CLARK'S FALL 2016 MATH 3100 MIDTERM IIA

Directions: All series are real series. Calculators are not permitted. You have 60 minutes. Good luck!

I. [30 points]
a) State what it means for a series $\sum_n a_n$ to converge absolutely and what it means to converge nonabsolutely.

Solution: The series $\sum_n a_n$ convergees absolutely if $\sum_n |a_n| < \infty$. The series $\sum_n a_n$ converges nonabsolutely if it converges and is not absolutely convergent.

Comments:
a) Some people included in their answer that absolute convergence implies convergence. I took off a point or two for that: I am asking for the *definition* of absolute convergence. That absolute convergence implies definition is *not* what absolute convergence *means*, it is an important fact about it.
b) Above we wrote $\sum_n |a_n| < \infty$ to mean that a series with non-negative terms converges. We should not write $\sum_n a_n < \infty$ when the terms are both positive and negative: in that case, O'Connor no longer implies and we don't know that the sum is either a real number or the series diverges to ∞.

b) Prove or disprove: every convergent series is absolutely convergent.

Solution: Certainly not: equivalently, there are series which converge nonabsolutely. Note though to get a disproof we need to exhibit an actual series which converges nonabsolutely. Well, in fact infinitely many of these appear later in the exam, but for illustrative purposes it is good to exhibit the alternating harmonic series $\sum_n \frac{(-1)^n}{n}$. The absolute series is the harmonic series which diverges by Oresme / Condensation / Integral Test. The Alternating Series Test applies to show convergence.

c) Let $\{a_n\}$ be increasing with $a_1 \geq 0$.
Prove or disprove: $\sum_n a_n$ diverges.

Solution: Disprove: the hypotheses allow $a_n = 0$ for all $n \in \mathbb{Z}^+$, in which case the series is $\sum_n 0 = 0$.

Although this is technically correct ("the best kind of correct"), it feels cheap not to add that in fact this is the only case in which the series can converge. Otherwise we have $a_N > 0$ for some $N \in \mathbb{Z}^+$, and since the sequence

$\{a_n\}$ is increasing, we have $a_n \geq a_N$ for all $n \geq N$, and thus

$$\lim_{n\to\infty} a_n \in [a_N, \infty],$$

so $a_n \not\to 0$ and we have divergence by the Nth Term Test.

Comment: A lot of people wrote that a_n is increasing hence unbounded above and diverges to ∞. That's certainly not true: of course it could just as well be bounded above. I was confused by that response. I wonder if you were thinking instead of the sequence of partial sums $S_n = a_1 + \ldots + a_n$? If we assume that $a_1 > 0$, then

$$S_n = a_1 + \ldots + a_n \geq a_1 + a_1 + \ldots + a_1 = n(a_1),$$

so indeed S_n is unbounded above. (Similarly if $a_N > 0$ for some $N \in \mathbb{Z}^+$.)

II. [20 points]

a) For which $x \in \mathbb{R}$ does the following series converge: $\sum_{n=0}^{\infty} \left(\frac{x^2}{x^2+1}\right)^n$?

Solution: The series is geometric with geometric ratio $r = \frac{x^2}{x^2+1}$. For all $x \in \mathbb{R}$ we have $x^2 < x^2+1$, so $0 \leq r < 1$. Thus $|r| < 1$ and $\sum_n \left(\frac{x^2}{x^2+1}\right)^n < \infty$.

b) For each $x \in \mathbb{R}$ for which the series converges, evaluate the sum.

Solution: For any r with $|r| < 1$ we know that $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. Thus

$$\sum_{n=0}^{\infty} \left(\frac{x^2}{x^2+1}\right)^n = \frac{1}{1 - \frac{x^2}{x^2+1}} = x^2 + 1.$$

Comment: I certainly accepted $\frac{1}{1-\frac{x^2}{x^2+1}}$ for full credit: in fact, I didn't even do the algebra myself but learned that the answer simplifies from a student exam. In general you do not need to algebraically simplify your final answer unless specifically directed to (e.g. if you were asked to show that the series converges to $x^2 + 1$).

Comment: Some people seem a bit shaky on how the formulas for the sum of a geometric series change depending upon where you start the sum. In my view the most basic formula to remember is the one given above: for $|r| < 1$ we have

$$\sum_{n=0}^{infty} r^n = \frac{1}{1-r}.$$

From this we easily (so easy that we need not remember the formula, just how to get it) the formula for $\sum_{n=N}^{\infty} r^n$ for any $N \in \mathbb{Z}^+$: we have

$$\sum_{n=N}^{\infty} r^N = r^N + r^{N+1} + \ldots = r^N \left(1 + r + r^2 + \ldots\right) = \frac{r^N}{1-r}.$$

III. [20 points] Let $q \geq 0$, and consider the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^q}.$$

a) For which q does the series converge absolutely?

Solution: The absolute series is $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^q}$. This is one of a class of series we studied in class and appear in the text. The key here is to use either the Condensation Test or the Integral Test. I will do it both ways here.

Condensation: The functions $n \mapsto n$, $n \mapsto n^q$ and $n \mapsto \log n$ are all famously increasing: for $n \mapsto n^q$ we use $q \geq 0$ here. Products and compositions of increasing functions are increasing, so $n \mapsto n(\log n)^q$ is increasing, and thus $n \mapsto \frac{1}{n(\log n)^q}$ is decreasing and positive. (The point here is that it is really rather *obvious* that this function is decreasing: we don't need to make a big deal about it.) So the Condensation Test says that our given series converges iff the following one converges:

$$\sum_n 2^n \frac{1}{2^n(\log 2^n)^q} = \sum_n \frac{1}{(n \log 2)^q} = \frac{1}{(\log 2)^q} \sum_n \frac{1}{n^q}.$$

So we got a q-series (times a nonzero constant, which does not affect convergence, of course), and we know – by Condensation! – that $\sum_n \frac{1}{n^q} < \infty$ if and only if $q > 1$. So our given series converges absolutely iff $q > 1$.

Integral Test: As above, the function $f : [2, \infty) \to (0, \infty)$ given by $f(x) = \frac{1}{x(\log x)^q}$ is decreasing, so by the Integral Test we have

$$\sum_n \frac{1}{n(\log n)^q} < \infty \iff \int_2^{\infty} \frac{dx}{x(\log x)^q} < \infty.$$

Taking $u = \log x$ we get

$$\int_2^{\infty} \frac{dx}{x(\log x)^q} = \int_{\log 2}^{\infty} \frac{du}{u^q}.$$

Here we will take it as already done that this integral converges iff $q > 1$: we have done this integral before when applying the Integral Test to the p-series. (Briefly, we just use the power rule when $q \neq 1$; when $q = 1$ the antiderivative is $\log x$, and $\lim_{x \to \infty} \log x = \infty$, a fact we used already in making the change of variables.) So the series converges iff $q > 1$.

b) For which q does the series converge nonabsolutely?

Solution: Again we know $n \mapsto \frac{1}{n(\log n)^q}$ is positive and decreasing. Since

$$0 \leq \frac{1}{n(\log n)^q} \leq \frac{1}{n}$$

and $\frac{1}{n} \to 0$, we have $\frac{1}{n(\log q)^n} \to 0$, so the Alternating Series Test applies to give convergence for all $q \geq 0$. Together with part a) we get nonabsolute convergence iff $q \in [0,1]$.

c) For which q does the series diverge?

Solution: After having done parts a) and b), the answer must be whatever values of $q \geq 0$ are left over. (That is why this problem was worth 20 points rather than 30 points, by the way.) In this case no values of $q \geq 0$ are left over, so for no $q \geq 0$ does the series diverge.

Comment: It is natural to wonder what happens when $q < 0$ and even to suspect that the series diverges: after all, that is what happens for $\sum_n \frac{1}{n^p}$ for $p < 0$. (Unfortunately, a few people didn't get that the "$q \geq 0$" was meant to apply to the entire problem. In retrospect I wish I had repeated "for $q \geq 0$" in each part.) Well, what happens for $q < 0$? We may as well take $Q = -q > 0$ and observe that

$$\sum_n \frac{1}{n(\log n)^q} = \sum_n \frac{(\log n)^Q}{n}.$$

In other words, when q is negative we really have a power of log in the numerator rather than the denominator. For all $n \geq 3$ we have $\frac{(\log n)^Q}{n} \geq \frac{1}{n}$, so the absolute series diverges for all $q < 0$ by comparison to the harmonic series. As for nonabsolute convergence: I claim that for each fixed $Q > 0$, the sequence $\frac{(\log n)^Q}{n}$ is *eventually decreasing* and converges to 0. If so, after throwing away finitely many terms – which, as ever, does not disturb convergence – the Alternating Series Test applies to show convergence. The difference here is that it's no longer obvious: we really have to do some calculus to show these things. First let us show that

$$f(x) = \frac{(\log x)^Q}{x}$$

is eventually decreasing. Indeed, we have

$$f'(x) = \frac{1}{x^2} \cdot \left(xQ(\log x)^{Q-1}\frac{1}{x} - (\log x)^Q\right) = \frac{(\log x)^{Q-1}}{x^2}\left(Q - \log x\right),$$

which is negative if and only if $\log x > Q$, i.e., if and only if $x > e^Q$. So f is eventually decreasing. Making the change of variables $y = \log x$, we get

$$\lim_{x \to \infty} \frac{(\log x)^Q}{x} = \lim_{y \to \infty} \frac{y^Q}{e^y}.$$

Exponential functions grow faster than power functions, as one can confirm using l'Hôpital's Rule again, so the limit is 0. (Alternately, $\lim_{x\to\infty} \frac{(\log x)^Q}{x} = \frac{\infty}{\infty}$, and one can apply L'H'ôpital's Rule repeatedly until the exponent Q becomes less than or equal to 0, at which point one sees that the limit is 0.)

Of course, the case $q < 0$ is much trickier! You're welcome that I excluded it, and I'm sorry if you didn't realize that I excluded it.

IV. [30 points] Classify each of the following series as absolutely convergent, nonabsolutely convergent or divergent.
a) $\sum_n (-1)^n \frac{n^{10}}{n!}$.

Solution: The ratio test limt is

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^{10}}{(n+1)!} \cdot \frac{n!}{n^{10}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{10} \frac{1}{n+1} = 1 \cdot 0 = 0.$$

Since $0 < 1$, the series is absolutely convergent.
b) $\sum_n (-1)^n \left(\frac{n^2+n+1}{5n^2-1}\right)^n$.

Solution: The root test limit is

$$\theta = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n^2+n+1}{5n^2-1}\right) = 1/5.$$

Since $\frac{1}{5} < 1$, the series is absolutely convergent.

c) $\sum_n \frac{n^n}{n!}$.

First Solution: The presence of $n!$ suggests applying the ratio test. This will succeed, after some work: we have

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)(n+1)^n n!}{(n+1)n!n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n.$$

This last limit is, famously, e. The evaluation is a little involved: set

$$L = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x.$$

Then

$$\log L = \lim_{x\to\infty} x \log(1 + \frac{1}{x}) = \lim_{x\to\infty} \frac{\log(1 + \frac{1}{x}}{\frac{1}{x}}.$$

Put $y = \frac{1}{x}$. Then using L'Hôpital's Rule, we get

$$\log L = \lim_{y\to 0} \frac{\log(1+y)}{y} = \frac{0}{0} = \lim_{y\to 0} \frac{1}{1+y} = 1.$$

So $L = e^1 = e$. Since $e > 1$, the series diverges.

Second Solution: In this case it is much easier to *look at the nth term*:

$$\frac{n^n}{n!} = \left(\frac{n}{n}\right)\left(\frac{n}{n-1}\right)\left(\frac{n}{n-2}\right)\cdots\left(\frac{n}{2}\right)\left(\frac{n}{1}\right) \geq 1 \cdot 1 \cdot 1 \cdots 1 \cdot 1 = 1.$$

So $\frac{n^n}{n!} \not\to 0$, and the series diverges by the nth term test.

Comment: It is important to remember that whenever the ratio or root test limit is applied to show divergence of a series, this divergence comes by showing that the Nth term of the series does not approach zero. Sometimes it is easier to show that directly than by considering ratios or roots.