PROF. CLARK'S MATH 3200 SPRING 2016 MIDTERM 1

No calculators are permitted. Please solve all the problems. At all times you should justify your work by correct reasoning. You have 60 minutes. Good luck.

1) [30 points] Let X, Y, A_1, A_2, B_1, B_2 be sets.
a) Show: $X \setminus Y = \varnothing \iff X \subseteq Y$.
b) Show: if $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$, then $A_1 \times B_1 \subseteq A_2 \times B_2$.
c) Let $X = \{1,2\} \cup [3,4]$ and $Y = [0,5]$. Sketch $X \times Y$ as a subset of $\mathbb{R}^2$.

Solution:
a) $X \setminus Y$ is the set of all objects x such that $x \in X$ and $x \notin Y$. So $X \setminus Y = \varnothing$ if and only if there is no object which lies in X and does not lie in Y. This holds if and only if every object which lies in X also lies in Y, which holds if and only if $X \subseteq Y$.

b) Let $x \in A_1 \times B_1$. Then $x = (a,b)$ with $a \in A_1$ and $b \in B_1$. Since $A_1 \subseteq A_2$ we have $a \in A_2$; similarly, since $B_1 \subseteq B_2$ we have $b \in B_2$. Thus $x = (a,b) \in B_1 \times B_2$.

c) At some point I hope to upload a picture, but for now a "prose sketch": the region consists of the vertical line segment from $(1,0)$ to $(1,5)$, the vertical line segment from $(2,0)$ to $(2,5)$, and the (shaded in) rectangle with vertices $(3,0)$, $(3,5)$, $(4,0)$ and $(4,5)$.

Comments: Part a) is an instance of proving both directions of a biconditional ($P \iff Q$) by giving a sequence of bi-implications: i.e., each statement holds if and only if the next holds. This is worth some style points, but though a solution showing $P \implies Q$ and $Q \implies P$ separately may take more space, it probably takes less time to write.

Is the converse to part b) true? (Lucky for you I didn't ask this: there's a trick!)

Honestly, a student who has mastered the material of Chapter 1 should be able to solve these problems with minimal effort. Neither of the proofs in parts a) or b) require any real ideas: you just unpack the definitions and follow your nose. Any points lost here are not a moral failing but indicate some misconceptions which you should seek to remedy ASAP. Please feel free to ask me about them.

2) [20 points] Show by means of truth tables that:
a) $(\neg P \implies \neg Q) \iff (Q \implies P)$ is a tautology.
(That is, show that it is true for all truth values of P and Q.)
b) $(\neg P \implies \neg Q) \iff (P \implies Q)$ is not a tautology.

Solution: The truth table is

P	Q	$P \implies Q$	$Q \implies P$	$\neg P \implies \neg Q$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Because the columns for $\neg P \implies \neg Q$ and $Q \implies P$ are identical, these two statements are logically equivalent. Because the columns for $\neg P \implies \neg Q$ and $P \implies Q$ are not identical, these two statements are not logically equivalent.

Comments: The vast majority of the students got full credit on this problem.

3) [20 points] Negate the following statements:
a) If x is even and y is even, then z is even.
b) If you come any closer, I'll scream.

Solution:
a) x is even and y is even and z is odd.
b) I accepted any of the following for full credit:
"You come closer and I won't scream."
"You come any closer and I won't scream."
"There exists a distance that you can come closer such that I won't scream."

Comments: This was the hardest question on the exam. I was disappointed with this, because it is similar to a problem from a previous year's midterm, which was posted on the course webpage with solutions, including a footnote saying "This was the hardest question on the exam."

The most important comment is that – as I tried to explore thoroughly in class and in the homework – the negation of an implication is not an implication! Namely, "$P \implies Q$" is logically equivalent to "P is false or Q is true," so its negation is "P is true and Q is false." Symbolically:

$$\neg(P \implies Q) \iff P \wedge \neg Q.$$

Maybe the following helps to nail this down: the conditional, the converse, the inverse and the contrapositve are all true "3/4 of the time": that is, three of the rows in their truth table have T and only one has F. Therefore the negation of an implication must be a statement which is true "1/4 of the time." So it can't be an implication! In general this "probabilistic approach" can help you see quickly that certain statements are *not* logically equivalent.

As for many natural language statements, there are some subtleties in the meaning of "If you come any closer, I'll scream." I think the most faithful interpretation of this – after all, familiar – statement construes the hypothesis as a binary action: just as soon as the speakee moves in the speaker's direction, the speaker will scream. (And of course, even if the speakee does not come any closer, there are surely other actions that would cause the speaker to scream: it is not a biconditional.) So the negation is that the speakee moves closer to the speaker and the speaker does not scream: the point is that this is precisely the conditions which make the speaker's original statement false. However, a few students interpeted "any" in a more refined way, namely as a universal quantifier. Thus they construed the original statement to be: "For all positive values of x, if you move x units closer to me then I will scream." In this case the negation is "There exists a positive value of x such that you move x units closer and I will not scream"; rewriting this in more normal English we get the third answer above. I think this is probably not what the speaker meant, but it is what a very careful, literal interpreation of the "any" in the statement would give. So it is technically correct: the best kind of correct (when it comes to questions on math exams).

One student contacted me to take issue with this question: she said that both parts test negating $P \implies Q$, so they test the same thing, and a student who misses this one thing will miss 20 points. My response to that is: first, they are not testing *exactly* the same thing, and the proof is that although the majority of students got – disappointingly! – 0 out of 20 points on this question, there were some students who got one part completely correct and another part wrong (and also a positive, though very small, number of students who got both parts completely correct). The second statement involves natural language, which always introduces complications, but since mathematics is usually written in carefully worded natural language rather than logical symbols, it is important to be able to do the (more difficult) problem b). Now I should say that I arrived at this problem by modifying a midterm problem from Fall 2009, which asked for the negation of:

"x is odd, and if y is even then z is odd."

I didn't want to include an identical problem from a past exam...so I tweaked it. In retrospect I think that this older problem is better, because the negation also involves use of DeMorgan's Laws, which did not get tested on this midterm. On the other hand this older problem is harder – try it!

Finally, I want to underscore that correctly negating statements is very important, and the logic of negations is very *very* important, so indeed it is not out of place to lose 20 out of 110 points for failing to negate an implication. Among other things, the negation of an implication is the entire logical basis of **proof by contradiction**, the next topic in the course. But more basically, making errors in basic logic is a bit like never learning the difference between sugar and salt: it doesn't matter what else you do right, because what you're trying to make will probably be ruined. So please take this opportunity to understand your mistake!

4) [30 points]
a) Explain what "$\forall x \in S$, $P(x) \implies Q(x)$ is vacuously true" means.
b) Explain what "$\forall x \in S$, $P(x) \implies Q(x)$ is trivially true" means.
c) Show: for all $x \in \mathbb{Z}$, if $9 \mid x^2+1$ then $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^3}{6}$.

Solution:
a) "Vacuously true" means that $P(x)$ is false for all $x \in S$, and thus the implication holds.
b) "Trivially true" means that $Q(x)$ is true for all $x \in S$, and thus the implication holds.
c) I claim that the implication is vacuously true: thus for all $x \in \mathbb{Z}$, $x^2 + 1$ is *not* divisible by 9. In fact I claim it is not divisible by 3, which is a stronger statement. To see this, we use the fact (coming from the Division Theorem) that for every $x \in \mathbb{Z}$ exactly one of the following holds: $x = 3k$ for some $k \in \mathbb{Z}$, $x = 3k+1$ for some $k \in \mathbb{Z}$ or $x = 3k+2$ for some $k \in \mathbb{Z}$.
Case 1: If $x = 3k$. Then

$$x^2 + 1 = (3k)^2 + 1 = 9k^2 + 1 = 3(3k^2) + 1$$

isn't divisible by 3.
Case 2: If $x = 3k+1$. Then

$$x^2 + 1 = (3k+1)^2 + 1 = 9k^2 + 6k + 1 + 1 = 3(3k^2 + 2k) + 2$$

isn't divisible by 3.
Case 3: If $x = 3k + 2$. Then

$$x^2 + 1 = (3k + 2)^2 + 1 = 9k^2 + 12k + 4 + 1 = 3(3k^2 + 4k + 1) + 2$$

isn't divisible by 3.

Comments: By the way, it is *not true* that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^3}{6}$. In fact $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. But this statement was chosen to (i) rather obviously have nothing to do with the divisibility properties of $x^2 + 1$ and (ii) be scary enough to want to ignore – thus the only reasonable way to proceed is to try to show that the hypothesis is never satisfied. A few people seemed to write down some link between $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^3}{6}$ and divisibility properties: I was very confused by this. Note that the variable x does not even appear in the second statement (and the n is a "dummy variable" which just tells us to sum the thing inside; it cannot be quanitifed over), so how can there be any logical relationship between the hypothesis and the conclusion?

That $\forall x \in \mathbb{Z},\ 3 \nmid x^2 + 1$ came up both in the homework and in class, and I was hoping that this would be familiar. It seems that there was enough else going on that many people did not make this connection. A few people split up the algebra into 9 cases instead of 3: correct but way more work than was intended. Also some people split up into two cases according to whether x is even or odd. That doesn't work to prove something about divisiblity by 3^2. I hope that more experience – and more work with congruences – will make this clear.

5) [10 points] Show: for all $x \in \mathbb{Z}$, $8 \mid (x + 1)(x^2 + 2)(x^3 + 3)(x^4 + 4)(x^5 + 5)(x^6 + 6)$.

Solution:
First observe that for all $n \in \mathbb{Z}^+$, $x^n = x \cdots x$ (n times) is even if and only if x is even.
Case 1: x is even. Then $x + 1$ is odd, $x^2 + 2$ is even + even = even, $x^3 + 3$ is even + odd = odd, $x^4 + 4$ is even + even = even, $x^5 + 5$ is even + odd = odd and $x^6 + 6$ is even + even = even. Of those six factors, three of them were even. Multiplying together three even integers and three odd integers we get

$$(2a)(2b)(2c)(2d + 1)(2e + 1)(2f + 1) = 8\,(abc(2d + 1)(2e + 1)(2f + 1))$$

is divisible by 8.
Case 2: x is odd. This proceeds the same as above except now $x + 1$, $x^3 + 3$ and $x^5 + 5$ are even whereas $x^2 + 2$, $x^4 + 4$ and $x^6 + 6$ are odd. Still we have a product of three even integers and three odd integers, so it is divisible by 8.

Comment: You had a whole chapter's worth of parity problems, and I felt I had to give you a parity problem different from all of these. This one unfortunately seems a bit too hard given the time allotted. (Though some people did solve it, and more had the right idea but not enough time to write out all the details.) Though it was worth a small enough number of points so as not to affect the scores too much, I think its difficulty somewhat compromised its diagnostic purpose: if you couldn't solve this parity problem in ten minutes or less, maybe you could do a more moderate problem given a more reasonable amount of time. This is the one problem on the exam that I feel that a student could miss and still have a full mastery of the course material so far.