MATH 3200 MIDTERM EXAM 1, WITH SOLUTIONS

Instructions: This is a closed book exam with calculators not permitted. You will have 55 minutes to complete the exam. The extra credit problem is truly difficult, so that it would be a poor strategy to try to solve it unless you have completed, checked and rechecked your work on the other six problems.

1) [5 points] a) Define the symmetric difference $A \triangle B$ of two sets A and B.

Solution: $A \triangle B = (A \setminus B) \cup (B \setminus A)$. In words, it consists of all elements which are either in A or B but are not in both A and B.

b) [15 points] Show that $A \triangle B \subseteq A$ if and only if $B \subseteq A$.
(Suggestion: A Venn diagram alone will not be sufficient, but drawing one will probably help you organize your thoughts.)

Solution: First assume that $A \triangle B \subseteq A$. Then since $B \setminus A \subset A \triangle B$, we have $B \setminus A \subset A$. But this happens if and only if $B \setminus A$ is empty, i.e., if and only if $B \subseteq A$. The converse is almost exactly the same: if $B \subseteq A$ then

$$A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \setminus B) \cup \emptyset = (A \setminus B) \subseteq A.$$

2) [10 points] a) Define the following terms: conditional, converse, inverse, contrapositive, biconditional, converse biconditional, inverse biconditional.

Solution: The basic conditional is $A \implies B$. The converse is $B \implies A$. The inverse is $\sim A \implies \sim B$. The contrapositive is $\sim B \implies \sim A$. The biconditional is $A \iff B$, i.e., $A \implies B$ and $B \implies A$. The converse biconditional is $B \iff A$. The inverse biconditional is $\sim A \iff \sim B$.

b) [10 points] Which are logically equivalent to each other? (Proof not required.)

Solution: The conditional and the contrapositive are logically equivalent.
The inverse and the converse are logically equivalent.
The biconditional, converse biconditional and inverse biconditional are all logically equivalent. And there are no further logical equivalences.

3) [10 points] For $A = \{a\}$ and $C = \{a, c\}$, exhibit B such that $\mathcal{P}(A) \subsetneq B \subsetneq \mathcal{P}(C)$.[1]

Solution: Since $\mathcal{P}(A) = \{\emptyset, \{a\}\}$ and $\mathcal{P}(C) = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$, acceptable choices for a set strictly in between would be either of the following:

$$\{\emptyset, \{a\}, \{c\}\},$$
$$\{\emptyset, \{a\}, \{a, c\}\}.$$

[1] Recall $\mathcal{P}(A)$ means the power set of A and $\subsetneq$ means "a proper subset of."

4) Negate the following sentences (avoid use of "it is not the case that...").
a) [5 points] x is odd, y is odd and z is even.

Solution: Either x is even, y is even or z is odd.

b) [5 points] Every prime number is odd.

Solution: There exists an even prime number. (True: 2.)

c) [5 points] Either you pay the bill or we're not going out to dinner.

Solution: You don't pay the bill, and we're going out to dinner.

d) [5 points] If I'm lying, I'm dying.

Solution: I am lying and I am not dying.[2]

5) Prove the following results.
a) [10 points] Let $n \in \mathbb{Z}$. If n^5 is even, then n is even.

Solution: By contrapositive, assume n is odd. Then n^5 is a product of odd numbers, so is odd.

b) [10 points] If n is an odd integer, then $5n^9 + 11$ is even.

Solution: Directly using a parity argument: if n is odd, then n^9 is a product of odd numbers so is odd. Since 5 is odd, $5n^9$ is odd. so $5n^9 + 11$ is odd plus odd equals even.

c) [10 points] Let $\lfloor x \rfloor$ be the greatest integer less than or equal to x (e.g. $\lfloor \pi \rfloor = 3$). Suppose $x \in \mathbb{Z}$ is such that $\lfloor \frac{\cos(e^{2\pi x^2})}{x^4+2009} \rfloor$ is an even integer. Show $2x^2 + 3$ is odd.

Solution: This holds trivially: for any integer x, $2x^2$ is even, so $2x^2 + 3$ is:
even + odd equals odd.

6) [20 points] Let $n \in \mathbb{Z}$. Show that $3 \mid (2n^2 + 1)$ if and only if $3 \nmid n$.
(Hint: For at least some of the proof, it is a good idea to consider the three cases $n = 3k$, $n = 3k + 1$, $n = 3k + 2$.)

Solution: To show $\Longrightarrow$ we will use the contrapositive: suppose $3 \mid n$. Then $2n^2$ is divisible by 3, so $2n^2 + 1$ is not, being one more than a multiple of 3.

Now for $\Longleftarrow$, there are two cases to consider: $n = 3k + 1$ and $n = 3k + 2$. In the first case, $2n^2 + 1 = 2(3k + 1)^2 + 1 = 2 \cdot (9k^2 + 6k + 1) + 1 = 3 \cdot (6k^2 + 2k + 1)$ is divisible by 3. In the second case, $2n^2 + 1 = 2(3k+2)^2 + 1 = 2 \cdot (9k^2 + 12k + 4) + 1 = 3 \cdot (3k^2 + 8k + 3)$ is divisible by 3.

[2] This was the hardest question on the exam! Less than half of you got it right, including two students who otherwise answered everything perfectly.

Remark: A more elegant solution, using the same calculations, would be to observe that the statement is logically equivalent to: if $3 \mid n$ then $3 \nmid 2n^2 + 1$, and if $3 \nmid n$, then $3 \mid 2n^2 + 1$. These individual implications are handled as above. (Many students wrote something like this.)

Extra Credit: Use congruences modulo 3 to show that if $x, y, z \in \mathbb{Z}$ are such that $3x^2 + 3y^2 = z^2$, then x, y and z are all divisible by 3. Then explain why this implies that the only integers (x, y, z) for which $3x^2 + 3y^2 = z^2$ are $(x, y, z) = (0, 0, 0)$.

Solution: Since $z^2 = 3(x^2 + y^2)$, $3 \mid z^2$. This implies that $3 \mid z$ (an easy argument by contraposition, considering the two cases $z = 3k + 1$ and $z = 3k + 2$). So we may write $z = 3z'$ for some $z' \in \mathbb{Z}$, so

$$3x^2 + 3y^2 = (3z')^2 = 9(z')^2,$$

or

$$x^2 + y^2 = 3(z')^2.$$

Reducing this last equation modulo 3, we get

$$x^2 + y^2 \equiv 0 \pmod 3.$$

Since $0^2 \equiv 0 \pmod 3$, $1^2 \equiv 1 \pmod 3$, $2^2 = 4 \equiv 1 \pmod 3$, the possible values for a square modulo 3 are 0 and 1. So the only way we can add two squares and get something which is 0 mod 3 is if $x^2 \equiv y^2 \equiv 0 \pmod 3$, which, as above, implies $3 \mid x$ and $3 \mid y$. This shows the first part: x, y, z are all divisible by 3. Now, having shown this, we can divide (x, y, z) all by 3 to get another integer solution $(\frac{x}{3}, \frac{y}{3}, \frac{z}{3})$. But the above argument again applies to show that all of the coordinates are divisible by 3; in other words, each of x, y and z can all be divided by 3 as many times as we like and still remain integers. The only way this can be is if $x = y = z = 0$.