MATH 3100: SEQUENCES AND SERIES
Fall 2014, Dr. Klipper

In-Depth Example for Week 14

A Function which Does Not Equal its Taylor Series

For any function f which is infinitely differentiable at 0, we can create its Taylor series

$$P^f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

This series may not converge on the entire domain of f: it definitely converges at 0, and in general its domain of convergence is an interval centered at 0. However, an important question is: what does the series converge *to*? Does it necessarily converge to $f(x)$?

To determine this, we introduced the concept of Taylor remainders in class: $R_n(x) = f(x) - P_n^f(x)$, so that $\lim R_n(x) = f(x) - P^f(x)$. Thus, f equals P^f iff R_n has limit 0. Frequently, we can use Taylor's theorem to estimate $|R_n(x)|$ and get a bound approaching 0, so in many cases, f does equal its Taylor series.

A function that equals its Taylor series is called *analytic*. Many famous functions are analytic, such as e^x, $\sin x$, $\cos x$, and $1/(1-x)$. Our job for this handout is to show that *non-analytic functions exist*; these are functions for which $P^f \neq f$ on the domain of convergence. In fact, since the Taylor series is the only possible power series a function can have (due to our uniqueness theorems), *non-analytic functions do not have power series representations*!

In this document, we will prove this theorem, giving us a non-analytic function:

Theorem. *The function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is infinitely differentiable at 0, and $f^{(n)}(0) = 0$ for every $n \geq 0$. Thus, $P^f(x) = \sum 0x^n = 0$, and $f(x) \neq P^f(x)$ for any $x \neq 0$ (even though $P^f(x)$ converges everywhere).

Roughly speaking, this function is extremely flat near $x = 0$; try drawing this with a calculator and zooming in really far at the origin. Now, normally x^n is fairly flat-looking near $x = 0$, with larger n values causing a slower rise in the function and hence a flatter shape. In effect, our function $f(x)$ is so flat that it's flatter than all the x^n functions!

<u>*Brainstorming:*</u>

For this function, I cannot find its Taylor series by reusing the known series of e^x and composing with an inner function $g(x) = -1/x^2$. The problem is that $g(x) = -1/x^2$ is not a continuous function at 0, and it certainly does not satisfy $g(0) = 0$. I have no choice but to use the definition of Taylor series by computing every derivative.

Three issues make this function particularly tricky to use when computing the derivatives:

1. The piecewise nature of the definition means that it's not even clear if the function is continuous! Before we take any derivatives at all, we should know whether f is continuous at 0. If not, there's no point in differentiating!

2. Due to the piecewise definition, each derivative will also have to be described piecewise! There will be a formula we can use for $f^{(n)}(x)$ when $x \neq 0$, and a separate result for $f^{(n)}(0)$. This brings up a concern: even if $\lim_{x\to 0} f^{(n)}(x)$ exists, that does not necessarily tell us that limit is $f^{(n)}(0)$! In other words, we shouldn't assume that $f^{(n)}$ is continuous at 0 either.[1] As a result, we're going to have to use the definition of derivative (using difference quotients) to figure out the derivatives at 0.

3. The derivatives don't follow a nice pattern: the first derivative uses the Chain Rule, and the second will use both the Chain and Product Rules. An exact formula for $f^{(n)}(x)$ is incredibly complicated. Instead, we'll have to figure out some rough idea of what form the derivative takes, show that this form gets us the limits as $x \to 0$ that we need, and we'll prove by induction that this form is correct.

Concerning this third point, let's get a sense of what this form might be! Here are the first few derivatives of f at nonzero values:

$$f'(x) = \frac{2}{x^3}e^{-1/x^2} \quad f''(x) = \frac{4}{x^6}e^{-1/x^2} - \frac{6}{x^4}e^{-1/x^2} \quad f'''(x) = \frac{8}{x^9}e^{-1/x^2} - \frac{36}{x^7}e^{-1/x^2} + \frac{24}{x^5}e^{-1/x^2}$$

From the looks of this, we can notice one important general trend: we get a few terms which each have the form $c/x^p \cdot e^{-1/x^2}$ for some constants c and p. Being more specific than this gets tricky, and it turns out this simple remark will suffice. In fact, if you factor out e^{-1/x^2}, you get expressions like

$$f'''(x) = e^{-1/x^2}\left(8(x^{-1})^9 - 36(x^{-1})^7 + 24(x^{-1})^5\right)$$

The expression in parentheses is sometimes called a *polynomial in* x^{-1}. (In other words, you substitute x^{-1} into a polynomial, getting a collection of constant multiples, powers, and sums or differences.)

We'll be able to show that

- Anything of the form $c/x^p \cdot e^{-1/x^2}$ vanishes at the origin, meaning it has limit 0 as $x \to 0$.
- Anything of the form $c/x^p \cdot e^{-1/x^2}$ will produce a derivative with terms that also have that form!

These will be the key steps in a proof by induction.

<u>*Solution:*</u>

To make our proof more manageable, we'll first prove a couple auxiliary results as lemmas. These will make it much easier to manipulate the derivatives we'll find:

Lemma 1. *For any $c \in \mathbb{R}$ and any $p \in \mathbb{N}$, we have*

$$\lim_{x\to 0} \frac{c}{x^p}e^{-1/x^2} = 0$$

[1]There is a theorem in analysis that does establish that if $f^{(n)}(x)$ has a limit as $x \to 0$, that limit must be $f^{(n)}(0)$. We will not be using this result, though, as its proof is too off-track from the main result.

Proof of lemma 1. First, write $-1/x^2$ as $-(1/x)^2$. We will perform a change of variable $u = 1/x$. However, the limit of u depends on whether x approaches 0 from positive or negative values, i.e. whether $x \to 0^+$ or $x \to 0^-$.

First, suppose $x \to 0^+$. Then $u \to \infty$, and

$$\lim_{x\to 0} c(1/x)^p e^{-1/x^2} = \lim_{u\to\infty} cu^p e^{-u^2} = \lim_{u\to\infty} \frac{cu^p}{e^{u^2}}$$

Since e^{u^2} grows faster than e^u as $u \to \infty$, and we already know that e^u dominates u^p (i.e. $u^p/e^u \to 0$), it follows $cu^p/(e^{u^2})$ also has limit 0. This proves the one-sided limit as $x \to 0^+$. The one-sided limit as $x \to 0^-$ is similar; apart from saying $u \to -\infty$ this time, all the other steps stay the same. □

We also introduce a result to help with derivatives:

Lemma 2. *If P is a polynomial, say $P(x) = \sum_{k=0}^n a_k x^k$, and $x \neq 0$, then the derivative of $e^{-1/x^2}P(1/x)$ has the form $e^{-1/x^2}Q(1/x)$ for some (possibly different) polynomial Q. In other words, there exists some $m \geq 0$ and constants b_k for $0 \leq k \leq m$ such that*

$$\frac{d}{dx}\left(e^{-1/x^2}P(1/x)\right) = \frac{d}{dx}\sum_{k=0}^{n}\frac{a_k}{x^k}e^{-1/x^2} = \sum_{k=0}^{m}\frac{b_k}{x^k}e^{-1/x^2} = e^{-1/x^2}Q(1/x)$$

Proof of lemma 2. For each k from 0 to n, we use the Product Rule and Chain Rule of derivatives to find

$$\frac{d}{dx}\left(\frac{a_k}{x^k}e^{-1/x^2}\right) = \frac{-ka_k}{x^{k+1}}e^{-1/x^2} + \frac{a_k}{x^k}\cdot\frac{2}{x^3}e^{-1/x^2} = \left(\frac{-ka_k}{x^k} + \frac{2a_k}{x^{k+3}}\right)e^{-1/x^2}$$

This has the form of a polynomial in $1/x$ times e^{-1/x^2}, because it can be written as e^{-1/x^2} times $-ka_k(1/x)^k + 2a_k(1/x)^{k+3}$. Adding up all these derivatives for k from 0 to n gives e^{-1/x^2} times a sum of finitely many polynomials with $1/x$ plugged in. Since the sum of any finite quantity of polynomials is a polynomial, our final derivative is e^{-1/x^2} times a polynomial in $1/x$. □

Now, we can prove our main theorem.

Proof of theorem. First, we prove by induction that for all $n \geq 0$ and all $x \neq 0$, there exists a polynomial Q_n such that

$$f^{(n)}(x) = e^{-1/x^2}Q_n(1/x)$$

(The subscript does not mean order here... the degree of Q_n may be much higher than n.) For the base case of $n = 0$, $f(x) = e^{-1/x^2}\cdot 1$, so $Q_0(x) = 1$ works. Now, let $n \geq 0$, and assume the claim is true for n as inductive hypothesis. Now, $f^{(n+1)}(x)$ is the derivative of $f^{(n)}(x)$. By Lemma 2, there exists some polynomial Q_{n+1} such that

$$\frac{d}{dx}e^{-1/x^2}Q_n(1/x) = e^{-1/x^2}Q_{n+1}(1/x)$$

finishing the inductive step.

Second, and finally, we prove that for all $n \geq 0$, $f^{(n)}(0) = 0$. (It follows, by the way, that f is infinitely differentiable at 0 and hence each derivative is also continuous at 0.) The proof is by induction on $n \geq 0$. For the base case of $n = 0$, $f^{(0)}(0) = f(0) = 0$ by definition of f. Now, let $n \geq 0$, and assume $f^{(n)}(0) = 0$ as inductive hypothesis. The $(n+1)$st derivative at 0 is, by the definition of derivative,

$$\lim_{x \to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x \to 0} \frac{f^{(n)}(x)}{x}$$

by the IH. From the first proof by induction, we know that when $x \neq 0$, $f^{(n)}(x) = e^{-1/x^2} Q_n(1/x)$ for some polynomial Q_n. Let's write $Q_n(x) = \sum_{k=0}^{m} a_k x^k$, where m is the degree of Q_n. Thus,

$$\lim_{x \to 0} \frac{f^{(n)}(x)}{x} = \lim_{x \to 0} \frac{\sum_{k=0}^{m} a_k (1/x)^k e^{-1/x^2}}{x} = \lim_{x \to 0} \sum_{k=0}^{m} \frac{a_k}{x^{k+1}} e^{-1/x^2}$$

By Lemma 1, each term of this sum has limit 0, so the sum has limit 0 and therefore $f^{(n+1)}(0) = 0$. This finishes the inductive step and the theorem! $\square$