In-Depth Example for Week 14

A Function which Does Not Equal its Taylor Series

For any function $f$ which is infinitely differentiable at 0, we can create its Taylor series

$$P_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

This series may not converge on the entire domain of $f$: it definitely converges at 0, and in general its domain of convergence is an interval centered at 0. However, an important question is: what does the series converge to? Does it necessarily converge to $f(x)$?

To determine this, we introduced the concept of Taylor remainders in class: $R_n(x) = f(x) - P^n_f(x)$, so that $\lim R_n(x) = f(x) - P^n_f(x)$. Thus, $f$ equals $P^n_f$ iff $R_n$ has limit 0. Frequently, we can use Taylor’s theorem to estimate $|R_n(x)|$ and get a bound approaching 0, so in many cases, $f$ does equal its Taylor series.

A function that equals its Taylor series is called analytic. Many famous functions are analytic, such as $e^x$, $\sin x$, $\cos x$, and $1/(1-x)$. Our job for this handout is to show that non-analytic functions exist; these are functions for which $P^n_f \neq f$ on the domain of convergence. In fact, since the Taylor series is the only possible power series a function can have (due to our uniqueness theorems), non-analytic functions do not have power series representations!

In this document, we will prove this theorem, giving us a non-analytic function:

**Theorem.** The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is infinitely differentiable at 0, and $f^{(n)}(0) = 0$ for every $n \geq 0$. Thus, $P^n_f(x) = \sum 0x^n = 0$, and $f(x) \neq P^n_f(x)$ for any $x \neq 0$ (even though $P^n_f(x)$ converges everywhere).

Roughly speaking, this function is extremely flat near $x = 0$; try drawing this with a calculator and zooming in really far at the origin. Now, normally $x^n$ is fairly flat-looking near $x = 0$, with larger $n$ values causing a slower rise in the function and hence a flatter shape. In effect, our function $f(x)$ is so flat that it’s flatter than all the $x^n$ functions!

**Brainstorming:**

For this function, I cannot find its Taylor series by reusing the known series of $e^x$ and composing with an inner function $g(x) = -1/x^2$. The problem is that $g(x) = -1/x^2$ is not a continuous function at 0, and it certainly does not satisfy $g(0) = 0$. I have no choice but to use the definition of Taylor series by computing every derivative.

Three issues make this function particularly tricky to use when computing the derivatives:

1. The piecewise nature of the definition means that it’s not even clear if the function is continuous! Before we take any derivatives at all, we should know whether $f$ is continuous at 0. If not, there’s no point in differentiating!
2. Due to the piecewise definition, each derivative will also have to be described piecewise! There will be a formula we can use for \( f^{(n)}(x) \) when \( x \neq 0 \), and a separate result for \( f^{(n)}(0) \). This brings up a concern: even if \( \lim_{x \to 0} f^{(n)}(x) \) exists, that does not necessarily tell us that limit is \( f^{(n)}(0) \)! In other words, we shouldn’t assume that \( f^{(n)} \) is continuous at 0 either.\(^1\) As a result, we’re going to have to use the definition of derivative (using difference quotients) to figure out the derivatives at 0.

3. The derivatives don’t follow a nice pattern: the first derivative uses the Chain Rule, and the second will use both the Chain and Product Rules. An exact formula for \( f^{(n)}(x) \) is incredibly complicated. Instead, we’ll have to figure out some rough idea of what form the derivative takes, show that this form gets us the limits as \( x \to 0 \) that we need, and we’ll prove by induction that this form is correct.

Concerning this third point, let’s get a sense of what this form might be! Here are the first few derivatives of \( f \) at nonzero values:

\[
\begin{align*}
f'(x) &= \frac{2}{x^3}e^{-1/x^2} & f''(x) &= \frac{4}{x^6}e^{-1/x^2} - \frac{6}{x^4}e^{-1/x^2} & f'''(x) &= \frac{8}{x^9}e^{-1/x^2} - \frac{36}{x^7}e^{-1/x^2} + \frac{24}{x^5}e^{-1/x^2}
\end{align*}
\]

From the looks of this, we can notice one important general trend: we get a few terms which each have the form \( c/x^p \cdot e^{-1/x^2} \) for some constants \( c \) and \( p \). Being more specific than this gets tricky, and it turns out this simple remark will suffice. In fact, if you factor out \( e^{-1/x^2} \), you get expressions like

\[
f''''(x) = e^{-1/x^2} \left( 8(x^{-1})^9 - 36(x^{-1})^7 + 24(x^{-1})^5 \right)
\]

The expression in parentheses is sometimes called a polynomial in \( x^{-1} \). (In other words, you substitute \( x^{-1} \) into a polynomial, getting a collection of constant multiples, powers, and sums or differences.)

We’ll be able to show that

- Anything of the form \( c/x^p \cdot e^{-1/x^2} \) vanishes at the origin, meaning it has limit 0 as \( x \to 0 \).

- Anything of the form \( c/x^p \cdot e^{-1/x^2} \) will produce a derivative with terms that also have that form!

These will be the key steps in a proof by induction.

Solution:

To make our proof more manageable, we’ll first prove a couple auxiliary results as lemmas. These will make it much easier to manipulate the derivatives we’ll find:

**Lemma 1.** For any \( c \in \mathbb{R} \) and any \( p \in \mathbb{N} \), we have

\[
\lim_{x \to 0} \frac{c}{x^p}e^{-1/x^2} = 0
\]

\(^1\) There is a theorem in analysis that does establish that if \( f^{(n)}(x) \) has a limit as \( x \to 0 \), that limit must be \( f^{(n)}(0) \). We will not be using this result, though, as its proof is too off-track from the main result.
Proof of lemma 1. First, write $-1/x^2$ as $-(1/x)^2$. We will perform a change of variable $u = 1/x$. However, the limit of $u$ depends on whether $x$ approaches 0 from positive or negative values, i.e. whether $x \to 0^+$ or $x \to 0^-$. 

First, suppose $x \to 0^+$. Then $u \to \infty$, and 

$$
\lim_{x \to 0} c(1/x)^p e^{-1/x^2} = \lim_{u \to \infty} cu^p e^{-u^2} = \lim_{u \to \infty} cu^p e^{-u^2}
$$

Since $e^{u^2}$ grows faster than $e^u$ as $u \to \infty$, and we already know that $e^u$ dominates $u^p$ (i.e. $u^p/e^u \to 0$), it follows $cu^p/(e^{u^2})$ also has limit 0. This proves the one-sided limit as $x \to 0^+$. The one-sided limit as $x \to 0^-$ is similar; apart from saying $u \to -\infty$ this time, all the other steps stay the same. 

We also introduce a result to help with derivatives:

Lemma 2. If $P$ is a polynomial, say $P(x) = \sum_{k=0}^n a_k x^k$, and $x \neq 0$, then the derivative of $e^{-1/x^2} P(1/x)$ has the form $e^{-1/x^2} Q(1/x)$ for some (possibly different) polynomial $Q$. In other words, there exists some $m \geq 0$ and constants $b_k$ for $0 \leq k \leq m$ such that 

$$
\frac{d}{dx} \left( e^{-1/x^2} P(1/x) \right) = \frac{d}{dx} \sum_{k=0}^n a_k \frac{x^k}{x^{k+1}} e^{-1/x^2} = \sum_{k=0}^m b_k \frac{x^k}{x^{k+1}} e^{-1/x^2} = e^{-1/x^2} Q(1/x)
$$

Proof of lemma 2. For each $k$ from 0 to $n$, we use the Product Rule and Chain Rule of derivatives to find 

$$
\frac{d}{dx} \left( \frac{a_k}{x^{k+1}} e^{-1/x^2} \right) = -\frac{ka_k}{x^{k+1}} e^{-1/x^2} + \frac{a_k}{x^{k+1}} \frac{2}{x^3} e^{-1/x^2} = \left( -\frac{ka_k}{x^k} + \frac{2a_k}{x^{k+3}} \right) e^{-1/x^2}
$$

This has the form of a polynomial in $1/x$ times $e^{-1/x^2}$, because it can be written as $e^{-1/x^2}$ times $-ka_k(1/x)^k + 2a_k(1/x)^{k+3}$. Adding up all these derivatives for $k$ from 0 to $n$ gives $e^{-1/x^2}$ times a sum of finitely many polynomials with $1/x$ plugged in. Since the sum of any finite quantity of polynomials is a polynomial, our final derivative is $e^{-1/x^2}$ times a polynomial in $1/x$. 

Now, we can prove our main theorem.

Proof of theorem. First, we prove by induction that for all $n \geq 0$ and all $x \neq 0$, there exists a polynomial $Q_n$ such that 

$$
f^{(n)}(x) = e^{-1/x^2} Q_n(1/x)
$$

(The subscript does not mean order here... the degree of $Q_n$ may be much higher than $n$.) For the base case of $n = 0$, $f(x) = e^{-1/x^2} \cdot 1$, so $Q_0(x) = 1$ works. Now, let $n \geq 0$, and assume the claim is true for $n$ as inductive hypothesis. Now, $f^{(n+1)}(x)$ is the derivative of $f^{(n)}(x)$. By Lemma 2, there exists some polynomial $Q_{n+1}$ such that 

$$
\frac{d}{dx} e^{-1/x^2} Q_n(1/x) = e^{-1/x^2} Q_{n+1}(1/x)
$$

finishing the inductive step.

Second, and finally, we prove that for all $n \geq 0$, $f^{(n)}(0) = 0$. (It follows, by the way, that $f$ is infinitely differentiable at 0 and hence each derivative is also continuous at 0.) The proof
is by induction on \( n \geq 0 \). For the base case of \( n = 0 \), \( f^{(0)}(0) = f(0) = 0 \) by definition of \( f \).

Now, let \( n \geq 0 \), and assume \( f^{(n)}(0) = 0 \) as inductive hypothesis. The \((n+1)\)st derivative at 0 is, by the definition of derivative,

\[
\lim_{x \to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x \to 0} \frac{f^{(n)}(x)}{x}
\]

by the IH. From the first proof by induction, we know that when \( x \neq 0 \), \( f^{(n)}(x) = e^{-1/x^2}Q_n(1/x) \) for some polynomial \( Q_n \). Let’s write \( Q_n(x) = \sum_{k=0}^{m} a_k x^k \), where \( m \) is the degree of \( Q_n \). Thus,

\[
\lim_{x \to 0} \frac{f^{(n)}(x)}{x} = \lim_{x \to 0} \sum_{k=0}^{m} a_k \frac{(1/x)^k e^{-1/x^2}}{x} = \lim_{x \to 0} \sum_{k=0}^{m} a_k \frac{x^{k+1} e^{-1/x^2}}{x}
\]

By Lemma 1, each term of this sum has limit 0, so the sum has limit 0 and therefore \( f^{(n+1)}(0) = 0 \). This finishes the inductive step and the theorem! \( \square \)