MATH 3200 PRACTICE PROBLEMS 1

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In all of the following questions, let x, y, z be objects and A, B, C be sets.

1) Let A and B be sets.
a) What is the meaning of $A = B$? Of $A \subset B$? Of $A \subsetneq B$?

Solution: $A = B$ means that the sets A and B have exactly the same elements: any object x is an element of A if and only if it is an element of B. I'm not sure what $A \subset B$ means! We agreed not to use this symbol. Instead, we should use $A \subseteq B$ for containment that could be equality and $A \subsetneq B$ for proper ontainment.

b) What is the meaning of $A \cup B$? Of $A \cap B$? Of $A \setminus B$?

Solution: $A \cup B$ is the set of all elements x such that $x \in A$ or $x \in B$ (or both). $A \cap B$ is the set of all elements x such that $x \in A$ and $x \in B$. $A \setminus B$ is the set of all elements x such that $x \in A, x \notin B$.

2) Define the **symmetric difference** of A and B as $(A \setminus B) \cup (B \setminus A)$.
In this problem, the symmetric difference will be denoted as $A \Delta B$.
a) Draw a Venn diagram indicating the symmetric difference.

Solution: Two of the four regions should be shaded: the one representing elements of A which are not in B, and the one representing elements of B which are not in A.

b) Let $A = \{2k \mid k \in \mathbb{Z}^+\}$ and let B be the set of prime numbers. What is $A \Delta B$?

Solution: First, recall that by a standard (among mathematicians) convention, 1 is not regarded as a prime. Now, $A \Delta B$ is the set of all positive integers which are either even or prime but not both. Thus, it is the collection of all odd prime numbers together with the collection of all even positive integers greater than 2.

c) Show that $(A \Delta B) \cup (A \cap B) = A \cup B$.

Solution: First we show $(A \Delta B) \cup (A \cap B) \subset A \cup B$. For this, it is most efficient to asume that $x \in (A \Delta B) \cup (A \cap B) \setminus A$ and deduce that $x \in B$. Since $x \notin A$, certainly $x \notin A \cap B$, so $x \in A \Delta B$. Since, again, $x \notin A$, we conclude $x \in B \setminus A$, so $x \in A \cup B$.

Second we show that $A \cup B \subset (A \Delta B) \cup (A \cap B)$. Suppose that $x \in A$. Then, if x is also in B, $x \in A \cap B$ hence is in $(A \Delta B) \cup (A \cap B)$. Otherwise $x \in A \setminus B \subset A \Delta B \subset (A \Delta B) \cup (A \cap B)$. The case of $x \in B$ is is the same – indeed, demonstrably so, because there is a symmetry between A and B in the statement. (So it is not necessary to repeat the argument with $x \in B$, although it would be perfectly permissible to do so.)

3) a) What is meant by a universal set?

Solution: A universal set is a set which contains every other set which is being considered in the context of a certain discussion. For instance, if one is working with sets of real numbers, then $\mathbb{R}$ is a universal set. Note that there is no true universal set – that is, there is no set X such that $x \in X$ for all objects X. b) What is meant by $\overline{A}$, and when is it defined?

Solution: This is the complement of A, and it is defined exactly when A is a subset of a fixed universal set X, in which case it means $X \setminus A$.

4) a) What is a partition of a set S?

Solution: A partition of a set S is a family $\mathcal{F}$ of subsets of S satisfying:
(i) For all $A \in \mathcal{F}$, $A \neq \emptyset$.
(ii) $\bigcup_{A \in \mathcal{F}} = S$.
(iii) If $A, B \in \mathcal{F}$ and $A \neq B$, then $A \cap B = \emptyset$.

b) Are there any sets S which have no partitions? Which sets S have exactly one partition?

Solution: No, every set admits at least one partition. We saw in class that the empty set admits the empty partition $\mathcal{F} = \emptyset$ (and it is easy to see that it admits no other partition). Any nonempty set admits a unique partition with one part, namely $\mathcal{F} = \{S\}$. If S has exactly one element, then this is the only possible partition of S. If S has at least two elements, then the partition $\mathcal{F} = \{\{x\}_{x \in S}$ into one-element sets is a different partition. Thus a set S has exactly one partition iff it is empty or has exactly one element.

5) Decide whether each of the following is true or false, and briefly explain.
a) Any partition of a finite set must have a finite number of elements.

Solution: True. The parts of a partition are elements of the power set 2^S of the set S. If S has n elements, then since each element of a partition has at least one element of S, the largest number of elements a partition can have is n (because if we had any more the sum of the sizes of the parts would be greater than the size of the whole!).

b) Any partition of an infinite set must have an infinite number of elements.

Solution: False. As we saw above, any nonempty set admits a partition with exactly one element, the set itself. Other examples:
• We partitioned $\mathbb{Z}$ into $E = \{\text{even numbers}\}$ and $O = \{\text{odd numbers}\}$. Division by $b \in \mathbb{Z}^+$ with remainder partitions $\mathbb{Z}$ into b different parts.

• (M. Dober) For instance, $\{(-\infty, 0), \{0\}, (0, \infty)\}$ is a partition of $\mathbb{R}$ into three parts.

6) a) Define: implication, contrapositive, converse, inverse, biconditional.

Solution: implication: $A \implies B$. If A is true, then B is true. More precisely, this means that A is false or B is true.
Contrapositive: $\neg B \implies \neg A$: If B is false, then A is false. More precisely, this means that A is false or B is true, hence it is equivalent to the implication.
Converse: $B \implies A$: If B is true, then A is true. More precisely, this means that B is false or A is true.

Inverse: $\neg A \implies \neg B$: if A is false, then B is false. More precisely, this means that B is false or A is true, hence it is equivalent to the inverse.

b) What is meant by the "converse fallacy"?

This is a term in logic, philosophy and rhetoric: it means confusion of $A \implies B$ with $B \implies A$. It is perhaps the most common logical mistake made in everyday reasoning.

c) Show that $A \implies B$ is logically equivalent to $\neg B \implies \neg A$ in two different ways: (i) using a truth table, and (ii) using an argument in plain English.

Solution: (i) This was done in class, and for sure it is in your text. We omit it here. (ii): Suppose $A \implies B$ holds and assume that B is false. Then it cannot be that A is true, because that would make the implication false, so it must be that A is false. That is:

$$(A \implies B) \implies (\neg B \implies \neg A).$$

Conversely, suppose $\neg B \implies \neg A$, and suppose that A is true. The argument is the same as above: it cannot be that B is false, for then $\neg B$ would be true and that would make the implication false, so it must be that B is true. That is:

$$(\neg B \implies \neg A) \implies (A \implies B).$$

7) True or false: Neither the inverse nor the converse of an implication is logically equivalent to the implication, but the inverse and the converse are logically equivalent to each other.

True. See the solution to 5).

8) Negate the following sentence: "You can fool some of the people all of the time, and all of the people some of the time, but you cannot fool all of the people all of the time."

The negation is: "At least one of the following holds: (i) There exists at least one time when you can't fool anyone, or (ii) At all times, there exists at least person that you can't fool, or (iii) At all times, you can fool all of the people."

9) What does it mean for an implication to hold trivially? To hold vacuously?

An implication of the form $\forall x \in S, P(x) \implies Q(x)$ holds trivially if for all $x \in S$, $Q(x)$ is true. It holds vacuously if for all $x \in S$, $P(x)$ is false.

10) For which of the following famous theorems does the converse also hold?[1]
a) (Pythagorean Theorem) Let a, b and c be the side lengths of a triangle T. If T is a right triangle and c is the length of the hypotenuse, then $c^2 = a^2 + b^2$.
b) (Rational Roots Theorem) Let $P(x) = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial with integer coefficients $a_0, \ldots, a_n$, and let $r = \frac{c}{d}$ be a rational number written in lowest terms. If $P(r) = 0$, then $d \mid a_n$ and $c \mid a_0$.

Solution: A better question: first state the converse and then determine whether the converse is also true. I will answer this better question.
a) The converse is: let a, b, c be the side lengths of a triangle T. If $c^2 = a^2 + b^2$, then T is a right triangle. This is true: it follows from the law of cosines.
b) The converse is: if $d \mid a_n$ and $c \mid a_0$ then $P(\frac{c}{d}) = 0$. This is false. For instance, for $P(x) = x^2 - 2$, it says that $P(2) = P(-2) = 0$, which is certainly not the case.

11) True or false: it is not possible to prove a theorem by giving a single example. Discuss.

Solution: False. It is possible, but unusual. Most theorems in mathematics are of the form $\forall x \in S, P(x) \implies Q(x)$, where S is some infinite set. In this case giving a single value of x for which $P(x) \implies Q(x)$ certainly does not prove the theorem. However, occasionally one does encounter theorems of the form $\exists x \in S \mid P(x)$, e.g.:

"In 1903, a mathematician by the name of Frank Nelson Cole (1861 - 1926) gave a 'lecture' to the American Mathematical Society entitled 'On the Factorisation of Large Numbers'. Without saying a word, Cole proceeded to write on a blackboard the calculations for 2 to the power of 67, then carefully subtracted 1. This was M67 and it is equal to 147,573,952,589,676,412,927. He then wrote out 193,707,721 x 761,838,257,287 on the other side of the board and proceeded to work out the multiplication by hand. The final product was written at the bottom and was seen to be identical to M67...He received a standing ovation."

In other words, Cole proved that $\exists x, y \in \mathbb{Z} \mid (x, y > 1 \wedge xy = 2^{67} - 1)$ by exhibiting $x = 193707721,\ y = 751838257287$.

12) Suppose x is an integer such that $x \equiv 3 \pmod 6$. Show that $3 \mid x$ and $2 \nmid x$.

Solution: The hypothesis means that $x = 6k + 3$ for some $k \in \mathbb{Z}$. Thus $x = 3(2k+1)$, so $3 \mid x$, and $x = 2(3k+1)+1$, so x is odd, which as we have seen, means that it is not even: $2 \nmid x$.

13) Let $x \in \mathbb{Z}$. Show that if $7 \mid x^2 + 1$, then $13 \mid x^3 + 5x^2 + 17x - 100$.

[1] Moral: when you see a theorem for the first time, ask yourself whether the converse is true!

Solution: I claim that it is *never* the case that for an integer x, x^2+1 is divisible by 7, so the implication holds vacuously. To see this, it's enough to consider the possible remainders upon division by 7, so there are 7 cases in all.
Case 1: $x = 7k$. Then $x^2+1 = 49k^2+1 = 7(7k^2)+1$ is not divisible by 7.
Case 2: $x = 7k+1$. Then $x^2+1 = 49k^2+14k+1+1 = 7(7k^2+2k)+2$ is not divisible by 7.
Case 3: $x = 7k+2$. Then $x^2+1 = 49k^2+28k+4+1 = 7(7k^2+4k)+5$ is not divisible by 7.
Case 4: $x = 7k+3$. Then $x^2+1 = 49k^2+42k+9+1 = 7(7k^2+6k+1)+2$ is not divisible by 7.
Case 5: $x = 7k+4$. Then $x^2+1 = 49k^2+56k+16+1 = 7(7k^2+8k+2)+3$ is not divisible by 7.
Case 6: $x = 7k+5$. Then $x^2+1 = 49k^2+70k+25+1 = 7(7k^2+10k+3)+5$ is not divisible by 7.
Case 7: $x = 7k+6$. Then $x^2+1 = 49k^2+84k+36+1 = 7(7k^2+12k+5)+2$ is not divisible by 7.
Yes, this problem is a bit out of hand for an exam question. As a number theorist, I happen to know that for any prime number p of the form $4k+3$, then for all integers x, x^2+1 is *not* divisible by p. Probably I was trying to sneak in too much number theory here.

14) Suppose that $3x+2$ is odd. Show that $x^4+2x+2009$ is even.

Solution: If $3x+2$ is odd, then $3x$ is odd minus 2, or odd minus even, or odd. Moreover, if $3x$ is odd, then x must be odd (since if x were even, so would $3x$ be even). Then $x^4+2x+2009$ is:

$$(\text{odd}\cdot\text{odd}\cdot\text{odd}\cdot\text{odd})+(\text{even}\cdot\text{odd})+\text{odd} = \text{odd}+\text{even}+\text{odd} = (\text{odd}+\text{even})+\text{odd}$$
$$= \text{odd}+\text{odd} = \text{even}\,.$$