CLARK'S FALL 2016 MATH 3100 MIDTERM III

Directions: Unless otherwise stated, answers must be accompanied by correct calculations and/or reasoning in order to get full credit. Calculators are not permitted. You have 60 minutes. Each part of each problem is worth 10 points. Good luck!

O.
a) Write your name on the exam sheet. When you turn in your exam, arrive with the exam sheet on top and the other sheets below.

Solution: Okay.

I.
a) Let $D \subseteq \mathbb{R}$, let $\{f_n : D \to \mathbb{R}\}_{n=1}^{\infty}$ be a sequence of functions, and let $f : D \to \mathbb{R}$. Define "f_n converges to f pointwise on D" and "f_n converges to f uniformly on D."

Solution: f_n converges to f pointwise on D if for all $x \in D$ and for all $\epsilon > 0$, there is $N \in \mathbb{N}$ suh that for all $n > N$, we have $|f_n(x)-f(x)| < \epsilon$.

f_n converges to f uniformly on D if for all $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for all $x \in D$ and all $n > N$, we have $|f_n(x) - f(x)| < \epsilon$. Alternately, f_n converges uniformly to f on D if for all $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$, we have

$$\sup_{x \in D} |f_n(x) - f(x)| < \epsilon.$$

b) Let I be an interval, a an interior point of I, and $f : I \to \mathbb{R}$ a smooth function. Define the Taylor series $T(x)$ centered at a. For $N \in \mathbb{N}$, define the Nth Taylor polynomial $T_N(x)$ centered at a and the remainder function $R_N(x)$.

Solution: We have

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$
$$R_N(x) = f(x) - T_N(x).$$

c) With notation as in part b), show: for $x \in I$, $\lim_{N\to\infty} R_N(x) = 0$ holds if and only if $f(x) = T(x)$.

Solution: We have $\lim_{N\to\infty} R_N(x) = 0$ if and only if

$$0 = \lim_{N\to\infty} R_N(x) = \lim_{N\to\infty} f - T_N(x) = \lim_{N\to\infty} f(x) - \lim_{N\to\infty} T_N(x) = f(x) - T(x),$$

i.e., if and only if $f(x) = T(x)$.

II.

a) Evaluate the sum $\sum_{n=0}^{\infty} \frac{1}{10^{n-6} n!}$.

Solution: We have

$$\sum_{n=0}^{\infty} \frac{1}{10^{n-6} n!} = \frac{1}{10^{-6}} \sum_{n=0}^{\infty} \frac{(1/10)^n}{n!} = 10^6 e^{1/10}.$$

b) (In memory of Leonard Cohen.)
Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-1975)^n}{41^n n^7}$.

Solution: The power series is centered at $a = 1975$. To find the radius of convergence we use the Ratio Test: we have

$$\rho = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{41^n n^7}{41^{n+1}(n+1)^7}$$
$$= \frac{1}{41} \lim_{n\to\infty} \frac{(n+1)^7}{n^7} = \frac{1}{41}.$$

We test the endpoints: if $x = 1975 + 41$, then we get

$$\sum_{n=1}^{\infty} \frac{(1975+41-1975)^n}{41^n n^7} = \sum_{n=1}^{\infty} \frac{1}{n^7} < \infty :$$

this is a convergent p-series. In fact it is absolutely convergent, so when we plug in the other endpoint we'll get $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^7}$ which is still convergent. So the interval of convergence is

$$[1975 - 41, 1975 + 41] = [1934, 2016.]$$

Again, this is in memory of Leonard Norman Cohen, CC GOQ (September 21, 1934 – November 7, 2016).

c) Dis/prove: there is a power series with domain of convergence $[0, \infty)$.

Solution: The domain of convergence of a power series $\sum_n a_n(x-a)^n$ is an interval centered at a, with radius of convergence $R \in [0, \infty]$. So if the domain is unbounded we have $R = \infty$ and the domain is $(-\infty, \infty) = \mathbb{R}$. So the domain cannot be $[0, \infty)$.

III.
For $n \in \mathbb{Z}^+$, let $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \frac{x^2}{x^2+n^3}$.
a) Show: f_n converges pointwise to 0 on $\mathbb{R}$.

Solution: If $x = 0$ then $f_n(x) = 0$ for all n, so certainly this converges to 0. Otherwise, for fixed x, the numerator is the constant function (of n) x^2 and the denominator is the degree 3 polynomial function (of n) $n^3 + x^2$. So we have a rational function of n in which the degree of the denominator exceeds the degree of the numerator, and we know that the limit as $n \to \infty$ is 0.

b) Show: f_n converges uniformly to 0 on $[0, 1000]$.

First Solution: For all $x \in [0, 1000]$, we have

$$\left|\frac{x^2}{x^2+n^3}\right| = \frac{x^2}{x^2+n^3} \leq \frac{1000^2}{x^3+n^3} \leq \frac{1000^2}{n^3}.$$

So $||f_n|| \leq \frac{10^6}{n^3} \to 0$ as $n \to \infty$ and $f_n \xrightarrow{u} 0$ on $[0, 1000]$.

Second Solution: There is a theorem of Dini (mentioned only in the review materials to the course; for a proof, see e.g. `http://math.uga.edu/~pete/3100supp.pdf`, pages 108–110) that says: if $f_n : [a, b] \to \mathbb{R}$ is a sequence of continuous functions converging pointwise to a continuous f and is a **decreasing sequence of functions**[1] in the sense that for all $x \in [a, b]$, the sequence $\{f_n(x)\}$ is decreasing, then f_n converges uniformly to f on $[a, b]$. Since for all $x \in [0, \infty)$ and $n \in \mathbb{Z}^+$ we have

$$\frac{x^2}{x^2+n^3} \geq \frac{x^2}{x^2+(n+1)^3},$$

then $\{f_n : [0, 1000] \to \mathbb{R}\}$ is a decreasing sequence of functions converging uniformly to 0.

[1]Beware: "decreasing sequence of functions" is not the same as "sequence of decreasing functions"; indeed, as we will see shortly, each f_n is increasing!

Interregnum: We claim that each $f_n : [0, \infty)$ is increasing (and non-negative). If so, that makes the computation of $||f_n||$ easy: we have

$$||f_n|| = \sup_{x \in [0,1000]} |f_n(x)| = f_n(1000) = \frac{1000^2}{1000^2 + n^3}.$$

Then $||f_n|| \to 0$ – the numerator is a constant and the denominator approaches infinity – so it will follow that $f_n \stackrel{u}{\to} f$ on $[0, 1000]$. The next two solutions give two ways of establishing that each f_n is increasing.

Third Solution: Using the quotient rule we find that

$$f_n'(x) = \frac{2n^3 x}{(n^3 + x^2)^2}.$$

This is non-negative for all $n \in \mathbb{Z}^+$ and all $x \in [0, \infty)$, so "by calculus" $f_n : [0, \infty) \to \mathbb{R}$ is increasing. Using the Interregnum, this gives a third solution. (Note that we didn't have to figure out in advance that we suspect f_n is increasing in order to do this. For any non-negative differentiable function, we can try to compute $||f_n||$ using calculus.)

Fourth Solution: Doing a little algebra, we get

$$f_n(x) \frac{1}{1 + \frac{n^3}{x^2}}.$$

From this we can stare at $f_n(x)$ to see that it is increasing on $[0, \infty)$: as x increases, so does x^2, so $\frac{n^3}{x^2}$ decreases, so the denominator gets smaller, so the entire fraction gets bigger. Using the Interregnum, this gives a fourth solution.

c) Show: f_n does not converge uniformly on $[0, \infty)$.
(Hint: For each $n \in \mathbb{Z}^+$, what is $\lim_{x \to \infty} f_n(x)$?)

Solution: Following the hint, we see that for each $n \in \mathbb{Z}^+$,

$$\lim_{x \to \infty} \frac{x^2}{x^2 + n^3} = 1.$$

Indeed, this time n is the constant and x is the variable, so the leading terms of both the numerator and denominator are x^2. Since $n^3 > 0$ for all $n \in \mathbb{Z}^+$, certainly $x^2 < x^2 + n^3$ so $0 \leq f_n(x) < 1$. From this we deduce that

$$\forall n \in \mathbb{Z}^+, ||f_n|| = \sup_{x \in [0, \infty)} |f_n(x)| = 1$$

for all n: if the limit at ∞ is 1, that means the function takes values arbitrarily close to (but smaller than, here) 1. Thus $||f_n||$ does not converge to 0, so the convergence to 0 is not uniform. And since $f_n \to 0$ pointwise, if it converges uniformly to anything, it must be the zero function. Thus $\{f_n\}$ is not uniformly convergent on $[0, \infty)$.

Alternate Solution: For $n \in \mathbb{Z}^+$, observe that

$$f_n(n^{3/2}) = \frac{(n^{3/2})^2}{(n^{3/2})^2 + n^3} = \frac{n^3}{n^3 + n^3} = \frac{1}{2},$$

so certainly $||f_n|| \geq \frac{1}{2}$ for all n. As before (and as usual!), this means that f_n does not converge uniformly.

Remark: The second solution is shorter and looks very clever. As for many things in mathematics that look very clever, much of the cleverness is hidden in the lack of explanation of what is being done. Once you look at the first solution you see that no matter what n is, there is some value x_n such that $f_n(x_n)$ is at least $1 - \delta$, for any $\delta > 0$ we want. An easy value of δ is $\delta = \frac{1}{2}$, and if you solve the equation $f_n(x_n) = \frac{x_n^2}{x_n^2+n^3} = \frac{1}{2}$ you get $x_n = n^{3/2}$.