PROF. CLARK'S MATH 3200 SPRING 2016 MIDTERM 2

1) [20 points] Show: for all $n \geq 1$ we have

$$(1+1)+(1+3)+(1+5)+\ldots+(1+(2n-1)) = n^2+n.$$

First Solution: We go by induction on n.
Base Case ($n=1$): Indeed $(1+1) = 2 = 1^2+1$.
Induction Step: Let $n \in \mathbb{Z}^+$ and suppose that

$$(1+1)+(1+3)+(1+5)+\ldots+(1+(2n-1)) = n^2+n.$$

Then

$$(1+1)+(1+3)+\ldots(1+(2n-1))+(1+(2(n+1)-1))$$

$$\stackrel{\text{IH}}{=} n^2+n+2n+2 = n^2+3n+2 = (n+1)^2+(n+1).$$

Second Solution: Our first example of an induction proof was

$$1+\ldots+n = \frac{n(n+1)}{2}.$$

Using this result we can deduce the current one. Indeed

$$(1+1)+(1+3)+\ldots+(1+(2n-1)) = 2+4+\ldots+2n$$

$$= 2(1+\cdots+n) = 2\frac{n(n+1)}{2} = n(n+1) = n^2+n.$$

2) [20 points] Show: for all $n \geq 5$ we have

$$(n!)^2 > 5^n.$$

(**For millennials who need calculators to do any arithmetic whatsoever:** you may use $(5!)^2 = 120^2 = 14400$ and $5^5 = 3125$.)
Solution: We go by induction on n.
Base Case ($n=5$): We have $(5!)^2 = 14400 > 3125 = 5^5$.
Induction Step: Let $n \geq 5$ be an integer and suppose that

$$(n!)^2 > 5^n.$$

Multiplying the above inequality by the positive number $(n+1)^2$ we get

$$((n+1)!)^2 = (n+1)^2(n!)^2 > (n+1)^2 5^n.$$

Since $n \geq 5$, $(n+1)^2 \geq (5+1)^2 \geq 36 > 5$. Thus

$$((n+1)!)^2 > (n+1)^2\ 5^n > 5 \cdot 5^n = 5^{n+1}.$$

3) [40 points] Prove or disprove each of the following statements:
a) For all $x \in \mathbb{R}$, if x is rational then x^2 is rational.
b) For all $x \in \mathbb{R}$, if x is irrational then x^2 is irrational.
c) For all $x \in \mathbb{R}$, if x^2 is rational then x is rational.
d) For all $x \in \mathbb{R}$, if x^2 is irrational then x is irrational.

Solution:
a) **Proof**: if x is rational then $x = \frac{p}{q}$ with $p, q \in \mathbb{Z}$ and $q \neq 0$. Then

$$x^2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2},$$

which is a rational number.
b) **Disproof**: Take $x = \sqrt{2}$. Then x is irrational but $x^2 = \sqrt{2}^2 = 2$ is rational.
c) Observing that this statement is the contrapositive of part b) gives a **disproof**.
d) Observing that this statement is the contrapositive of part a) gives a **proof**.

4) [20 points] **Do any two of a), b), c). Clearly indicate which two parts you're doing. You may use the result from the part you do not attempt.**
a) Prove: for all $x, y \in \mathbb{Z}$, $x + y$ and $x - y$ have the same parity.
b) Prove: for $a \in \mathbb{Z}$, $a \equiv 2 \pmod 4 \iff (2 \mid a \text{ and } 4 \nmid a)$.
c) **Dis**prove: there are integers x and y such that $x^2 - y^2 \equiv 2 \pmod 4$.

Solution:
a) Two integers have the same parity iff their difference is even. Since $(x + y) - (x - y) = 2y$ is even, we're done.
b) We are asked to prove an equivalence, so we must separately show $\Rightarrow$ and $\Leftarrow$.

First suppose $a \in \mathbb{Z}$ and $a \equiv 2 \pmod 4$. Thus, $4 \mid a - 2$, so $a - 2 = 4k$ for some $k \in \mathbb{Z}$ and finally $a = 4k + 2$. Since $a = 2(2k + 1)$ and $2k + 1 \in \mathbb{Z}$, we have $2 \mid a$. If $4 \mid a = 4k + 2$ then $a = 4A$ and $4A = 4k + 2$, so $2 = 4(A - K)$ and $4 \mid 2$: contradiction. So $4 \nmid a$.

Now suppose $2 \mid a$ and $4 \nmid a$. By division with remainder, we know exactly one of the following occurs:
Case 1: $a = 4k$ for some $k \in \mathbb{Z}$.
Case 2: $a = 4k + 1$ for some $k \in \mathbb{Z}$.
Case 3: $a = 4k + 2$ for some $k \in \mathbb{Z}$.
Case 4: $a = 4k + 3$ for some $k \in \mathbb{Z}$.
Clearly Case 1 cannot occur: it directly contradicts our assumption $4 \nmid a$. In Case 2 we have $a = 2(2k) + 1$ is odd, so $2 \nmid a$. Similarly in Case 4 we have $a = 2(2k+1) + 1$ is odd so $2 \nmid a$. So we must be in Case : $a = 4k + 2$.[1]
c) We have $x^2 - y^2 = (x + y)(x - y)$. Moreover, by part a) we have that $x + y$ and $x - y$ have the same parity.
Case 1: Both $x + y$ and $x - y$ are odd. Hence so is $(x + y)(x - y) = x^2 - y^2$, so – by part b) – it is not congruent to 2 modulo 4.
Case 2: Both $x + y$ and $x - y$ are even: $x + y = 2A$, $x - y = 2B$ for $A, B \in \mathbb{Z}$. Then

$$(x + y)(x - y) = (2A)(2B) = 4AB$$

is divisible by 4, so – by part b) – is not congruent to 2 modulo 4.

Comment: A previous year's midterm exam included the question: show that there are no integers x and y such that $x^2 - y^2 = 2$. This is an immediate consequence of this problem....but I think it is much harder! The point of having three parts is to step you through the ideas of the proof.

[1] I wrote this up in great detail. You would have gotten full credit for writing less: e.g. I would assume that at this point everyone knows how to prove that $4k + 1$ and $4k + 3$ are odd.