have

$$\rho_{\ell^\infty}(\mathfrak{g}_{FK}) \subset \iota_\ell(T_\ell(\mathcal{O})^\times).$$

Moreover, \mathfrak{g}_F -equivariance of ι_ℓ^0 gives

$$\rho_{\ell^\infty}(\mathfrak{g}_F) \subset N\mathcal{C}_\ell^0.$$

This recovers a standard result of Serre [Se66, Theorem 5].

Lemma 3.12. *Let $G_{\ell^\infty} = \rho_{\ell^\infty}(\mathfrak{g}_F)$ be the image of the ℓ -adic Galois representation. The following are equivalent:*

- (i) G_{ℓ^∞} lies in the Cartan subgroup.
- (ii) G_{ℓ^∞} is commutative.
- (iii) $K \subset F$.

We now deduce a stronger version of a result of Serre [CCS13, Lemma 15]. Let us first note the following in the case $\Lambda = \mathcal{O}$.

Lemma 3.13. *Let $K = \mathbb{Q}(\sqrt{\Delta_0})$ be an imaginary quadratic field, and let \mathcal{O} be an order in K of discriminant $\Delta = \mathfrak{f}^2 \Delta_0$: thus $\mathcal{O} = \mathbb{Z} \left[\frac{\Delta + \sqrt{\Delta}}{2} \right]$. Let c be the nontrivial element of $\text{Aut}(K/\mathbb{Q})$. Let $N \geq 2$, put $\mathcal{O}_N = (1/N)\mathcal{O}/\mathcal{O}$ and $i\mathcal{O}_N = (\frac{i}{N})\mathcal{O}/(i\mathcal{O})$.*

a) If Δ is even or N is odd, then $\frac{1}{N}, \frac{\sqrt{\Delta}}{N}$ (resp. $\frac{i}{N}, \frac{\sqrt{-\Delta}}{N}$) is a $\mathbb{Z}/N\mathbb{Z}$ -basis for \mathcal{O}_N (resp. $i\mathcal{O}_N$). The corresponding matrix of c is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (resp. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$).

b) In all cases $\frac{1}{N}, \frac{\Delta + \sqrt{\Delta}}{2N}$ (resp. $i \left(\frac{\Delta + \sqrt{\Delta}}{2N} \right)$) is a $\mathbb{Z}/N\mathbb{Z}$ -basis for \mathcal{O}_N (resp. $i\mathcal{O}_N$). The corresponding matrix of c is $\begin{bmatrix} 1 & \Delta \\ 0 & -1 \end{bmatrix}$ (resp. $\begin{bmatrix} -1 & -\Delta \\ 0 & 1 \end{bmatrix}$).

Corollary 3.14. *a) If $N \geq 3$, then c acts nontrivially on \mathcal{O}_N .
b) If $N = 2$, then c acts nontrivially on \mathcal{O}_N iff Δ is odd.*

Lemma 3.15. *Let K be an imaginary quadratic field, and let \mathcal{O} be an order in K of discriminant $\Delta = \mathfrak{f}^2 \Delta_0$. Let F be a field of characteristic 0, and let E/F be an \mathcal{O} -CM elliptic curve. Let $N \in \mathbb{Z}^+$, and suppose at least one of the following holds:*

- $N \geq 3$;
- $N = 2$ and Δ is odd.

Then $F(E[N]) \supset K$.

Proof. We may certainly assume $K \not\subset F$. Let $\sigma \in \mathfrak{g}_F$ be any element which restricts nontrivially to KF . Then by Corollary 3.14, σ acts nontrivially on \mathcal{O}_N . Since $\iota_N : \mathcal{O}_N \hookrightarrow \text{End } E[N]$ is injective and \mathfrak{g}_F -equivariant, it follows that σ acts nontrivially on $\text{End } E[N]$. It can be shown that for any G -module M , if $\sigma \in G$ acts nontrivially on $\text{End}(M)$ then σ acts nontrivially on M . \square

3.5. A Result on Torsion Fields.

Theorem 3.16. *Let \mathcal{O} be an order in an imaginary quadratic field K . Let F be a field of characteristic 0, and let E/F be an \mathcal{O} -CM elliptic curve. Let $N \in \mathbb{Z}^+$. Let $\mathfrak{h}_{/FK} : E \rightarrow \mathbb{P}^1$ be a **Weber function** for E : that is, \mathfrak{h} is the composition of the quotient map $E \rightarrow E/(\text{Aut } E)$ with an isomorphism $E/(\text{Aut } E) \cong \mathbb{P}^1$. Then the field $FK(\mathfrak{h}(E[N]))$ contains the N -ray class field $K^{(N)}$ of K .*

Remark 3.17. *When $\mathcal{O} = \mathcal{O}_K$, the equality $K(j(\mathbb{C}/\mathcal{O}_K), \mathfrak{h}(E[N])) = K^{(N)}$ is one of the central results of the classical theory of complex multiplication [Sil94, Theorem II.5.6]. We believe that the general case is also “classically known”, but for lack of a suitable reference we include a proof. In fact we include two proofs: the first one is intended for readers who are using Silverman’s text [Sil94] as a primary reference and shows how to modify the proof given there for \mathcal{O}_K to the general case. The second uses the adelic perspective [Lan87].*

Proof. The fields generated by values of a Weber function \mathfrak{h} are independent of the F -rational model of E [Sil94, Example II.5.5.2]. Since E has a model over $\mathbb{Q}(j(E))$, we may and shall assume $F = \mathbb{Q}(j(E))$.

FIRST PROOF: Let $L = FK(\mathfrak{h}(E[N]))$. When $\mathcal{O} = \mathcal{O}(\Delta_K)$, we consider the proof of $L = K^{(N)}$ given in Silverman [Sil94, Theorem II.5.6]. The argument proceeds by showing that for all but finitely many degree 1 prime ideals \mathfrak{p} of \mathcal{O}_K , $\mathfrak{p} \in P(N)$ (that is, $\mathfrak{p} = (\pi)$ for some $\pi \equiv 1 \pmod{N\mathcal{O}_K}$) if and only if \mathfrak{p} splits completely in L . The “only if” direction uses that $K(j(E)) = K^{(1)}$ is the Hilbert class field of

K . In general

$$L \supset K(j(E)) = K_D \supset K^{(1)}$$

so this implication is not valid for (most) non-maximal orders. However, the “if” direction, which relates Frobenius elements of K_Δ to the Frobenius on reductions of $\mathcal{O}(\Delta)$ -CM elliptic curves, goes through essentially verbatim with one proviso (to be addressed shortly).

Granting that, we get that for all but finitely many primes \mathfrak{p} of \mathcal{O} , if \mathfrak{p} splits completely in L then it splits completely in K_D , and then Chebotarev Density implies that $L \supset K_D$. Recall that if $\iota : \mathcal{O} \hookrightarrow \mathcal{O}_K$ is the natural inclusion, then $\iota_* : \mathfrak{p} \mapsto \mathfrak{p}\mathcal{O}_K$ induces a bijection between maximal ideals of \mathcal{O} prime to \mathfrak{f} and maximal ideals of \mathcal{O}_K prime to \mathfrak{f} which preserves degrees.

The proviso is that the proof given [Sil94, Proposition II.5.3] is only valid for $\Delta = \Delta_K$. But it is easy to reduce the general case to the $\mathcal{O}(\Delta_K)$ -CM case: there is a cyclic degree \mathfrak{f} F -isogeny $\varphi : E \rightarrow E'$, where $\text{End } E' = \mathcal{O}_K$ [CCS13, Proposition 25]. Let $\sigma_{\mathfrak{p}} = (\mathfrak{p}, K_D/K)$ be the Frobenius element. Then the composite map

$$E' \xrightarrow{\lambda} (E')^{\sigma_{\mathfrak{p}}} \xrightarrow{\widehat{\varphi}^{\sigma_{\mathfrak{p}}}} E^{\sigma_{\mathfrak{p}}}$$

factors through E : to see this, it suffices to show that the kernel K of $\widehat{\varphi}$ is contained in the kernel of the composite map: the first map takes K to $K^{\sigma_{\mathfrak{p}}}$, and the kernel of the second map is $K^{\sigma_{\mathfrak{p}}}$. So we get an isogeny

$$\lambda : E \rightarrow E^{\sigma_{\mathfrak{p}}}$$

whose reduction modulo any prime of K_D lying over \mathfrak{p} is the p th power Frobenius. SECOND PROOF: We use the results and notation of Lang [Lan87, §10.3]. Applying Theorem 7 first with $\mathfrak{a} = \mathcal{O}$ and $u = \frac{1}{N}$ and then with $\mathfrak{a} = \mathcal{O}_K$ and $u = \frac{1}{N}$. We observe that for an ideal b , $b\mathcal{O} = \mathcal{O} \implies b\mathcal{O}_K = \mathcal{O}_K$. This is much as in Theorem 6. We conclude

$$L \supset K(j(\mathbb{C}/\mathcal{O}), h(\frac{1}{N} + \mathcal{O})) \supset K(j(\mathbb{C}/\mathcal{O}_K), h(\frac{1}{N} + \mathcal{O}_K)).$$

But as an \mathcal{O}_K -module, $\frac{1}{N}\mathcal{O}_K/\mathcal{O}_K$ is generated by $\frac{1}{N} + \mathcal{O}_K$ [Lan87, p. 135], so

$$K(j(\mathbb{C}/\mathcal{O}_K), h(\frac{1}{N} + \mathcal{O}_K)) = K(j(\mathbb{C}/\mathcal{O}_K), h(E[N])) = K^{(N)}. \quad \square$$

4. RESULTS ON TORSION POINTS ON CM ELLIPTIC CURVES

Throughout this section we will use the following setup: \mathcal{O} is an imaginary quadratic order with fraction field K . Let Δ_K be the discriminant of \mathcal{O}_K , \mathfrak{f} the conductor of \mathcal{O} , and Δ the discriminant of \mathcal{O} , so $\Delta = \mathfrak{f}^2\Delta_K$. Let F be a subfield of \mathbb{C} , and let E/F be an \mathcal{O} -CM elliptic curve. Again $\mathfrak{h}_{/FK}$ will denote a Weber function.

4.1. Points of Order 2.

Let \mathcal{O} be an imaginary quadratic order of discriminant $\Delta < -4$, with fraction field K . Let E/\mathbb{C} be an \mathcal{O} -CM elliptic curve. Let $F = \mathbb{Q}(j(E))$, and let $L = F(E[2])$, so L/F is Galois of degree dividing 6. By Fact 1, the isomorphism class of F depends only on Δ . Since $\Delta < -4$, the x -coordinate is a Weber function on E , and thus L does not depend upon the chosen Weierstrass model (a fortiori, any two \mathcal{O} -CM elliptic curves with the same j -invariant are *quadratic* twists of each other, and

2-torsion points are invariant under quadratic twist). Thus as an abstract number field and a Galois extension thereof, F and L depend only on Δ .

Theorem 4.1. (*Parish*) For all $\Delta < -4$, $F \subsetneq L$.

Proof. Equivalently, E does not have full 2-torsion defined over $F = \mathbb{Q}(j(E))$. This follows from more precise results of Parish [Pa89, Table 1]. \square

Theorem 4.2. Let \mathcal{O} be an imaginary quadratic order of discriminant $\Delta < -4$ and with fraction field K . Let $E_{/C}$ be an elliptic curve with \mathcal{O} -CM. Let $F = \mathbb{Q}(j(E))$, and let $L = F(E[2])$.

- a) We have $K \subset L$ iff Δ is odd.
- b) If $\Delta \equiv 1 \pmod{8}$, then $L = FK$ and thus $[L : F] = 2$.
- c) If $\Delta \equiv 5 \pmod{8}$, then $[L : F] = 6$.
- d) If Δ is even, then $[L : F] = 2$.

Proof. a) If Δ is odd, then Lemma 3.15 gives $K \subset L$. Suppose Δ is even. Then Lemma 3.13 implies $E[2](\mathbb{R}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since $K \not\subset \mathbb{R}$, the result follows.

b) If $\Delta \equiv 1 \pmod{8}$, then the mod 2 Cartan subgroup is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\times \times (\mathbb{Z}/2\mathbb{Z})^\times$, better known as the trivial group. It follows that $FK(E[2]) = FK$. Together with part a) this shows $L = F(E[2]) = FK$.

c) By part a) we have $K \subset L$, and thus by Theorem 3.16 we have $L \supset K^{(2)}$, the 2-ray class field of K . Since $\Delta \equiv 5 \pmod{8}$, we have $[K^{(2)} : K^{(1)}] = 3$. Since for any elliptic curve $E_{/F}$ we have $[F(E[2]) : F] \mid 6$, the result follows.

d) Since Δ is even, the mod 2 Cartan subgroup is cyclic of order $2^2 - 2 = 2$, and thus $[FK(E[2]) : FK] \mid 2$. It follows by using the result of part a) that $K \not\subset F(E[2])$, or just the fact that $[L : F] \mid 6$ so we cannot have $[L : F] = 4$, that $[L : F] \mid 2$. Combining with Theorem 4.1 we get the result. \square

Corollary 4.3. Suppose $\Delta \neq -4$. Let F be a number field, and let $E_{/F}$ be an $\mathcal{O}(\Delta)$ -CM elliptic curve. If $\mathbb{Z}/2 \times \mathbb{Z}/2\mathbb{Z} \subset E(F)$, then $2 \mid [F : \mathbb{Q}]$.

Proof. Step 1: Suppose $\Delta = -3$. Then E has an equation of the form $y^2 = x^3 + B$. Since $E(F)$ has full 2-torsion, $x^3 + B$ splits in F and $\mathbb{Q}(\sqrt{-3}) \subset F$.

Step 2: Suppose $\Delta < -4$. Since $\mathbb{Q}(E[2]) = \mathbb{Q}(\mathfrak{h}(E[2]))$, the 2-torsion field is independent of the model of E . We may thus assume without loss of generality that E is obtained by base extension from an elliptic curve $E_{/\mathbb{Q}(j(E))}$. Applying Theorem 4.2 we get

$$2 \mid [\mathbb{Q}(j(E), E[2]) : \mathbb{Q}(j(E))] \mid [F : \mathbb{Q}]. \quad \square$$

Remark 4.4. The elliptic curve $E_{/\mathbb{Q}} : y^2 = x^3 - x$ shows that the hypothesis $\Delta \neq -4$ in Corollary 4.3 is necessary.

Corollary 4.5. If $[F : \mathbb{Q}]$ is odd, $E_{/F}$ is a CM elliptic curve and $(\mathbb{Z}/2\mathbb{Z})^2 \subset E(F)$ then $E(F)[12] = (\mathbb{Z}/2\mathbb{Z})^2$.

Proof. By Corollary 4.3 we may assume $\Delta = -4$. It is enough to show that $E(F)$ has no subgroup isomorphic to $\mathbb{Z}/6\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. This is immediate from Table 2. \square

4.2. A refinement of the SPY bounds.

Theorem 4.6. *Let \mathcal{O} be an imaginary quadratic order of discriminant Δ ; put $K = \mathbb{Q}(\sqrt{\Delta})$. Let F be a number field, and let E/F an elliptic curve with \mathcal{O} -CM. Let $w(K) = \#\mathcal{O}_K^\times$. Suppose $E(F)[\text{tors}]$ contains a point of prime order $\ell > 2$.*

a) *If $(\frac{\Delta}{\ell}) = -1$, then*

$$\left(\frac{2(\ell^2 - 1)}{w(K)}\right) h(K) \mid [FK : \mathbb{Q}].$$

b) *If $(\frac{\Delta}{\ell}) \neq -1$, then*

$$\left(\frac{2(\ell - 1)}{w(K)}\right) h(K) \mid [FK : \mathbb{Q}].$$

When $\mathcal{O} = \mathcal{O}_K$ and $\ell \nmid \Delta$, Theorem 4.6 was proved in part by the last two authors [CCS13, Theorem 2]. Further, the hypothesis $\mathcal{O} = \mathcal{O}_K$ comes into the proof only via the statement that $K(j(E))(\mathfrak{h}(E[N])) = K^{(N)}$, the N -ray class field of K . In fact the argument uses only that the former field contains the latter field, and this holds by Theorem 3.16.

Proof. Suppose $\ell > 2$ ramifies in K . Thus

$$\mathcal{O}_\ell = \mathcal{O} \otimes \mathbb{Z}/\ell\mathbb{Z} \cong \mathbb{F}_\ell[t]/(t^2).$$

Thus its image $C_\ell = \iota(\mathcal{O}_\ell) \subset \text{End } E[\ell] \cong M_2(\mathbb{F}_\ell)$ is generated over the scalar matrices by a single nonzero nilpotent matrix g . Since the eigenvalues of g are \mathbb{F}_ℓ -rational we can put it in Jordan canonical form over \mathbb{F}_ℓ , and thus we get a choice of basis e_1, e_2 of $E[\ell]$ such that

$$C_\ell \cong \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix} \mid \alpha, \beta \in \mathbb{F}_\ell \right\}.$$

We will show that under the hypotheses of Theorem 4.6, we have

$$(3) \quad \frac{2(\ell - 1)h(K)}{w(K)} \mid [FK : \mathbb{Q}].$$

Let $x = ae_1 + be_2 \in E(F)$ have order ℓ . For all $S = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix} \in \rho_\ell(\mathfrak{g}_{FK})$ we have

$$(\alpha a + \beta b)e_1 + (\alpha b)e_2 = Sx = x = ae_1 + be_2,$$

and thus

$$(\alpha - 1)b = (\alpha - 1)a + \beta b = 0.$$

If $\alpha \neq 1$, then $b = 0$ and thus also $a = 0$ – contradiction – so $\alpha = 1$ and $\rho_\ell(\mathfrak{g}_{FK})$ consists of elements of the form $\begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}$ and $\rho_\ell(\mathfrak{g}_{FK})$ has size 1 or ℓ .

Case 1: Suppose $\#\rho_\ell(\mathfrak{g}_{FK}) = 1$. By Theorem 3.16 we have $FK \supset K(j(E), \mathfrak{h}(E[\ell]))$, and so [CCS13, Corollary 9]

$$[K(j(E), \mathfrak{h}(E[\ell])) : \mathbb{Q}] = \left(\frac{2(\ell - 1)h(K)}{w(K)}\right) \ell \mid [FK : \mathbb{Q}].$$

This gives (3), in fact with an extra factor of ℓ .

Case 2: If $\#\rho_\ell(\mathfrak{g}_{FK}) = \ell$, then trivialize ρ_ℓ by passing to a degree ℓ field extension F' . Applying Case 1 to F' , the extra factor of ℓ over F' gives (3) for F . \square

4.3. The Real Cyclotomy Theorems.

Lemma 4.7. *Let Δ be an imaginary quadratic discriminant, and let $K = \mathbb{Q}(\sqrt{\Delta})$. Let F be a number field, and let E/F be an elliptic curve. Let $N \geq 3$, and suppose $(\mathbb{Z}/N\mathbb{Z})^2 \subset E(FK)$. Then:*

- a) *We have $[\mathbb{Q}(\zeta_N) : \mathbb{Q}(\zeta_N) \cap F] \in \{1, 2\}$.*
- b) *Suppose F is real. Then $\mathbb{Q}(\zeta_N) \cap F = \mathbb{Q}(\zeta_N)^+$.*
- c) *Suppose $\gcd(N, \Delta_K) = 1$. Then $\mathbb{Q}(\zeta_N) \subsetneq FK$.*
- d) *If $K \not\subset F(\zeta_N)$, then $\mathbb{Q}(\zeta_N) \subset F$.*
- e) *Suppose N is an odd prime power. Then $\mathbb{Q}(\zeta_N)^+ \subset F$.*
- f) *Suppose $N = 2^a$ with $a \geq 3$. Then $\mathbb{Q}(\zeta_{N/2})^+ \subsetneq F$.*

Proof. a) Let $\chi_N : \mathfrak{g}_F \rightarrow (\mathbb{Z}/N\mathbb{Z})^\times$ be the mod N cyclotomic character, and let $H_F = \chi_N(\mathfrak{g}_F)$. As usual the Weil pairing gives $\mathbb{Q}(\zeta_N) \subset FK$, so $\chi_N(\mathfrak{g}_{FK}) \equiv 1$, and thus $\#H_F \leq 2$. Moreover $F = F(\zeta_N)^{H_F} \supset \mathbb{Q}(\zeta_N)^{H_F}$, and the result follows.

b) Since $N \geq 3$, $\mathbb{Q}(\zeta_N) \not\subset F$ and by part a) we have $[\mathbb{Q}(\zeta_N) : \mathbb{Q}(\zeta_N) \cap F] = 2$. Further, $\mathbb{Q}(\zeta_N) \cap F \subset \mathbb{Q}(\zeta_N) \cap \mathbb{R} = \mathbb{Q}(\zeta_N)^+$.

c) We have $\mathbb{Q}(\zeta_N) \subset FK$; if equality held, then $K \subset \mathbb{Q}(\zeta_N)$. But K is ramified at some prime $\ell \nmid N$ and $\mathbb{Q}(\zeta_N)$ is ramified only at primes dividing N .

d) The hypothesis implies that FK and $F(\zeta_N)$ are linearly disjoint over F , $\chi_N|_{\mathfrak{g}_{FK}} \equiv 1$ implies $\#H_F = \{1\}$ and $\mathbb{Q}(\zeta_N) \subset F$.

e) If N is an odd prime power, then $(\mathbb{Z}/N\mathbb{Z})^\times$ is cyclic, so either $H_F = 1$ and $\mathbb{Q}(\zeta_N) = \mathbb{Q}(\zeta_N)^{H_F} \subset F$ or $H_F = \{\pm 1\}$ and $\mathbb{Q}(\zeta_N)^+ = \mathbb{Q}(\zeta_N)^{H_F} \subset F$.

f) Since $N = 2^a$ with $a \geq 3$, $(\mathbb{Z}/N\mathbb{Z})^\times$ has three elements of order 2: -1 and $2^{a-1} \pm 1$. So we have $H_F \subsetneq \{\pm 1, 2^{a-1} \pm 1\}$, and thus

$$F \supset \mathbb{Q}(\zeta_N)^{H_F} \supsetneq \mathbb{Q}(\zeta_N)^{\{\pm 1, 2^{a-1} \pm 1\}} = \mathbb{Q}(\zeta_{N/2})^+. \quad \square$$

Theorem 4.8. *(Real Cyclotomy I) Let Δ be an imaginary quadratic discriminant, and let $K = \mathbb{Q}(\sqrt{\Delta})$. Let $N \in \mathbb{Z}^+$ be such that $\gcd(N, \Delta) = 1$. Let $F \not\supset K$ be a number field, and let E/F be an $\mathcal{O}(\Delta)$ -CM elliptic curve. Suppose that $E(F)$ contains a point of order N .*

- a) *We have $(\mathbb{Z}/N\mathbb{Z})^2 \subset E(FK)$.*
- b) *F contains an index 2 subfield of $\mathbb{Q}(\zeta_N)$.*
- c) *If N is an odd prime power, then $\mathbb{Q}(\zeta_N)^+ \subsetneq F$. If $N \geq 8$ is an even prime power, then $\mathbb{Q}(\zeta_{N/2})^+ \subsetneq F$.*
- d) *If F is real and $N \geq 3$, then $\mathbb{Q}(\zeta_N)^+ \subsetneq F$.*

Proof. a) We immediately reduce to the case that $N = \ell^a$ is a power of a prime number ℓ . Let $T_\ell(\mathcal{O}) = \mathcal{O} \otimes \mathbb{Z}_\ell$, and identify $T_\ell(\mathcal{O})$ with its isomorphic image in $\text{End } T_\ell E$. For $b \in \mathbb{Z}^+$, let $\mathcal{O}_{\ell^b} = T_\ell(\mathcal{O})/\langle \ell^b \rangle$. The hypothesis $\gcd(N, \Delta) = 1$ implies that $T_\ell(\mathcal{O})$ is the maximal \mathbb{Z}_ℓ -order in $K_\ell = K \otimes \mathbb{Q}_\ell$. We know that $T_\ell(E)$ is free of rank one as a $T_\ell(\mathcal{O})$ -module.

Case 1: Suppose $\left(\frac{\Delta}{\ell}\right) = 1$. Then $T_\ell(\mathcal{O}) \cong \mathbb{Z}_\ell \oplus \mathbb{Z}_\ell$. Put $\iota = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $NT_\ell(\mathcal{O})^\times = \langle T_\ell(\mathcal{O})^\times, \iota \rangle$. Because F does not contain K , there is $\sigma \in \mathfrak{g}_F$ such that $\rho_{\ell^\infty}(\sigma) \in NT_\ell(\mathcal{O})^\times \setminus T_\ell(\mathcal{O})^\times$. We may choose a \mathbb{Z}_ℓ -basis \tilde{e}_1, \tilde{e}_2 of $T_\ell(E)$ and represent the $T_\ell(\mathcal{O})$ -action on $T_\ell(E)$ via $\left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \mid \alpha, \beta \in \mathbb{Z}_\ell \right\}$. Let

$$\tilde{I}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{I}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in T_\ell(\mathcal{O}).$$

For $i = 1, 2$, let $\tilde{V}_i = \langle \tilde{e}_i \rangle_{\mathbb{Z}_\ell}$, and observe that each \tilde{V}_i is an $T_\ell(\mathcal{O})$ -submodule of $T_\ell(E)$. For $i = 1, 2$, put $V_i = \tilde{V}_i \pmod{\ell} \subset E[\ell](\bar{F})$. Because we may write $\rho_{\ell^\infty}(\sigma)$ as ιM with $M \in T_\ell(\mathcal{O})^\times$, we have

$$\rho_{\ell^\infty}(\sigma)(\tilde{V}_1) = \tilde{V}_2, \quad \rho_{\ell^\infty}(\sigma)(\tilde{V}_2) = \tilde{V}_1$$

and thus also

$$\rho_\ell(\sigma)(V_1) = V_2, \quad \rho_\ell(\sigma)(V_2) = V_1.$$

Lift $P \in E[\ell^n](F)$ to a point $\tilde{P} = a\tilde{e}_1 + b\tilde{e}_2 \in T_\ell(E)$. We claim $a, b \in \mathbb{Z}_\ell^\times$: if not, $P' = [\ell^{n-1}]P \in V_1 \cup V_2 \setminus \{0\}$, and $\rho_\ell(\sigma)(P') = P'$ gives a contradiction. It follows that for $i = 1, 2$, $\langle \tilde{I}_i \tilde{P} \rangle_{\mathbb{Z}_\ell} = \tilde{V}_i$, so the $T_\ell(\mathcal{O})$ -submodule generated by \tilde{P} is $T_\ell(E)$. Going modulo ℓ^a we get that the \mathcal{O}_{ℓ^a} -submodule generated by P is $E[\ell^a]$, and thus $(\mathbb{Z}/\ell^a\mathbb{Z})^2 \subset E(FK)$. Case 2: Suppose $\left(\frac{\Delta}{\ell}\right) = -1$. Then $T_\ell(\mathcal{O})$ is a discrete valuation ring with uniformizing element ℓ and fraction field K_ℓ and thus \mathcal{O}_{ℓ^a} is a finite principal ring with maximal ideal $\langle \ell \rangle$. The elements of $(\mathcal{O}_{\ell^a}, +)$ of order ℓ^a are precisely the units, so $\mathcal{O}_{\ell^a}^\times$ acts transitively on the order ℓ^a elements of $E[\ell^a]$ and thus the \mathcal{O}_{ℓ^a} -submodule of $E[\ell^a]$ generated by P is $E[\ell^a]$.

b) This follows from Lemma 4.7a).

c) If $N \geq 3$ is an odd prime power then by Lemma 4.7e) we have $\mathbb{Q}(\zeta_N)^+ \subset F$. Applying Lemma 4.7c) we get

$$1 < [FK : \mathbb{Q}(\zeta_N)] = [F : \mathbb{Q}(\zeta_N)^+].$$

The case of an even prime power $N \geq 8$ is similar but easier, since the strictness in the containment $\mathbb{Q}(\zeta_{N/2})^+ \subsetneq F$ comes from Lemma 4.7f). For part d) we apply Lemma 4.7b) and deduce the strictness of the containment as above. \square

In the case when F is real, we can dispense with the hypothesis $\gcd(\Delta, N) = 1$.

Theorem 4.9. (*Real Cyclotomy II*) *Let \mathcal{O} be an order of discriminant Δ in an imaginary quadratic field K , let F be a real number field, and let $E_{/F}$ be an \mathcal{O} -CM elliptic curve. Let $N \geq 1$, and suppose $E(F)$ contains a point of order N . Then:*

a) $\mathbb{Q}(\zeta_N) \subset FK$ and $\mathbb{Q}(\zeta_N)^+ \subset F$.

b) If $\gcd(N, \Delta_K) = 1$ and $N \geq 3$, then $\mathbb{Q}(\zeta_N)^+ \subsetneq F$.

Proof. To establish $\mathbb{Q}(\zeta_N) \subset FK$ we reduce to the case in which $N = \ell^n$ is a prime power. It will then follow that $\mathbb{Q}(\zeta_N)^+ = \mathbb{Q}(\zeta_N)^c \subset (FK)^c = F$. The proof of part b) is the same use of Lemma 4.7 as in the proof of Theorem 4.8.

Let Λ be the real lattice associated to E , unique up to \mathbb{R} -homothety. There is $r \in \mathbb{R}^\times$ such that $\Lambda' = r\Lambda$ is a primitive proper \mathcal{O} -ideal.

First suppose $\Delta = 4D$. Then $\mathcal{O} = [1, \sqrt{D}]$, and Λ' is of type I or II as in Theorem 3.7 above. If it is of type I, it follows that

$$\Lambda = \left[\frac{t}{r}, \frac{\sqrt{D}}{r} \right], \quad \text{where } t \in \mathbb{N}, t|D \text{ and } \left(t, \frac{D}{t} \right) = 1.$$

With respect to this basis the action of complex conjugation is given by

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and the action of \mathcal{O} on Λ is given by

$$\alpha + \beta\sqrt{D} \mapsto \begin{bmatrix} \alpha & \beta \left(\frac{D}{t} \right) \\ \beta t & \alpha \end{bmatrix}.$$

We may choose a \mathbb{Z}_ℓ -basis \tilde{e}_1, \tilde{e}_2 for $T_\ell(E)$ such that the image of the Cartan subgroup in $\mathrm{GL}_2(\mathbb{Z}_\ell)$ is

$$\mathcal{C}_\ell^\times = \left\{ \begin{bmatrix} \alpha & \beta \left(\frac{D}{t}\right) \\ \beta t & \alpha \end{bmatrix} \mid \alpha^2 - \beta^2 D \in \mathbb{Z}_\ell^\times \right\}$$

and the element $c \in \mathfrak{g}_F$ induced by complex conjugation corresponds to $\rho_{\ell^\infty}(c) = T$. Let $e_i = \tilde{e}_i \pmod{\ell^n}$. For $x \in E(F)[\ell^n] \setminus E(F)[\ell^{n-1}]$, we may choose $a, b \in \mathbb{Z}/\ell^n\mathbb{Z}$ such that $x = ae_1 + be_2$. Then

$$ae_1 + be_2 = x = Tx = ae_1 - be_2,$$

so $2b \equiv 0 \pmod{\ell^n}$. We assume for the moment that ℓ is odd, so it follows that $b \equiv 0 \pmod{\ell^n}$. Thus $a \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times$.

For $S = \begin{bmatrix} \alpha & \beta \left(\frac{D}{t}\right) \\ \beta t & \alpha \end{bmatrix} \in \mathcal{C}_\ell^\times \cap G_\ell$, we have

$$(4) \quad ae_1 + be_2 = x = Sx = \left(\alpha a + \beta b \left(\frac{D}{t}\right) \right) e_1 + (\beta at + \alpha b) e_2.$$

Modulo ℓ^n this becomes

$$ae_1 = x = Sx = \alpha ae_1 + \beta at e_2.$$

It follows that $\alpha \equiv 1 \pmod{\ell^n}$ and $\beta t \equiv 0 \pmod{\ell^n}$, and thus

$$S \equiv \begin{bmatrix} 1 & \beta \left(\frac{D}{t}\right) \\ 0 & 1 \end{bmatrix} \pmod{\ell^n}.$$

Let $\sigma \in \mathfrak{g}_{FK}$. Then there exists $\beta_0 = \beta_0(\sigma)$ such that

$$\rho_{\ell^n}(\sigma) = \begin{bmatrix} 1 & \beta_0 \left(\frac{D}{t}\right) \\ 0 & 1 \end{bmatrix}.$$

By Galois equivariance of the Weil pairing, $\sigma \zeta_{\ell^n} = \zeta_{\ell^n}^{\det \rho_{\ell^n}(\sigma)} = \zeta_{\ell^n}$, so $\zeta_{\ell^n} \in FK$.

If $\ell = 2$, assume without loss of generality that $n \geq 2$ because if $N = 2M$ with M odd then $\mathbb{Q}(\zeta_N) = \mathbb{Q}(\zeta_M)$ and $\mathbb{Q}(\zeta_N)^+ = \mathbb{Q}(\zeta_M)^+$. We must adjust our approach by working mod 2^{n-1} . Indeed, $Tx = x$ will only imply $b \equiv 0 \pmod{2^{n-1}}$ and $a \in (\mathbb{Z}/2^n\mathbb{Z})^\times$. For $S \in \mathcal{C}_2^\times \cap G_2$, $Sx = x$ gives $\alpha \equiv 1 \pmod{2^{n-1}}$, $\beta t \equiv 0 \pmod{2^{n-1}}$. In fact, $\beta t \equiv 0 \pmod{2^n}$ as well. Indeed, by (4), $b = \beta at + \alpha b$, which means

$$a^{-1}b(1 - \alpha) \equiv \beta t \pmod{2^n}.$$

As $2^{n-1} \mid b$ and $2^{n-1} \mid (1 - \alpha)$, the claim follows since $n \geq 2$. Thus

$$\det S = \alpha^2 - \beta^2 D = \alpha^2 - \beta^2 t \frac{D}{t} \equiv \alpha^2 \equiv 1 \pmod{2^n},$$

and $\det \circ \rho_{2^n}|_{\mathfrak{g}_{FK}}$ is trivial. We conclude $\zeta_{2^n} \in FK$.

If Λ' is of type II, then

$$\Lambda = \left[\frac{t}{r}, \frac{t + \sqrt{\Delta}}{2r} \right], \text{ where } t \in \mathbb{N}, 4t|t^2 - \Delta \text{ and } \left(t, \frac{t^2 - \Delta}{4t} \right) = 1.$$

With respect to this basis the action of complex conjugation is given by

$$T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},$$

and the action of \mathcal{O} on Λ is given by

$$\alpha + \beta\sqrt{D} \mapsto \begin{bmatrix} \alpha - \beta \left(\frac{t}{2}\right) & -\beta \left(\frac{t^2 - \Delta}{4t}\right) \\ \beta t & \alpha + \beta \left(\frac{t}{2}\right) \end{bmatrix}.$$

This gives rise to the Cartan subgroup

$$\mathcal{C}_\ell^\times = \left\{ \begin{bmatrix} \alpha - \beta \left(\frac{t}{2}\right) & -\beta \left(\frac{t^2 - \Delta}{4t}\right) \\ \beta t & \alpha + \beta \left(\frac{t}{2}\right) \end{bmatrix} \mid \alpha^2 - \beta^2 \frac{t^2}{4} + \beta^2 \left(\frac{t^2 - \Delta}{4}\right) \in \mathbb{Z}_\ell^\times \right\}$$

As before, we may choose a \mathbb{Z}_ℓ -basis \tilde{e}_1, \tilde{e}_2 for $T_\ell(E)$ such that $\rho_{\ell^\infty}(\mathfrak{g}_{FK}) \subset \mathcal{C}_\ell^\times$ and the element $c \in \mathfrak{g}_F$ induced by complex conjugation corresponds to $\rho_{\ell^\infty}(c) = T$.

Let $e_i = \tilde{e}_i \pmod{\ell^n}$. For $x \in E(F)[\ell^n] \setminus E(F)[\ell^{n-1}]$, we may choose $a, b \in \mathbb{Z}/\ell^n\mathbb{Z}$ such that $x = ae_1 + be_2$. Then

$$ae_1 + be_2 = x = Tx = (a + b)e_1 - be_2,$$

so $b \equiv 0 \pmod{\ell^n}$ and $a \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times$. For $S = \begin{bmatrix} \alpha - \beta \left(\frac{t}{2}\right) & -\beta \left(\frac{t^2 - \Delta}{4t}\right) \\ \beta t & \alpha + \beta \left(\frac{t}{2}\right) \end{bmatrix} \in \mathcal{C}_\ell^\times \cap G_\ell$, we have

$$ae_1 = x = Sx = \left(\alpha - \beta \left(\frac{t}{2}\right) \right) ae_1 + \beta t e_2 \pmod{\ell^n}.$$

Thus $\alpha - \beta \left(\frac{t}{2}\right) \equiv 1 \pmod{\ell^n}$ and $\beta t \equiv 0 \pmod{\ell^n}$. It follows that

$$\alpha + \beta \left(\frac{t}{2}\right) \equiv \alpha + \beta \left(\frac{t}{2}\right) - \beta t = \alpha - \beta \left(\frac{t}{2}\right) \equiv 1 \pmod{\ell^n}.$$

Hence

$$S \equiv \begin{bmatrix} 1 & -\beta \left(\frac{t^2 - \Delta}{4t}\right) \\ 0 & 1 \end{bmatrix} \pmod{\ell^n}.$$

We conclude $\zeta_{\ell^n} \in FK$ as before. Finally, we consider the case when $\Delta \equiv 1 \pmod{4}$. Then $\mathcal{O} = \left[1, \frac{1 + \sqrt{\Delta}}{2}\right]$ and Λ' is of type II as in Theorem 3.7. Thus

$$\Lambda = \left[\frac{t}{r}, \frac{t + \sqrt{\Delta}}{2r} \right], \text{ where } t \in \mathbb{N}, 4t|t^2 - \Delta \text{ and } \left(t, \frac{t^2 - \Delta}{4t} \right) = 1.$$

Following the method used above, we may choose a \mathbb{Z}_ℓ -basis \tilde{e}_1, \tilde{e}_2 for $T_\ell(E)$ such that the image of the Cartan subgroup in $\mathrm{GL}_2(\mathbb{Z}_\ell)$ consists of matrices of the form

$$\begin{bmatrix} \alpha - \beta \left(\frac{t-1}{2}\right) & \beta \left(\frac{\Delta - t^2}{4t}\right) \\ \beta t & \alpha + \beta \left(\frac{t+1}{2}\right) \end{bmatrix}$$

and the element $c \in \mathfrak{g}_F$ induced by complex conjugation corresponds to

$$T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Let $e_i = \tilde{e}_i \pmod{\ell^n}$. For $x \in E(F)[\ell^n] \setminus E(F)[\ell^{n-1}]$, we again choose $a, b \in \mathbb{Z}/\ell^n\mathbb{Z}$ such that $x = ae_1 + be_2$. Then $Tx = x$ gives $b \equiv 0 \pmod{\ell^n}$ and hence

$a \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times$. For $S = \begin{bmatrix} \alpha - \beta \left(\frac{t-1}{2}\right) & \beta \left(\frac{\Delta - t^2}{4t}\right) \\ \beta t & \alpha + \beta \left(\frac{t+1}{2}\right) \end{bmatrix} \in \mathcal{C}_\ell^\times \cap G_\ell$, we have

$$ae_1 = x = Sx = \left(\alpha - \beta \left(\frac{t-1}{2}\right) \right) ae_1 + \beta t e_2 \pmod{\ell^n}.$$

As before, this implies $\alpha - \beta \left(\frac{t-1}{2}\right) \equiv 1 \pmod{\ell^n}$ and $\beta t \equiv 0 \pmod{\ell^n}$. Since

$$\alpha + \beta \left(\frac{t+1}{2}\right) \equiv \alpha + \beta \left(\frac{t+1}{2}\right) - \beta t = \alpha - \beta \left(\frac{t-1}{2}\right) \equiv 1 \pmod{\ell^n},$$

we have

$$S \equiv \begin{bmatrix} 1 & \beta \left(\frac{\Delta-t^2}{4t}\right) \\ 0 & 1 \end{bmatrix} \pmod{\ell^n}.$$

It follows that $\zeta_{\ell^n} \in FK$. \square

Remark 4.10. *When $\ell^n = 2$, Theorem 4.9a) is vacuous. Part b) holds in this case unless $F = \mathbb{Q}$ and $j \in \{0, 54000, -15^3, 255^3\}$. These curves have CM by $\mathcal{O}(\Delta)$ for $\Delta \in \{-3, -12, -7, -28\}$, and a point of order 2.*

Corollary 4.11. *a) Let $N \geq 5$, let F be an odd degree number field, and let $E_{/F}$ be a CM elliptic curve such that $E(F)$ contains a point of order N . Then there is a prime $p \equiv 3 \pmod{4}$ and a positive integer a such that $N \in \{p^a, 2p^a\}$.*

b) Let \mathcal{S} be the set of positive integers N such that there is an odd degree number field F and a CM elliptic curve $E_{/F}$ such that $E(F)$ contains a point of order N . Then \mathcal{S} has density 0.

Proof. a) By Real Cyclotomy II (Theorem 4.9) we have $\mathbb{Q}(\zeta_N)^+ \subset F$, so $\frac{\varphi(N)}{2} \mid [F : \mathbb{Q}]$ and thus $4 \nmid \varphi(N)$. Since $\varphi(N)$ is divisible by 4 if $N > 4$ is divisible by 4, by a prime $p \equiv 1 \pmod{4}$, or by two odd primes, the result follows.

b) This follows easily from part a). More precisely, the Prime Number Theorem gives $\mathcal{S} \cap [1, X] = O\left(\frac{X}{\log X}\right)$. \square

4.4. Square-Root SPY Bounds.

Proposition 4.12. *([Coh00, Corollary 3.2.4]) Let K be a number field, and let \mathfrak{m} a nonzero ideal of \mathcal{O}_K , hence also a modulus in the sense of class field theory. Let $U = \mathcal{O}_K^\times$ and $U_{\mathfrak{m}} = \{\alpha \in U : \text{ord}_{\mathfrak{p}}(\alpha - 1) \geq \text{ord}_{\mathfrak{p}} \mathfrak{m} \text{ for all } \mathfrak{p} \mid \mathfrak{m}\}$. Then*

$$[K_{\mathfrak{m}} : K] = \frac{[K_1 : K]}{[U : U_{\mathfrak{m}}]} [\mathcal{O}_K : \mathfrak{m}] \cdot \prod_{\mathfrak{p} \mid \mathfrak{m}} (1 - [\mathcal{O}_K : \mathfrak{p}]^{-1}).$$

Theorem 4.13. *(Square-Root SPY Bounds) Let \mathcal{O} be an imaginary quadratic order of discriminant Δ and fraction field K , let F be a number field, and let $E_{/F}$ be an \mathcal{O} -CM elliptic curve. Let $N \geq 3$.*

a) If $(\mathbb{Z}/N\mathbb{Z})^2 \subset E(FK)$, then

$$\varphi(N) \leq \sqrt{\frac{[F : \mathbb{Q}] w_K}{h_K}}.$$

b) The hypothesis of part a) is satisfied when $K \not\subseteq F$ and $\gcd(\Delta, N) = 1$.

Proof. a) We have

$$\prod_{\mathfrak{p} \mid N \mathcal{O}_K} (1 - [\mathcal{O}_K : \mathfrak{p}]^{-1}) = \prod_{p \mid N} \prod_{\mathfrak{p} \mid p \mathcal{O}_K} (1 - [\mathcal{O}_K : \mathfrak{p}]^{-1}).$$

Further, we have

$$\prod_{\mathfrak{p} \mid p \mathcal{O}_K} (1 - [\mathcal{O}_K : \mathfrak{p}]^{-1}) = \begin{cases} (1 - \frac{1}{p})^2, & \left(\frac{\Delta_K}{p}\right) = 1 \\ (1 - \frac{1}{p}), & \left(\frac{\Delta_K}{p}\right) = 0 \\ (1 - \frac{1}{p^2}), & \left(\frac{\Delta_K}{p}\right) = -1 \end{cases},$$

so

$$\prod_{\mathfrak{p}|N\mathcal{O}_K} (1 - [\mathcal{O}_K : \mathfrak{p}]^{-1}) \geq \prod_{p|N} \left(1 - \frac{1}{p}\right)^2.$$

Applying Lemma 3.15 and Theorem 3.16, we get

$$FK = F(E[N]) \supset K^{(N)}$$

and thus

$$\begin{aligned} [F : \mathbb{Q}] &= \frac{[FK : \mathbb{Q}]}{2} \geq \frac{[K^{(N)} : \mathbb{Q}]}{2} = [K^{(N)} : K] \\ &= \frac{h_K}{[U : U_{N\mathcal{O}_K}]} [\mathcal{O}_K : N\mathcal{O}_K] \prod_{\mathfrak{p}|N\mathcal{O}_K} (1 - [\mathcal{O}_K : \mathfrak{p}]^{-1}) \\ &\geq \frac{h_K}{w_K} N^2 \prod_{p|N} \left(1 - \frac{1}{p}\right)^2 = \frac{h_K}{w_K} \varphi(N)^2. \end{aligned}$$

b) This is Theorem 4.8a). □

Example 4.14. *Let $N \geq 3$, let F be a cubic number field, and let $E_{/F}$ be a CM elliptic curve such that $E(F)$ contains a point of order N . The SPY Bounds give $\varphi(N) \leq [F : \mathbb{Q}]w_K \leq 18$; in particular the largest prime value of N permitted is 19. The odd order torsion subgroup of $E(F)$ has size at most 13 [CX08, Corollary 2]; in particular, the largest prime value permitted is 13. Real Cyclotomy II (Theorem 4.9) gives $\varphi(N) \mid 6$ and thus $N \mid 4$, $N \mid 14$ or $N \mid 18$; in particular, the largest prime value permitted is 7. Theorems 2.1 and 1.4 show that all of these values of N are actually attained except for $N = 18$.*

Suppose now that $E_{/F}$ is $\mathcal{O}(\Delta)$ -CM and $\gcd(\Delta, N) = 1$. Then the Square Root SPY Bounds give

$$\varphi(N) \leq \lfloor \sqrt{18} \rfloor = 4, \quad K = \mathbb{Q}(\sqrt{-3}),$$

$$\varphi(N) \leq \lfloor \sqrt{12} \rfloor = 3, \quad K = \mathbb{Q}(\sqrt{-1}),$$

$$\varphi(N) \leq \lfloor \sqrt{6} \rfloor = 2, \quad K \notin \{\mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-1})\}.$$

Combining with Real Cyclotomy II, we get that the only prime values of N permitted in this case are the ‘‘Olson primes’’ 2 and 3. The four elliptic curves in rows 13 through 16 of the table in Theorem 1.4 do not satisfy the Square Root SPY Bounds. It follows from Theorem 4.13 that for $N = 9$ and $N = 14$ we do not have $(\mathbb{Z}/N\mathbb{Z})^2 \subset E(FK)$. This shows that the hypothesis $\gcd(\Delta, N) = 1$ in Real Cyclotomy I (Theorem 4.8) is necessary in order for this stronger form of real cyclotomy to hold. On the other hand, we have

$$\mathbb{Q}[b]/(b^3 - 15b^2 - 9b - 1) \cong \mathbb{Q}[b]/(b^3 + 105b^2 - 33b - 1) \cong \mathbb{Q}(\zeta_9)^+,$$

$$\mathbb{Q}[b]/(b^3 - 4b^2 + 3b + 1) \cong \mathbb{Q}[b]/(b^3 - 186b^2 + 3b + 1) \cong \mathbb{Q}(\zeta_{14})^+,$$

in accordance with Real Cyclotomy II (Theorem 4.9).

5. NUMBER FIELDS OF PRIME DEGREE

5.1. Parish's Theorem.

The proof of Theorem 1.4 will make use of the following striking result.

Theorem 5.1. (*Parish [Pa89, Theorem 2]*) *Let $E_{/F}$ be a CM elliptic curve defined over a number field. If $F = \mathbb{Q}(j(E))$, then $E(F)[\text{tors}]$ is an Olson group.*

The following consequence is immediate.

Corollary 5.2. *If F is a number field of prime degree and $E_{/F}$ is a CM elliptic curve with $j(E) \notin \mathbb{Q}$, then $E_{/F}$ is Olson.*

5.2. Proof of Theorem 1.4.

Let F be a number field with $[F : \mathbb{Q}] = p$ a prime number: so F is real if $p \neq 2$. Let $E_{/F}$ be an elliptic curve with CM by an order \mathcal{O} of discriminant $D = \mathfrak{f}^2 D_0$ in the imaginary quadratic field K . Suppose $E_{/F}$ is not Olson.

Step 1: Suppose $p = 2$. In this case, the result is in principle a very special case of work of Müller-Ströher-Zimmer [MSZ89] shows that there are finitely many pairs (E, F) with F a quadratic number field and $E_{/F}$ a non-Olson elliptic curve with *integral moduli* – i.e., $j(E) \in \mathcal{O}_F$ – and lists all of them. In practice, we used Parish's Theorem, the SPY Bounds, and Table 2 to rederive the classification.

Step 2: Suppose $p = 3$. Work of Pétho-Weis-Zimmer [PWZ97] shows that as $E_{/F}$ ranges over all elliptic curves over cubic number fields with integral moduli, the only non-Olson group which arises infinitely many times is $\mathbb{Z}/5\mathbb{Z}$. By Corollary 4.11, there is no CM elliptic curve over a cubic field with a point of order 5. As above, although these results suffice in principle, in practice we used Parish's Theorem, the SPY Bounds, work of Clark-Xarles [CX08, Corollary 2], and Table 2 to rederive the classification.

Step 3: Suppose $p \geq 5$. Suppose $E(F)[\text{tors}]$ has a point of prime order $\ell \geq 5$. By Corollary 5.2, $j(E) \in \mathbb{Q}$. Thus \mathcal{O} has class number 1, so

$$(5) \quad \Delta \in \{-3, -4, -7, -8, -11, -12, -16, -19, -27, -28, -43, -67, -163\}.$$

If ℓ is unramified in K then by Real Cyclotomy II (Theorem 4.9), $\mathbb{Q}(\zeta_\ell)^+ \subsetneq F$ and thus $\frac{\ell-1}{2}$ properly divides p . Since $\ell \geq 5$, this is a contradiction. If ℓ is ramified in K then $\ell \in \{7, 11, 19, 43, 67, 163\}$. Real Cyclotomy II gives $\frac{\ell-1}{2} \mid p$, so $p = \frac{\ell-1}{2}$.

- If $\ell = 7$ then $p = 3$.
- If $\ell = 11$ then $p = 5$.
- If $\ell \in \{19, 43, 67, 163\}$, then $\frac{\ell-1}{2}$ is not a prime.

Step 4: It follows from Table 2 that no CM elliptic curve E with rational j -invariant defined over number field F of prime degree $p \geq 5$ can have any of the following as subgroups of $E(F)$:

$$\mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/9\mathbb{Z}, \mathbb{Z}/12\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}.$$

Therefore $p \leq 5$, and if $p = 5$ then $\Delta = -11$, $F = \mathbb{Q}(\zeta_5)^+$ and $E(F)$ has a point of order 11. It follows that there is such an elliptic curve [CCRS14, §4.5]. The same kind of computation shows that this curve is unique. Alternately, any other such

elliptic curve would be a quadratic twist of $E_{/F}$ by $F(\sqrt{d})/F$, say. For any odd $N \geq 3$, if $E_{/F}$ and the quadratic twist $E_{/F}^{(d)}$ each have points of order N , then E has full N -torsion over $F(\sqrt{d})$. This would force $F(\sqrt{d}) = FK = \mathbb{Q}(\zeta_{11})$ and thus, by Theorem 3.16, $\mathbb{Q}(\zeta_{11}) \supset K^{(11)}$, but $[\mathbb{Q}(\zeta_{11}) : \mathbb{Q}] = 10$ and $[K^{(11)} : \mathbb{Q}] = 110$.

Remark 5.3. *Theorem 1.4 is proved by ruling out non-Olson torsion in prime degree $p \geq 7$ and then computing all non-Olson CM elliptic curves over number fields of degree $p \in \{2, 3, 5\}$. These calculations were done en route to previous results [CCRS14, §4.2, §4.3, §4.5], but each group structure which arose in a given degree was recorded only once [CCRS14, Algorithm 3.2]. The proof explains how one could extract the needed calculations from tables appearing in previous work [MSZ89, PWZ97, CCRS14], Corollary 4.11 and some modest calculations with genus zero torsion structures.*

However, we did not feel that this was a good approach. Rather, the last two authors knew from our prior work that it would not be overly onerous to recalculate all non-Olson torsion in degrees 2, 3 and 5 from scratch. To achieve the most meaningful corroboration, this recalculation was done by the first author. They are fully consistent with (but more detailed than) the results of [CCRS14].

6. BEYOND PRIME DEGREES

6.1. Proof of Theorem 1.5. We give the proof of Theorem 1.5. For the reader's convenience we begin by recalling the statement of Schinzel's Hypothesis H.

Conjecture 6.1. *(Schinzel's Hypothesis H) Let $f_1, \dots, f_r \in \mathbb{Q}[t]$ be irreducible and integer-valued. Suppose: for all $m \geq 2$ there is $n \in \mathbb{Z}^+$ such that $m \nmid f_1(n) \cdots f_r(n)$. Then $\{n \in \mathbb{Z}^+ \mid |f_1(n)|, \dots, |f_r(n)| \text{ are all prime numbers}\}$ is infinite.*

Theorem 6.2. *Assume Schinzel's Hypothesis H, and let $d \in \mathbb{Z}^+$. Then*

$$\limsup_{p \in \mathcal{P}} \#\mathcal{T}_{\text{CM}}^{\text{new}}(2dp) \geq 1.$$

Proof. Applying Schinzel's Hypothesis H with $f_1(x) = x$, $f_2(x) = 6dx + 1$, we get infinitely many prime numbers p such that $N = 6dp + 1$ is prime. Thus $\frac{N-1}{3} = 2dp$. In particular $N \equiv 1 \pmod{3}$, so N splits in $K = \mathbb{Q}(\sqrt{-3})$, and then there is an \mathcal{O}_K -CM elliptic curve E defined over a number field F of degree $\frac{N-1}{3} = 2dp$ with an F -rational point of order N [CCS13, Theorem 3]. We claim that for sufficiently large N , $E(F)[\text{tors}] \in \mathcal{T}_{\text{CM}}^{\text{new}}(2dp)$: if so, the result follows. Now there is a positive integer N_0 (as yet inexplicit) such that for all primes $N \geq N_0$, if $E_{/F}$ is a CM elliptic curve over a number field F with an F -rational point of order N , then $[F : \mathbb{Q}] \geq \frac{N-1}{3}$ [CCS13, Theorem 1]. Thus for all primes $N \geq N_0$, $E(F)[\text{tors}]$ is a torsion subgroup that does not occur in any degree smaller than $[F : \mathbb{Q}]$, which certainly implies $E(F)[\text{tors}] \in \mathcal{T}_{\text{CM}}^{\text{new}}(2dp)$. \square

6.2. Unboundedness of Odd Degree Torsion Points on Elliptic Curves.

Theorem 6.3. *Let $d, N \in \mathbb{Z}^+$ with $d \geq 2$. The set of algebraic numbers $j \in \overline{\mathbb{Q}}$ such that there is a number field K with $\gcd([K : \mathbb{Q}], d) = 1$ and an elliptic curve $E_{/K}$ with $j(E) = j$ and a point of order N in $E(K)$ is infinite.*

Proof. Let $S \subset \overline{\mathbb{Q}}$ be finite. Identifying $Y_1(1)$ with \mathbb{A}^1 , we view S as a finite set of $\overline{\mathbb{Q}}$ -valued points of $Y_1(1)$. Let Z_1 be a finite closed \mathbb{Q} -subscheme of $X_1(1)_{/\mathbb{Q}}$ containing (the image in $X_1(N)$ of) S and all the cusps, and let Z_N be the preimage of Z_1 under $\pi : X_1(N) \rightarrow X_1(1)$, so Z_N is a finite closed \mathbb{Q} -subscheme of $X_1(N)$. Let $U = X_1(N) \setminus Z_N$. The least positive degree of a divisor on U is the least positive degree of a divisor on $X_1(N)$ [Cl07, Lemma 12]. Since the cusp at ∞ is a \mathbb{Q} -rational point on $X_1(N)$, this common quantity is 1, and thus there is a divisor $\sum_i n_i [P_i]$ on U such that $\sum_i n_i [\mathbb{Q}(P_i) : \mathbb{Q}] = 1$. It follows that for at least one i we have $d \nmid [\mathbb{Q}(P_i) : \mathbb{Q}]$. The point P_i corresponds to at least one pair $(E, x)_{/\mathbb{Q}(P_i)}$ where E is an elliptic curve and $x \in E(\mathbb{Q}(P_i))$ has order N [DR73, Proposition VI.3.2]. \square

6.3. Number Fields of S_d -Type.

Let G be a finite group. A number field F is **of G -type** if the automorphism group of the normal closure of F/\mathbb{Q} is isomorphic to G .

Theorem 6.4. *Let d be an odd positive integer, and let F be a degree d number field of S_d -type. Then every CM elliptic curve $E_{/F}$ is Olson.*

Proof. Step 1: Let M be the normal closure of F/\mathbb{Q} , and choose an isomorphism $S_d \cong \text{Aut}(M/\mathbb{Q})$. Let A (resp. B) be the maximal abelian subextension of F/\mathbb{Q} (resp. of M/\mathbb{Q}). Then

$$B = M^{[S_d, S_d]} = M^{A_d},$$

so $[B : \mathbb{Q}] = 2$. Since $\mathbb{Q} \subset A \subset B \cap F$ and $[F : \mathbb{Q}] = d$ is odd, we have $A = \mathbb{Q}$.

Step 2: Let $E_{/F}$ be a CM elliptic curve. Since F has odd degree, it is real, so if $E(F)$ contains a point of order N , by Real Cyclotomy II (Theorem 4.9) F contains the abelian number field $\mathbb{Q}(\zeta_N)^+$. Thus $\mathbb{Q}(\zeta_N)^+ = \mathbb{Q}$ and $N \in \{1, 2, 3, 4, 6\}$. Because d is odd, we have $\mathbb{Q}(\zeta_3) \not\subseteq F$ and thus $(\mathbb{Z}/3\mathbb{Z})^2 \not\subseteq E(F)$. Applying Corollary 4.5, we conclude that $E_{/F}$ is Olson. \square

APPENDIX: TABLE OF DEGREE SEQUENCES

Let $m \mid n$ be positive integers; we exclude the pairs $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 2)$. Then the modular curve $Y(m, n)$ classifying (roughly: the precise description of the moduli problem involves a Cartier-equivariant isomorphism and is omitted here) $(\mu_m \times \mathbb{Z}/n\mathbb{Z})$ -structures on elliptic curves is a fine moduli space. For every $j \in \overline{\mathbb{Q}}$, the fiber of the morphism $Y(m, n) \rightarrow Y(1)$ over j is a finite $\mathbb{Q}(j)$ -subscheme. The reduced subscheme of the fiber is therefore isomorphic to a finite product $\prod_{i=1}^N K_i$ of number fields. By the **degree sequence** for (m, n) and j we mean the sequence of degrees of the number fields $K_i(\zeta_m)$, written in non-decreasing order. These are the degrees of the (unique minimal) fields of definition K such that there is an elliptic curve $E_{/K}$ with $\mathcal{O}(\Delta)$ -CM and an injection $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \hookrightarrow E(K)$.

In the table below we list the degree sequences for the 13 class number one imaginary quadratic discriminants for certain pairs (m, n) . The results of this table are used in the proofs of Corollary 4.5 and Theorem 1.4.

	-3	-4	-7	-8	-11	-12	-16
(1, 4)	2	1,2	2,2,2	2,4	6	2,4	1,1,4
(1, 5)	4	2,4	12	12	4,8	12	4,8
(1, 6)	1,3	2,4	4,8	2,2,4,4	6,6	1,2,3,6	4,8
(1, 7)	2,6	12	3,21	24	24	6,18	24
(1, 8)	8	4,8	4,4,8,8	8,16	24	8,16	4,4,16
(1, 9)	3,9	18	36	6,12,18	6,12,18	9,27	36
(1, 10)	12	2,4,4,8	12,24	12,24	12,24	12,24	4,8,8,16
(1, 11)	20	30	10,50	10,50	5,55	60	60
(1, 12)	4,12	8,16	16,16,16	8,8,16,16	24,24	4,8,12,24	8,8,32
(1, 13)	4,24	6,36	84	84	84	12,72	12,72
(2, 4)	4	2,2,2	2,2,2,2,4	2,2,4,4	12	4,4,4	2,2,4,4
(2, 6)	2,6	4,4,4	8,8,8	4,4,4,4,4,4	6,6,12	2,2,2,6,6,6	8,8,8
(2, 8)	16	8,8,8	4,4,4,4,8,8,16	8,8,16,16	48	8,8,16,16	4,4,8,16,16
(3, 3)	2,2,2	4,4	8,8	4,4,4,4	4,4,4,4	6,6,6	8,8

	-19	-27	-28	-43	-67	-163
(1, 4)	6	6	2,4	6	6	6
(1, 5)	4,8	12	12	12	12	12
(1, 6)	12	3,9	4,8	12	12	12
(1, 7)	6,18	6,18	3,21	24	24	24
(1, 8)	24	24	4,4,16	24	24	24
(1, 9)	36	3,6,27	36	36	36	36
(1, 10)	12,24	36	12,24	36	36	36
(1, 11)	10,50	60	10,50	10,50	60	60
(1, 12)	48	12,36	16,32	48	48	48
(1, 13)	84	12,72	84	12,72	84	84
(2, 4)	12	12	4,4,4	12	12	12
(2, 6)	24	6,18	8,8,8	24	24	24
(2, 8)	48	48	8,8,16,16	48	48	48
(3, 3)	8,8	6,6,6	8,8	8,8	8,8	8,8

TABLE 2

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