CLARK'S SPRING 2018 MATH 4400 MIDTERM EXAM

Instructions: Calculators are not permitted. In problems 3 and 5, be sure to include reasoning that justifies your answer.

Problem 1. *State the Quadratic Reciprocity Law and its first and second supplements.*

Solution:
• Quadratic Reciprocity: Let $p \neq q$ be odd primes. Then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}.$$

• First Supplement: Let p be an odd prime. Then

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}.$$

• Second Supplement: Let p be an odd prime. Then

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}.$$

Problem 2. *State the Full Two Squares Theorem.*

Solution: A positive integer n is of the form $x^2 + y^2$ for $x, y \in \mathbb{Z}$ iff $\operatorname{ord}_p(n)$ is even for all primes $p \equiv 3 \pmod 4$.

Problem 3. *a)* ***Use the Euclidean algorithm*** *to compute the gcd of* 17 *and* 29.
(Note: clearly the gcd is 1*, but you must use the Euclidean algorithm. You'll thank me later.)*
b) Find ***an*** *integer solution* (x, y) *to the equation* $17x + 29y = 1$.
c) Find ***all*** *integer solutions* (x, y) *to the quation* $17x + 29y = 1$.

Solution:
a) We have

$$\begin{aligned} 29 &= 1 \cdot 17 + 12, \\ 17 &= 1 \cdot 12 + 5, \\ 12 &= 2 \cdot 5 + 2, \\ 5 &= 2 \cdot 2 + 1, \end{aligned}$$

so the gcd is – surprise!! – 1.
b) The point is that we can solve this by reversing the steps of the Euclidean Algorithm. (You're welcome for part a).) First write:

$$\begin{aligned} 1 &= 5 - 2 \cdot 2, \\ 2 &= 12 - 2 \cdot 5, \\ 5 &= 17 - 1 \cdot 12, \\ 12 &= 29 - 1 \cdot 17. \end{aligned}$$

Now we back substitute:

$$1 = 5 - 2 \cdot \mathbf{2} = 5 - 2 \cdot (\mathbf{12} - \mathbf{2} \cdot \mathbf{5})$$
$$5 \cdot \mathbf{5} - 2 \cdot 12 == 5 \cdot (\mathbf{17} - \mathbf{1} \cdot \mathbf{12}) - 2 \cdot 12$$
$$= 5 \cdot 17 - 7 \cdot \mathbf{12}$$

$$= 5 \cdot 17 - 7 \cdot (\mathbf{29} - \mathbf{1} \cdot \mathbf{17})$$
$$= 12 \cdot 17 - 7 \cdot 29.$$

So one solution is $(x_0, y_0) = (12, -7)$.
c) Given one solution (x_0, y_0), since $\gcd(17, 29) = 1$, we know that the general solution is

$$(x_n, y_n) = (12 + 29n, -7 - 17n) \text{ for } n \in \mathbb{Z}.$$

Problem 4. *Compute the following Legendre*[1] *symbols. You may use any (correct!) method.*

$$\left(\frac{1,000,000}{4,111}\right), \left(\frac{10,100}{2,017}\right), \left(\frac{2}{1,111,111,111,111,111,111,111}\right).$$

Solution: • Since $1,000,000 = 10^6 = (10^3)^2$ is coprime to $4,111$, we have

$$\left(\frac{1,000,000}{4,111}\right) = \left(\frac{10^3}{4,111}\right)^2 = 1.$$

• We have

$$10,100 = 5 \cdot 2,017 + 15,$$

so

$$\left(\frac{10,100}{2,017}\right) = \left(\frac{15}{2017}\right).$$

Since $2017 \equiv 1 \pmod 4$, we have

$$\left(\frac{15}{2017}\right) = \left(\frac{2017}{15}\right).$$

We have

$$2017 = 134 \cdot 15 + 7,$$

we have

$$\left(\frac{2017}{15}\right) = \left(\frac{7}{15}\right).$$

Since $7 \equiv 15 \equiv 3 \pmod 4$, we have

$$\left(\frac{7}{15}\right) = -\left(\frac{15}{7}\right) = -\left(\frac{1}{7}\right) = -1.$$

• Since $8 \mid 1,000 = 2^3 5^3$, what an integer is modulo 8 can be determined from its last three digits. So by the Second Supplement to Quadratic Reciprocity we have

$$\left(\frac{2}{1,111,111,111,111,111,111,111}\right) = \left(\frac{2}{111}\right).$$

We have

$$111 = 13 \cdot 8 + 7,$$

so $111 \equiv 7 \pmod 8$, and thus by the Second Supplement we have

$$\left(\frac{2}{1,111,111,111,111,111,111,111}\right) = 1.$$

Problem 5. *For each of the following equations, either find an integral solution (x, y) or show that there isn't one.*
a) $x^2 - 5y^2 = 3$.
b) $x^2 - 5y^2 = 4$.

[1] The denominators are indeed all primes.

Solution:
a) As we learned, if p is a prime, then if $\pm p$ is of the form $x^2 - Dy^2$ then D must be a square modulo p. (This just comes from reducing modulo p.) Here $D = 5$, $p = 3$ and $\left(\frac{D}{p}\right) = \left(\frac{5}{3}\right) = -1$, so there is no solution.
b) Well, $(2, 0)$ works, as does $(3, 1)$.