CLARK'S MATH 3200 THIRD MIDTERM EXAM, SPRING 2016

Directions: You have 60 minutes. Do all four problems. A fully correct answer must include correct reasoning / justification. No calculators are permitted.

1) Define: **inverse relation**, **equivalence relation**, **partial ordering**.

Solution: If $R \subset X \times Y$ is a relation, then

$$R^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in R\}.$$

An equivalence relation is a relation $R \subset X \times X$ which is:
- reflexive: $(x, x) \in R$ for all $x \in X$,
- symmetric: for all $x, y \in X$, $(x, y) \in R \implies (y, x) \in R$, and
- transitive: for all $x, y, z \in R$, $(x, y) \in R$ and $(y, z) \in R$ implies (x, z) in R.

A partial ordering is a relation $R \subset X \times X$ which is reflexive, transitive and
- anti-symmetric: for all $x, y \in X$, $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$.

2) For each of the following functions $f : \mathbb{R} \to \mathbb{R}$, determine whether f is injective and whether f is surjective.

$$f_1(x) = \prod_{i=1}^{2016} (x - i) = (x - 1)(x - 2) \cdots (x - 2016).$$

$$f_2(x) = \prod_{i=1}^{2017} (x - i) = (x - 1)(x - 2) \cdots (x - 2016)(x - 2017).$$

$$f_3(x) = 7^{8^{9^x}}.$$

$$f_4(x) = \arctan x.$$

(Suggestion: Think before you differentiate. In some cases there is an easier way...)

Solution:

$\mathbf{f_1}$: We have $f_1(1) = f_2(2) = 0$, so f_1 is not injective. Moreover, because f_1 is continuous and $\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = \infty$ (this is because f is a polynomial of even degree), f_1 has a **global minimum** m and thus its image is of the form $[m, \infty)$ for some real number m. It is not surjective. (Note that we are not saying what m is. For that we would need to exactly find all the roots of the derivative. Doing so *exactly* may be difficult or impossible.)

$\mathbf{f_2}$: Again we have $f_1(1) = f_2(1) = 0$, so f_2 is not injective. Because f_2 is continuous and we have $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to -\infty} f(x) = -\infty$ (this is because f is a polynomial of odd degree), by the Intermediate Value Theorem f is surjective.

$\mathbf{f_3}$: For any $a > 1$, the function a^x takes only positive values, so f_3 is not surjective. I claim that it is injective. For this we could just take the derivative directly, but that seems unnecessarily messy. I find it cleaner to observe first that for any $a > 1$, the function $g(x) = a^x$ has derivative $g'(x) = \ln a a^x$ which is always positive, so g is increasing. Thus f_3 is the composition of three increasing functions – it is $7^x \circ 8^x \circ 9^x$ – hence it is also increasing and thus injective.

$\mathbf{f_4}$: Because arctan is the inverse function of $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$, it is injective and has range $(-\pi/2, \pi/2)$, so is not surjective.

3) Let $f : X \to Y$ and $g : Y \to X$ be functions, and suppose $g \circ f = \mathbf{1}_X$: that is, for all $x \in X$, $g(f(x)) = x$.
a) Show: f is injective and g is surjective.[1]

Solution: The Green and Brown Fact is that if $g \circ f$ is injective, then f is injective, and if $g \circ f$ is surjective then g is surjective. Here we have $g \circ f = \mathbf{1}_X$, and the identity function is certainly both injective and surjective, so indeed we deduce that f is injective and g is surjective.

b) Show: f need not be surjective and g need not be injective.

Solution: We give a counterexample: Let $X = \{1\}$ and $Y = \{1, 2\}$, let $f : 1 \to 1$ and $g : 1 \to 1$, $2 \to 1$. Then $g \circ f : 1 \to 1$ so is the identity function on X. Nevertheless f is injective but not surjective – 2 is not in the image – and g is surjective but not injective: $g(1) = g(2)$.

c) Now suppose that g is injective. Show: $f \circ g = \mathbf{1}_Y$ (that is, for all $y \in Y$, $f(g(y)) = y$) and thus f and g are inverse functions.
(Hint: let $y \in Y$, and put $x = f(g(y))$. Apply g to both sides and show $x = y$.)

Solution: As advertised, let $y \in Y$, and put $x = f(g(y))$. We want to show that $x = y$. Apply g:

$$g(x) = g(f(g(y)) = g(y),$$

the last equality is because we know that $g \circ f = \mathbf{1}_X$. Since we have assumed that g is injective, we deduce $x = y$. Thus we have shown that f and g are compositional inverses, and as we know that means $g = f^{-1}$.

4) Let $\mathbb{R}^{\mathbb{R}}$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. I will give three relations on $\mathbb{R}^{\mathbb{R}}$. One of them is an equivalence relation, one is a partial ordering, and one is neither. Determine (with proof!) which is which.

$$R = \{(f, g) \in \mathbb{R}^{\mathbb{R}} \times \mathbb{R}^{\mathbb{R}} \mid f(2016) = g(2016)\}.$$
$$S = \{(f, g) \in \mathbb{R}^{\mathbb{R}} \times \mathbb{R}^{\mathbb{R}} \mid f(2016) \leq g(2016)\}.$$
$$T = \{(f, g) \in \mathbb{R}^{\mathbb{R}} \times \mathbb{R}^{\mathbb{R}} \mid f(x) \leq g(x) \ \forall x \in \mathbb{R}\}.$$

Solution: I claim R is an equivalence relation. For all $f, g, h : \mathbb{R} \to \mathbb{R}$, we have:
• $f(2016) = f(2016)$, giving reflexivity;
• if $f(2016) = g(2016)$, then $g(2016)$, giving symmetry;

[1] You may use the "Green and Brown Fact."

• if $f(2016) = g(2016)$ and $g(2016) = h(2016)$, then $f(2016) = h(2016)$, giving transitivity.

I claim S is neither symmetric nor anti-symmetric, hence neither an equivalence relation nor a partial ordering.
• If $f(2016) \leq g(2016)$, it certainly does not follow that $g(2016) \leq f(2016)$. For instance, let f be the constant function 0 and g be the constant function 1.
• If $f(2016) \leq g(2016)$ and $g(2016) = f(2016)$, what follows is that $f(2016) = g(2016)$, not that $f = g$ as functions. E.g. let $f(x) = x$ and let $g(x) = 2016$. Then $f(2016) = g(2016) = 2016$, but f and g are not equal at any other point.

I claim T is a partial ordering (and really is the most reasonable sense in which to construe $f \leq g$ for functions: it means that the graph of f lies entirely below or touching the graph of g). For all $f, g, h : \mathbb{R} \to \mathbb{R}$, we have:
• $f(x) \leq f(x)$ for all $x \in \mathbb{R}$, giving reflexivity.
• If we have $f(x) \leq g(x)$ and $g(x) \leq f(x)$ for all $x \in \mathbb{R}$, then $f(x) = g(x)$ for all $x \in \mathbb{R}$, i.e., $f = g$, giving anti-symmetry.
• If we have $f(x) \leq g(x)$ and $g(x) \leq h(x)$ for all $x \in \mathbb{R}$, then we have $f(x) \leq h(x)$ for all $x \in \mathbb{R}$, giving transitivity.

Comments on the provenance of the midterm problems:

All of the definitions from #1 appeared prominently in the lectures and in the online lecture notes. (Partial orderings were not discussed in the course text; the other two concepts were.)

Problems like #2 are featured prominently in the review materials and on previous midterms. A function very similar to f_3 appeared as problem 4b) on the spring 2009 midterm (with solutions). The functions f_1 and f_2 are similar to the function from problem 2c) on the fall 2009 midterm. In Thursday night's review session we mentioned that one can take the domain of the tangent function to be $(-\pi/2, \pi/2)$, in which case it is surjective and has an inverse function. The arctangent is that inverse function.

Problem #3 is, in its entirety, contained in Review Problem 5 (which also includes a fourth statement). We solved the first two parts in Thursday's review session.

Problem #4 has a large overlap with Problem 5 from the spring 2009 midterm (with solutions).