LECTURE NOTES ON SETS

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Contents

1. Introducing Sets

Sets are the first of the three languages of mathematics. They are the most basic kind of mathematical structure; all other structures are built out of them.[1]

A set is a collection of mathematical objects. This is not a careful definition; it is an informal description meant to convey (shortly) the correct intuition to you. We begin with some familiar examples.

Example 1. *One can think of a set as a kind of club; some things are members (already a little lie to be fixed later!); some things are not. So for instance current UGA students form a set. You are a member; I am not. Past or present presidents of the United States form a set. Barack Obama is a member. Mitt Romney is not.*

Example 2. *For any whole number* $n \geq 1$*,* $\{1, 2, \ldots, n\}$ *is a set, whose elements are indeed* $1, 2, 3, \ldots, n$*. Let us denote this set by* $[n]$*. So for instance*

$$5 \in [9] = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

and

$$9 \notin [5] = \{1, 2, 3, 4, 5\}.$$

(For whole numbers $a, b \geq 1$*, we have* $a \in [b]$ *precisely when* $a \leq b$*.)*

[1]Like most broad, sweeping statements made at the beginning of courses, this one is not *completely* true. Mathematics is at least 2500 years old: Pythagoras died circa 495 BCE. The practice of describing all mathematical objects in terms of sets dates from approximately 1900. Many mathematicians have at least contemplated basing mathematics on other kinds of objects; something called "categories," first introduced in the 1940's by Eilenberg and Mac Lane, have long had a significant (though minority) popularity. Nevertheless every student or practitioner of mathematics must speak the language of sets, which is what we are now introducing.

Example 3. *The positive integers*

$$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$$

are a set. The positive integer 1 *is an* ***element****, or* ***member*** *of* $\mathbb{Z}^+$*: we write this statement as*

$$1 \in \mathbb{Z}^+.$$

So is the positive integer 2*: we write*

$$2 \in \mathbb{Z}^+.$$

Similarly,

$$3 \in \mathbb{Z}^+, \ 4 \in \mathbb{Z}^+, \text{ and so forth.}$$

The negative integer -3 *is* not *an element of* $\mathbb{Z}^+$*. We write this as*

$$-3 \notin \mathbb{Z}^+.$$

The integer 0*, which is not positive (this is an explanation of terminology, not a mathematical fact), is not a member of* $\mathbb{Z}^+$*:*

$$0 \notin \mathbb{Z}^+.$$

Also $4/5 \notin \mathbb{Z}$*,* $\sqrt{2} \notin \mathbb{Z}^+$ *and Barack Obama* $\notin \mathbb{Z}^+$*. Of course* lots of things *are not in* $\mathbb{Z}^+$*: we had better move on.*

Example 4. *The non-negative integers, or* ***natural numbers***

$$\mathbb{N} = \{0, 1, 2, 3, \ldots, \}$$

are a set. The only difference between $\mathbb{Z}^+$ *and* $\mathbb{N}$ *is that* $0 \in \mathbb{N}$ *whereas* $0 \notin \mathbb{Z}^+$*. (This may seem silly, but it is actually useful to have both* $\mathbb{Z}^+$ *and* $\mathbb{N}$ *around.)*

Example 5. *The integers, both positive and negative*

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

form a set. This time $-3 \in \mathbb{Z}$*, but still* $4/5 \notin \mathbb{Z}$*,* $\sqrt{2} \notin \mathbb{Z}$ *and Barack Obama* $\notin \mathbb{Z}$*.*

Example 6. *A* ***rational number*** *is a quotient of two integers* $\frac{a}{b}$ *with* $b \neq 0$*. Rational numbers have many such representations, but* $\frac{a}{b} = \frac{c}{d}$ *exactly when* $ad = bc$*. The rational numbers form a set, denoted* $\mathbb{Q}$*.* $\sqrt{2} \notin \mathbb{Z}^+$ *(this is an important theorem of ancient Greek mathematics that we will discuss later); Barack Obama* $\notin \mathbb{Q}$*.*

Example 7. *The real numbers form a set, denoted* $\mathbb{R}$*. A real number can be represented as an integer followed by an infinite decimal expansion. Still Barack Obama* $\notin \mathbb{R}$*.*

Example 8. *A* ***complex number*** *is an expression of the form* $a + bi$*, where* $i^2 = -1$*. The set of complex numbers is denoted by* $\mathbb{C}$*.* i *is a member; still Barack Obama* $\notin \mathbb{C}$*.*

Example 9. *The Euclidean plane forms a set, denoted* $\mathbb{R}^2$*. Its elements are the points in the plane, i.e., ordered pairs* (x, y) *with* x, y *real numbers: we write* $x, y \in \mathbb{R}$*. For a positive integer* n*,* n*-dimensional Euclidean space forms a set, whose elements are ordered tuples* $(x_1, \ldots, x_n)$ *of real numbers. It is denoted* $\mathbb{R}^n$*.*

Example 10. *a) The lines in the Euclidean plane form a set.*
b) The planes in Euclidean space form a set.

Example 11. *The continuous functions* $f : [0, 1] \to \mathbb{R}$ *form a set.*

We admit that some of these examples were an excuse to introduce common mathematical notation. But the idea of a set is clear: it is a collection of objects. Practically speaking, this amounts to the following: if S is a set and x is any object, then exactly one of the following must hold: $x \in S$ or $x \notin S$. That's the point of a set: if you know exactly what is and is not a member of a set, then you know the set. Thus a set is like a bag of objects...but not a red bag or a cloth bag. The bag itself has no features: it is no more and no less than the objects it contains.

Remark 12. *We have included the last two examples in an attempt to drive home that the elements of a set need not (i) be numbers or (ii) "pointlike" in any geometric sense. (Sometimes it is helpful to think of the elements of an arbitrary set as "points," but this is just a way of thinking: they need not be points.)*

Example 13. *The empty set, denoted $\varnothing$, is a set. This is a set which contains no objects whatsoever: for any object x, we have $x \notin \varnothing$. Not only is this a fully kosher set, in some circles it is the most important example of a set.*

The following is the basic principle of sets: two sets S and T are **equal** precisely when they contain exactly the same objects: that is, for any object x, if $x \in S$ then $x \in T$, and conversely if $x \in T$ then $x \in S$.

An important consequence of this basic principle is that whereas above we said that the empty set $\varnothing$ is *a* set which contains no objects whatsoever, in fact it is *the* set which contains no such objects: any two sets which contain nothing contain exactly the same things!

A **finite list** of elements is something of the form $x_1, x_2, \ldots, x_n$, where n is a positive integer, and for each $1 \leq i \leq n$, x_i is an object. We allow the **empty list** when $n = 0$ (no objects!). Note that we do not require these objects to be different: e.g. $1, 1, 1, 1, 1, 1$ is a finite list of objects. A set is **finite** if it is of the form

$$S = \{x_1, \ldots, x_n\}$$

for some finite list $x_1, \ldots, x_n$: that is, for any object x, we have $x \in S$ precisely when $x = x_i$ for some i. A set is **infinite** if it is not finite. The **cardinality** of a finite set is the least number n of elements such that the set is associated to a list of n elements: in other (perhaps simpler) terms, it is the number of elements of a defining finite list that has no repetitions. I will denote the cardinality of a finite set by $\#S$.

Example 14. *a) The empty set $\varnothing$ is finite, and $\#\varnothing = 0$.*
b) The set $[n] = \{1, 2, \ldots, n\}$ is finite, and $\#[n] = n$.
c) The sets, $\mathbb{Z}^+$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are all infinite. (In fact most interesting mathematical sets are infinite.)

A finite list has more structure than the finite set it determines: the objects of a set come in a certain **order**, a notion which allows them to occur more than once. The set determined by a finite set list ignores both of these things. Thus

$$\{1, 2, 3\} = \{1, 3, 2\} = \{2, 1, 3\} = \{2, 3, 1\} = \{3, 1, 2\} = \{3, 2, 1\}$$

and

$$\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4\} = \{1, 2, 3, 4\}.$$

We call will the process of defining a set using a finite list an **extensional** definition of a set. The other way of giving a set, called **intentional**, is by giving a defining **property** of the set. When we write

$$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$$

it looks like we're giving an extensional definition, but there is an "ellipsis" . . .: what does this mean? The only honest answer to give now is that the ellipsis stands for "and so on" and is thus a shorthand for the *intentional* concept of a positive whole number. Which is a fancy way of saying that I am assuming that you are familiar with the concept of a positive whole number and I am just referring to it, rather than giving some kind of precise, comprehensive description of it.

Thus the intentional description of a set is as the collection of objects satisfying a certain property. This description however must be taken with a grain of salt: for any set S there is a corresponding property of objects...namely the property of being in that set! Thus being an element of $\{17, 2016, \frac{7}{4}, \pi, \text{blue}\}$ defines a property, although in the everyday sense there is certainly no evident rule that is being used to form this set. Again, think of a set as *any* collection of objects; the difficulties we have in describing or specifying a set – especially, an infinite set – are "our problem". They do not restrict the notion of a set.

Example 15. *Here are some more examples of sets:*
(i) $\{\varnothing\}$.
(ii) $\{\varnothing, \{\varnothing\}\}$.
(iii) $\{\varnothing, 1, 2\}$.
(iv) $\{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.

The sets above have a new feature: the elements are themselves sets! This is absolutely permissible. While we have not given a definition of an object, a set absolutely qualifies. Starting with the empty set and using our extensional method in a recursive way, we can swiftly build a large family of sets...of a sort which is actually a bit confusing and needs to be thought about carefully. Thus for instance, the sets $\varnothing$ and $\{\varnothing\}$ are certainly *not* equal: the first set has zero elements and the second set has one element, which happens to be the set which has zero elements. In other terms (not guaranteed to be less confusing!), we must distinguish a bag which is empty from a bag which contains, precisely, an empty bag. Part (ii) shows how this madness can be continued. You should think carefully about the difference between the sets in parts (iii) and (iv): the set in part (iii) has some elements which are sets and some elements which are numbers. It also has 3 elements. Every element of the set in part (iv) is itself a set, and there are 4 elements.

We call a set **pure** if all its elements are sets. Although I will not try to justify this now, in fact all of mathematics could be done only with pure sets. This means that everything in sight can be defined to be a set of some kind. So for instance numbers like 0 and 1 would have to be defined to be sets. I will not say anything more about this now: if this interests you, you might want to think of a reasonable definition for 0, 1, 2,. . . in terms of the empty set and lots of brackets. If this troubles you: never mind, we move on!

2. Subsets

Let S and T be sets. We say that S is a **subset** of T if every element of S is also an element of T. Otherwise put, for all objects x, if $x \in S$ then also $x \in T$. The symbol for this is

$$S \subseteq T.$$

It is useful to have vocabulary to describe $S \subseteq T$ "from T's perspective." And we do: if $S \subseteq T$, we say that T **contains** S. However this comes with a

WARNING!!! If $x \in T$, then we often say "S contains x." However, if $S \subseteq T$ we also say "T contains S." So if the object x happens to be a set, then saying "S contains x" is ambiguous: it could mean $x \in S$ or also $x \subseteq S$. These need not be the same thing! Thus we should not say "S contains x" when x is a set unless the context makes completely clear what is intended; if necessary we could say "S contains x as an element" to mean $x \in S$.

A subset S of T is **proper** if $S \neq T$: every element of S is an element of T, but at least one element of T is not an element of S. We denote this by $S \subsetneq T$.

Exercise 16. *The empty set is a subset of every set S: $\varnothing \subseteq S$. We have $\varnothing \subsetneq S$ precisely when S is nonempty.*

Example 17. *With regard to our previously defined sets of numbers, we have*

$$\mathbb{Z}^+ \subsetneq \mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}.$$

The complex numbers are not "the end of the line" in any set-theoretic sense: we could take for instance the set of things which are either complex numbers or lines in the plane, and that would be bigger. In fact there are even "number systems" which extend the complex numbers...but they are not as ubiquitous in undergraduate mathematics as the number systems we have given above.

Exercise 18. *Let S and T be sets. Then $S = T$ precisely when $S \subseteq T$ and $T \subseteq S$.*

Although the preceding result is almost obvious, it is also very useful: in practice, it can be much easier to each "one-sided containment" $S \subseteq T$ and $T \subseteq S$ then to show $S = T$ all at once. This is analogous to the method of showing that two real numbers x and y are equal by showing $x \leq y$ and then that $y \leq x$. (The set-theoretic method comes up more often.)

3. Power Sets

For a set X, the **power set** of X is the set of all subsets of X. We denote the power set of X by 2^X.

Example 19. *0) The set $\varnothing$ has 0 elements. Its power set is $2^{\varnothing} = \{\varnothing\}$, which has $1 = 2^0$ elements.*
(i) The set $[1] = \{1\}$ has 1 element. Its power set is $2^{[1]} = \{\varnothing, \{1\}\}$, which has $2 = 2^1$ elements.
(ii) The set $[2] = \{1, 2\}$ has 2 elements. Its power set is $\{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$, which has $4 = 2^2$ elements.

Proposition 20. *Let S be a finite set of cardinality n. Then the power set 2^S is finite of cardinality 2^n.*

Proof. A set of cardinality n can be given as $\{x_1, \ldots, x_n\}$. To form a subset, we must *choose* whether to include x_1 or not: that's two options. Then, independently, we choose whether to include x_2 or not: two more options. And so forth: all in all, we get a subset precisely by decidind, independently, whether to include or exclude each of the n elements. This gives us $2 \cdots 2$ (n times) $= 2^n$ options altogether. □

We hope the previous result gives some explanation for our notation 2^S.

Observe that for a set S, we have $x \in 2^S$ precisely when $x \subseteq S$. Thus one feature of the power set is to convert the relation $\subseteq$ to the relation $\in$.

4. Operations on Sets

We wish here to introduce some – rather familiar, I hope – operations on sets.

For sets S and T, we define their **union** $S \cup T$ to be the set of all objects x which are elements of S, elements of T or both. (As we will see in the next chapter, in mathematics, the term "or" is always used inclusively.) We define their **intersection** $S \cap T$ to be the set of all objects which are elements of both S and T. Two sets S and T are **disjoint** if $S \cap T = \varnothing$; i.e., they have no objects in common.

For sets S and T, we define their **set-theoretic difference**

$$S \setminus T = \{x \mid x \in S \text{ and } x \notin T\}.$$

If we are only considering subsets of a fixed set X, then for $Y \subseteq X$ we define its **complement** Y^c to be $X \setminus Y$.

Example 21. *Let $X = \mathbb{Z}$, the integers. Let E be the set of even integers, i.e., integers of the form $2n$ for $n \in \mathbb{Z}$. Let O be the subset of odd integers, i.e., integers of the form $2n + 1$ for $n \in \mathbb{Z}$. Then:*
a) We have $E \cap O = \varnothing$: that is, no integer is both even and odd. Indeed, if $2m = x = 2n + 1$, then $1 = 2(m - n)$, and thus $m - n = \frac{1}{2}$. But that's ridiculous: if m, n are integers, so is $m - n$, and $\frac{1}{2} \notin \mathbb{Z}$.
b) We have $E \cup O = \mathbb{Z}$. First note that if $x \in E$ then $x = 2m$, so $-x = -2m = 2(-m) \in E$; similarly if $x \in O$ then $x = 2n+1$, so $-x = -2n-1 = -2n-2+2-1 = 2(-n-1) + 1 \in O$. Moreover $0 \in E$ and $1 \in O$, so it is enough to show that every integer $n \geq 2$ is either even or odd. The key observation is now that if for any $k \in \mathbb{Z}^+$, if $x - 2k \in E$ then $x \in E$, and if $x - 2k \in O$ then $x \in O$. Now consider $x - 2$. Since $x \geq 2$, $x - 2 \geq 0$. If $x - 2 \in \{0, 1\}$, then $x - 2$ is either even or odd, so x is either even or odd. Otherwise $x - 2 \geq 2$, so consider $x - 4$. We may continue in this way: in fact, there is a unique positive integer k such that $x - 2k \in \{0, 1\}$: if we keep subtracting 2, then eventually we will get something negative, and if we add back 2 then we must have either 0 or 1. This shows what we want.
c) Taking complements with respect to the fixed set X, we have $O^c = E$ and $E^c = O$. We say that E and O are complementary subsets of the integers.

Proposition 22. *(DeMorgan's Laws for Sets)*
Let A and B be subsets of a fixed set X. Then:
a) We have $(A \cup B)^c = A^c \cap B^c$.
b) We have $(A \cap B)^c = A^c \cup B^c$.

Proof.
a) Since $A \cup B$ consists of all elements of X which lie in A or in B (or both), the complement $(A \cup B)^c$ consists of all elements of X which lie in neither A nor B. That is, it consists precisely of elements which do not lie in A and do not lie in B, hence of elements which lie in the complement of A and in the complement of B.
b) Since $A \cap B$ consists of all elements of X which in both A and B, the complement $(A \cap B)^c$ consists of all elements of X which do not lie in both A and B. An element of X does not lie in both A and B precisely when it does not lie in A or it does not lie in B (or both), i.e., we get precisely the elements of $A^c \cup B^c$. □

Although these things can be converted to "word problems" and sounded out with little trouble, many people prefer a more visual approachl For this **Venn diagrams** are useful. A Venn diagram for two subsets A and B of a fixed set X consists of a large rectangle (say) representing X and within it two smaller, overlapping circles, representing A and B. Notice that this divides the rectangle X into four regions:

- $A \cap B$ is the common intersection of the two circles.
- $A \setminus B$ is the part of A which lies outside B.
- $B \setminus A$ is the part of B which lies outside A.
- $(A \cup B)^c$ is the part of X which lies outside both A and B.

Exercise 23. *Use Venn diagrams to prove DeMorgan's Laws for Sets.*

Proposition 24. *(Distributive Laws) Let A, B, C be sets. Then:*
a) We have $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
b) We have $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.
That is: intersection distributes over union and union distributes over intersection.

Proof. a) We will use the technique of showing that two sets are equal by showing that each contains the other. Suppose $x \in (A \cup B) \cap C$. Then $x \in C$ and $x \in A \cup B$, so either $x \in A$ or $x \in B$. If $x \in A$ then $x \in A \cap C$, whereas if $x \in B$ then $x \in B \cap C$, so either way $x \in (A \cap C) \cup (B \cap C)$. Thus

$$(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C).$$

Conversely, suppose $x \in (A \cap C) \cup (B \cap C)$. Then $x \in A \cap C$ or $x \in B \cap C$. Since both A and B are subsets of $A \cup B$, either way we have $x \in (A \cup B) \cap C$, so

$$(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C.$$

b) This is similar; I leave it to you as an exercise. □

Exercise 25. *A Venn diagram for three subsets A, B, C of a fixed set X consists of three circles inside a rectangle X positioned so as to divide X into $8 = 2^3$ regions in all (this comes from being in A vs. not being in A, being in B vs. not being in B, and being in C vs. not being in C). This is no problem to achieve: just draw three circles with the same radius and noncolinear centers sufficiently close together. Use this kind of Venn diagram to prove the distributive laws.*

Remark 26. *The more familiar distributive law is that multiplication – say of complex numbers – distributes over addition: for all $x, y, z \in \mathbb{C}$ we have $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$. It is interesting that in the set theoretic context each of union and intersection distributes over the other. This is a pleasant symmetry which is not present in the case of numbers: for most $x, y, z \in \mathbb{C}$ we do not have $(x \cdot y) + z = (x + z) \cdot (y + z)$. For instance try it with $x = y = z = 1$.*

5. Families of Sets

We can define unions and intersections for more than two sets. For now we will restrict to finitely many sets: let $A_1, \ldots, A_n$ be subsets of a fixed set X. Then we define $A_1 \cap \ldots \cap A_n$ to be the set of all objects which lie in all of the A_i's, and we define $A_1 \cup \ldots \cup A_n$ to be the set of all objects which lie in at least one of the A_i's.

Exercise 27. *Show that DeMorgan's Laws extend to n sets (for any integer $n \geq 2$):*

$$(A_1 \cup \ldots \cup A_n)^c = A_1^c \cap \ldots \cap A_n^c$$

and

$$(A_1 \cap \ldots \cap A_n)^c = A_1^c \cup \ldots \cup A_n^c.$$

Exercise 28. *Formulate and prove an extension of the distributive laws to n sets.*

Exercise 29. *Are there Venn diagrams for n sets with $n \geq 4$?*
(Hint: yes, but you cannot use circles.)

There is another, rather more sophisticated perspective to take on the expression $A_1, \ldots, A_n$: namely that it is a **family of sets indexed by** the set $[n] = \{1, \ldots, n\}$. Really this is a kind of **function** (although functions will not be formally defined and considered until much later in the course), by which I mean that it is an assignment of a set A_i to each $i \in \{1, \ldots, n\}$: we write

$$1 \mapsto A_1, 2 \mapsto A_2, \ldots, n \mapsto A_n.$$

More generally, a **family of sets indexed by a set I** is just a set I – let's say it's nonempty; nothing very interesting can happen otherwise – and an assignment of each $i \in I$ a set A_i. This is a construction that comes up naturally in mathematics, but we don't have too much to say about it now: mainly that it makes sense to take unions and intersections over an indexed family of sets: we define the **union** $\bigcup_{i \in I} A_i$ to be the set of all elements x which lie in A_i for at least one $i \in I$ and the **intersection** $\bigcup_{i \in I} A_i$ to be the set of all elements x which lie in A_i for all $i \in I$. Thus the union is the set of elements which lie in *some* set of the family and the intersection is the set of elements which lie in *every* set in the family. This generalizes the kind of union and intersection we studied before when I has two elements or has finitely many elements.

Example 30. *a) If $I = \mathbb{Z}^+$ then we have a* ***sequence of sets***

$$A_1, A_2, \ldots.$$

b) Suppose $I = \mathbb{Z}$ and for all $n \in I$ we put $A_n = \{n\}$. Then

$$\bigcup_{n \in \mathbb{Z}} \{n\} = \mathbb{Z}$$

and

$$\bigcap_{n \in \mathbb{Z}} \{n\} = \varnothing.$$

c) More generally, let I be any set which contains more than one element, and for $i \in I$ put $A_i = \{i\}$. Then

$$\bigcup_{i \in I} A_i = I$$

and

$$\bigcap_{i \in I} A_i = \varnothing.$$

(Why is it important here that I have more than one element?)

Example 31. *(Monotone Sequences of Sets)*
a) For $n \in \mathbb{Z}^+$ we put

$$A_n = [-n, n] \subseteq \mathbb{R}.$$

Then we have

$$A_1 = [-1, 1] \subseteq A_2 = [-2, 2] \subseteq \ldots \subseteq A_n = [-n, n] \subseteq \ldots.$$

In this case we have

$$\bigcup_{n \in \mathbb{Z}^+} A_n = \mathbb{R},$$

since every real number has absolute value less than n for some integer n. More easily, we have

$$\bigcap_{n \in \mathbb{Z}^+} A_n = [-1, 1].$$

*This sequence of sets has the interesting property that $A_n \subseteq A_{n+1}$ for all n. For any such sequence of sets, the common intersection of all the sets is just A_1. One might call this an **increasing sequence of sets**.*
b) For $n \in \mathbb{Z}^+$ we put

$$B_n = (\frac{-1-n}{n}, \frac{n+1}{n}) \subseteq \mathbb{R}.$$

Thus we have

$$B_1 = (-2, 2) \supseteq B_2 = (\frac{-3}{2}, \frac{3}{2}) \supseteq B_3 = (\frac{-4}{3}, \frac{4}{3}) \supseteq \ldots \supseteq B_n \supseteq \ldots$$

This time we have

$$\bigcup_{n \in \mathbb{Z}^+} B_n = B_1 = (-2, 2)$$

and the more interesting case is

$$\bigcap_{n \in \mathbb{Z}^+} B_n = [-1, 1].$$

*Thus the intersection of an infinite sequence of open intervals turns out to be a closed interval. This sequence has the interesting property that $B_n \supseteq B_{n+1}$ for all n: we call this a **decreasing sequence of sets** or a **nested sequence of sets**. For any nested sequence of sets, the union is the first set B_1.*

A family $\{A_i\}_{i \in I}$ of sets is **pairwise disjoint** if for all $i \neq j$ we have $A_i \cap A_j = \varnothing$.

Example 32. *If $I = \mathbb{Z}$ then we can take $A_n = \mathbb{R}$ for all $n \in \mathbb{Z}$. This is a family of sets indexed by $\mathbb{Z}$ each element of which is the set of real numbers. This example illustrates something that the course text seems to soft-pedal: that an indexed family of sets is more than just a* set *of sets; it consists of an assignment of a set to each element of an index set I. We are allowed to assign the same set to two different elements of I.*

6. Partitions

Let X be a nonempty set. A **partition** of X is, roughly, an exhaustive division of X into nonoverlapping nonempty pieces. More precisely, a partition of X is a set $\mathcal{P}$ of subsets of X satisfying all of the following properties:

(P1) $\bigcup_{S \in \mathcal{P}} S = X$.
(P2) For distinct elements $S \neq T$ in $\mathcal{P}$, we have $S \cap T = \varnothing$.
(P3) If $S \in \mathcal{P}$ then $S \neq \varnothing$.

Example 33. *Let $X = [5] = \{1,2,3,4,5\}$. Then:*
a) The set $\mathcal{P}_1 = \{\{1,3\},\{2\},\{4,5\}\}$ is a partition of X.
b) The set $\mathcal{P}_2 = \{\{1,2,3\},\{4\}\}$ is not a partition of X: $5 \in X$, but 5 is not an element of any element of $\mathcal{P}_2$, so (P1) fails. However, (P2) and (P3) both hold.
c) The set $\mathcal{P}_3 = \{\{1,2,3\},\{3,4,5\}\}$ is not a partition of X because $\{1,2,3\}$ and $\{3,4,5\}$ are not disjoint sts. However, (P1) and (P3) both hold.
d) The set $\mathcal{P}_4 = \{\{1,2,3,4,5\},\varnothing\}$ is not a partition of X because it contains the empty set, so (P3) fails. However, (P1) and (P2) both hold.

Example 34. *a) Let $X = [1] = \{1\}$. There is exactly one partition, $\mathcal{P} = \{X\}$.*
b) Let $X = [2] = \{1,2\}$. There are two partitions on X,

$$\mathcal{P}_1 = \{\{1,2\}\},\ \mathcal{P}_2 = \{\{1\},\{2\}\}.$$

c) Let X be any set with more than one element. Then the analogues of the above partitions exist: namely there is the ***trivial partition*** *(or* ***indiscrete partition****)*

$$\mathcal{P}_t = \{X\}$$

and the ***discrete partition***

$$\mathcal{P}_D = \{\{x\} \mid x \in X\}.$$

I hope the notation does not distract you from the simplicity of what's happening here: in the trivial partition we "break X up into one piece" (or in another words, we don't break it up at all). In the discrete partition we "break X up into the largest possible number of pieces," i.e., one-element sets.
d) If X has more than two elements then there are partitions on X other than the trivial partition and the discrete partition. For instance on $X = [3] = \{1,2,3\}$ there are three more:

$$\{\{1\},\{2,3,\}\},\ \{\{2\},\{1,3\},\},\ \{\{3\},\{1,2\}\}.$$

Note that these three different partitions share a common feature: namely for each positive integer n, they have the same number of pieces of size n. If we count partitions on a set altogether, we find ourselves counting a lot of similar-looking decompositions which are labelled differently, as are the above three guys. It is a classic number theory problem to count partitions on $[n]$ up to the various sizes of the pieces. *In other words, in this sense 3 has 3 partitions:*

$$3 = 3 = 2 + 1 = 1 + 1.$$

Similarly, in this sense 4 has 5 partitions:

$$4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

For a positive integer n, define $P(n)$ to be the number of partitions of n in this sense, so what we've seen so far is

$$P(1) = 1,\ P(2) = 2,\ P(3) = 3,\ P(4) = 5.$$

There is an enormous amount of deep 20th century mathematics studying the asymptotic behavior of the partition function $P(n)$, which is a fancy way of saying studying how quickly it grows.

Example 35. *Let E be the set of even integers, and let O be the set of odd integers. Then $\{E, O\}$ is a partition of $\mathbb{Z}$. This serves to illustrate why partitions of sets are important: one can think of elements of the same set in a partition as sharing a common* property*, in this case the property that they are both even (if they are both in E) or that they are both odd (if they are both in O). Later we will see that conversely, a certain type of property of objects of a set X – called an* ***equivalence relation*** *– determines a partition of X and that conversely every partition on X arises from an equivalence relation on X.*

7. Cartesian Products

Let X and Y be sets. Then the **Cartesian product** $X \times Y$ is the set of ordered pairs (x, y) where $x \in X$ and $y \in Y$.

Example 36. *The main example (trope-namer, in contemporary internet slang) is of course $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, the Cartesian plane.*

More generally, if $X_1, \ldots, X_n$ are sets then the Cartesian product $X_1 \times \ldots \times X_n$ is the set of all ordered n-tuples $(x_1, \ldots, x_n)$ with $x_1 \in X_1, \ldots, x_n \in X_n$.

One can ask (although I didn't think to do so until after I got my PhD in mathematics) what an ordered pair "really is." And one can give an actual set-theoretic definition: this was done by various people in the early 20th century, and nowadays most like Kuratowski's definition best. This is pursued in one of the typed problems on the second problem set. More seriously though, it doesn't matter what kind of object (x, y) is; what matters is when two ordered pairs are equal, and the answer is that $(x_1, x_2) = (y_1, y_2)$ precisely when $x_1 = y_1$ and $x_2 = y_2$. Similarly, two ordered n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are equal precisely when $x_1 = y_1$, $x_2 = y_2$,...,$x_n = y_n$.

Even more generally, if I is a nonempty set and $\{X_i\}_{i\in I}$ is an indexed family of sets, then we can consider the Cartesian product $\prod_{i\in I} X_i$. An element of this is an object $\{x_i\}_{i\in I}$: that is, for each $i \in I$, an element $x_i \in X_i$.

Exercise 37. *a) Let $X_1, \ldots, X_n$ be sets. Show that the Cartesian product $X_1 \times \ldots \times X_n$ is empty precisely when at least one X_i is empty.*
b) Let $X_1, \ldots, X_n$ be finite sets. Show that the number of elements of $\prod_{i=1}^n X_i$ is $(\#X_1) \cdot (\#X_2) \cdot \ldots \cdot (\#X_n)$, i.e., the cardinality of the Cartesian product is the product of the cardinalities of the "factor" sets. (This is just the principle of "independent choices" that we have used above to count elements of power sets.)