REVIEW FOR THIRD 3200 MIDTERM

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1) Show that for all integers $n \geq 2$ we have

$$1^3 + \ldots + (n-1)^3 < \frac{1}{4}n^4 < 1^3 + \ldots + n^3.$$

2) The distributive law for the real numbers states that for any real numbers a, b, c, $a \cdot (b + c) = a \cdot b + a \cdot c$. Assuming this, show by induction that for all $n \in \mathbb{Z}^+$ and real numbers $a, b_1, \ldots, b_n$, $a \cdot (b_1 + \ldots + b_n) = a \cdot b_1 + \ldots + a \cdot b_n$.

3) State the principle of mathematical induction and the principle of strong/complete induction.

4) Define the following terms: a **relation** R between sets X and Y, the **domain** of a relation; the **inverse relation** R^{-1}; an **equivalence relation**; a **function**; the **domain of a function**; the **codomain of a function**; the **range of a function**; an **injective function**; a **surjective function**; a **bijective function**.

5) $f : X \to Y$ and $g : Y \to X$ be functions.
a) Say what it means for f and g to be inverse functions.
b) Suppose $\forall x \in X$, $g(f(x)) = x$. Prove/disprove: f and g are inverse functions.
c) Same question as part b), but now assume that f is surjective.
d) Same question as part b), but now assume that g is injective.

6) Let $f : X \to Y$ and $g : Y \to Z$ be functions.
a) Define $g \circ f$.
b) Suppose that f and g are injective. Show that $g \circ f$ is injective.
c) Suppose that f and g are surjective. Show that $g \circ f$ is surjective.
d) Suppose that f and g are bijective. Show that $g \circ f$ is bijective.

7) Let X be a set. Prove or disprove: there does not exist any function $f : X \to \varnothing$.

8) Let $n \in \mathbb{Z}^+$ and $b \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $x \mapsto x^n + b$. Determine the range of f. Is f injective? Surjective?
(Your answer may depend on n and/or b.)

9)a) Prove/disprove: if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and such that $f'(x) \geq 0$ for all x, then f is injective.
b) Same as part a), except with the assumption that $f'(x) > 0$ for all x.
c) Which of the following functions are injective (on their usual domains): $\sin x$, $\cos x$, $\tan x$, e^x, $\ln x$?
d) Which of the functions of part c) are surjective onto $\mathbb{R}$?

10) Let $f : X \to Y$ be a function.
a) Define a relation on X by $x \sim x'$ if $f(x) = f(x')$. Show: $\sim$ is an equivalence relation.
b) For any $y \in Y$, the **fiber over y in X** is the set

$$f^{-1}(y) = \{x \in X \mid f(x) = y\}.$$

Show that for $y \neq y'$ the fibers over y and y' are disjoint subsets of X.
c) Show that the set of fibers $\{f^{-1}(y) \mid y \in Y\}$ gives a partition of x if and only if f is surjective. Show that in this case the corresponding equivalence relation on X is the same as the $\sim$ relation of part a).

11) Let X be the set of all functions $f : \mathbb{R} \to (0, \infty)$. Define a relation $\sim$ on X by $f \sim g$ if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$. Show that $\sim$ is an equivalence relation on X. (It is called **asymptotic equality**).

12) For each of the following, give an example or prove no such example exists.
a) A relation on a set which is symmetric and transitive but not reflexive.
b) A relation $R \subset \mathbb{R} \times \mathbb{R}$ such that each vertical line $x = c$ intersects R in exactly one point, but R is not a function.
c) A function $R \subset \mathbb{R} \times \mathbb{R}$ such that each horizontal line $y = d$ intersects R in at most one point, but which is not invertible.

13) Suppose $f : X \to Y$ is a surjective function. Show that there exists a function $g : Y \to X$ such that for all $y \in Y$, $f(g(y)) = y$.