Instructions: Calculators are not permitted. In problems 3 and 5, be sure to include reasoning that justifies your answer.

Problem 1. State the Quadratic Reciprocity Law and its first and second supplements.

Solution:
• Quadratic Reciprocity: Let \( p \neq q \) be odd primes. Then
  \[
  \left( \frac{p}{q} \right) \left( \frac{q}{p} \right) = (-1)^{\frac{(p-1)(q-1)}{4}}.
  \]
• First Supplement: Let \( p \) be an odd prime. Then
  \[
  \left( \frac{-1}{p} \right) = (-1)^{\frac{p-1}{2}}.
  \]
• Second Supplement: Let \( p \) be an odd prime. Then
  \[
  \left( \frac{2}{p} \right) = (-1)^{\frac{p^2-1}{8}}.
  \]

Problem 2. State the Full Two Squares Theorem.

Solution: A positive integer \( n \) is of the form \( x^2 + y^2 \) for \( x, y \in \mathbb{Z} \) iff \( \text{ord}_p(n) \) is even for all primes \( p \equiv 3 \pmod{4} \).

Problem 3. a) Use the Euclidean algorithm to compute the gcd of 17 and 29. (Note: clearly the gcd is 1, but you must use the Euclidean algorithm. You’ll thank me later.)

b) Find an integer solution \((x, y)\) to the equation \(17x + 29y = 1\).

c) Find all integer solutions \((x, y)\) to the equation \(17x + 29y = 1\).

Solution:
a) We have

\[
29 = 1 \cdot 17 + 12,
17 = 1 \cdot 12 + 5,
12 = 2 \cdot 5 + 2,
5 = 2 \cdot 2 + 1,
\]
so the gcd is – surprise!! – 1.

b) The point is that we can solve this by reversing the steps of the Euclidean Algorithm. (You’re welcome for part a.) First write:

\[
1 = 5 - 2 \cdot 2,
2 = 12 - 2 \cdot 5,
5 = 17 - 1 \cdot 12,
12 = 29 - 1 \cdot 17.
\]

Now we back substitute:

\[
1 = 5 - 2 \cdot 2 = 5 - 2 \cdot (12 - 2 \cdot 5)
5 \cdot 5 - 2 \cdot 12 = 5 \cdot (17 - 1 \cdot 12) - 2 \cdot 12
= 5 \cdot 17 - 7 \cdot 12
\]
\[
= 5 \cdot 17 - 7 \cdot (29 - 1 \cdot 17)
= 12 \cdot 17 - 7 \cdot 29.
\]

So one solution is \((x_0, y_0) = (12, -7)\).

c) Given one solution \((x_0, y_0)\), since \(\gcd(17, 29) = 1\), we know that the general solution is
\[(x_n, y_n) = (12 + 29n, -7 - 17n)\] for \(n \in \mathbb{Z}\).

**Problem 4.** Compute the following Legendre\(^1\) symbols. You may use any (correct!) method.

\[
\left( \frac{1,000,000}{4,111} \right), \quad \left( \frac{10,100}{2,017} \right), \quad \left( \frac{2}{1,111,111,111,111,111} \right).
\]

**Solution:** \(\bullet\) Since \(1,000,000 = 10^6 = (10^3)^2\) is coprime to \(4,111\), we have
\[
\left( \frac{1,000,000}{4,111} \right) = \left( \frac{10^3}{4,111} \right)^2 = 1.
\]

\(\bullet\) We have
\[
10,100 = 5 \cdot 2,017 + 15,
\]
so
\[
\left( \frac{10,100}{2,017} \right) = \left( \frac{15}{2017} \right).
\]

Since \(2017 \equiv 1 \pmod{4}\), we have
\[
\left( \frac{15}{2017} \right) = \left( \frac{2017}{15} \right).
\]

We have
\[
2017 = 134 \cdot 15 + 7,
\]
we have
\[
\left( \frac{2017}{15} \right) = \left( \frac{7}{15} \right).
\]

Since \(7 \equiv 15 \equiv 3 \pmod{4}\), we have
\[
\left( \frac{7}{15} \right) = - \left( \frac{15}{7} \right) = - \left( \frac{1}{7} \right) = -1.
\]

\(\bullet\) Since \(8 \mid 1,000 = 2^35^3\), what an integer is modulo 8 can be determined from its last three digits.

So by the Second Supplement to Quadratic Reciprocity we have
\[
\left( \frac{2}{1,111,111,111,111,111} \right) = \left( \frac{2}{111} \right).
\]

We have
\[
111 = 13 \cdot 8 + 7,
\]
so \(111 \equiv 7 \pmod{8}\), and thus by the Second Supplement we have
\[
\left( \frac{2}{1,111,111,111,111,111} \right) = 1.
\]

**Problem 5.** For each of the following equations, either find an integral solution \((x, y)\) or show that there isn’t one.

\(a)\) \(x^2 - 5y^2 = 3\).

\(b)\) \(x^2 - 5y^2 = 4\).

\(^1\)The denominators are indeed all primes.
Solution:
a) As we learned, if \( p \) is a prime, then if \( \pm p \) is of the form \( x^2 - Dy^2 \) then \( D \) must be a square modulo \( p \). (This just comes from reducing modulo \( p \).) Here \( D = 5, p = 3 \) and \( \left( \frac{D}{p} \right) = \left( \frac{5}{3} \right) = -1 \), so there is no solution.
b) Well, \( (2, 0) \) works, as does \( (3, 1) \).