Remember from Section 2.5 that to vertically stretch a curve by $a$, you multiply $a$ on the outside. (If $a$ is negative, you reflect the curve too.) To horizontally stretch by $b$, you DIVIDE $b$ on the inside. (When $b$ is negative, that causes a reflection as well.) Also, to vertically shift by $d$, add $d$ to the outside.

For instance, consider the graphs of $y = \sin(x)$ and $y = \cos(x)$. Both have range $[-1, 1]$ and period $2\pi$.

If we use the three transforms described earlier to sin and cos, we get these curves:
\[
y = a \sin(bx) + d \quad \text{and} \quad y = a \cos(bx) + d \quad \text{have range} \quad [d - |a|, d + |a|] \quad \text{and period} \quad 2\pi/|b|
\]

We say that $y = d$ is the axis of the wave; it tells us the horizontal line down the middle of the wave. The value $|a|$ is called the amplitude of the wave; it measures how far it deviates from its central axis.

Some useful notes:

- The axis is always the average of the min and max values of the curve!
- The amplitude is $(\text{max} - \text{min})/2$, i.e. half the distance between crest and trough.
- The period measures how long the wave takes to repeat. To measure it on a graph, look for two points of the same type, like two crests or two troughs, and measure the horizontal distance between them.
- To tell sin and cos apart:
  - sin(x) starts at its axis and then rises. cos(x) starts at a crest (maximum turning point) and then falls.

**Ex 1:** Find the amplitude, period, and range of each curve. Also make a quick sketch.

(a) $y = 3 \sin(\pi x) + 2$  
(b) $y = -2 \cos(x/2)$

**NOTE:** For (b), since $a$ is negative, you’ll want to draw the wave upside-down. Instead of starting at the top and falling, it starts at the bottom and rises!

**Ex 2:** Let $f(x) = a \cos(2x) + d$, where $a$ is positive. If $f(x)$ oscillates between $-17$ and $5$, find $a$ and $d$.

**NOTE:** Since this question only talks about range, the 2 on the inside is irrelevant (it only affects period). What is important is that you have the min and max; you need axis and amplitude.

**Ex 3:** Determine the ranges of the following curves:

(a) $10 + 2 \sin(-8x + 9)$  
(b) $10 + 10 \cos^2(x)$

**Ex 4:** Determine the coordinates of the first maximum and minimum turning points on the graph of $y = 6 \sin(13x)$ on the interval $[0, 2\pi]$.

**NOTE:** Normally, the first maximum turning point of sin is at $(\pi/2, 1)$, and the first minimum is $(3\pi/2, -1)$. How are they stretched?

**Phase Shift**

It is also possible to shift a wave horizontally. This is a little harder to get correct. The amount a wave has been moved horizontally is called its **phase shift**.

The most important part of the wave here is sin($bx + c$) or cos($bx + c$). Here’s how I think of it: normally, the wave “starts” when its angle $\theta$ is 0. Thus, to find where my transformed wave “starts”, I should set $\theta = bx + c = 0$ and solve for $x$. When you do that, you find the new “start” location is $x = -c/b$.

Here’s a summary:

<table>
<thead>
<tr>
<th>Form of wave</th>
<th>Amplitude</th>
<th>Period</th>
<th>Phase Shift</th>
<th>Axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = a \sin(bx + c) + d$ or $y = a \cos(bx + c) + d$</td>
<td>$</td>
<td>a</td>
<td>$</td>
<td>$2\pi/</td>
</tr>
</tbody>
</table>

It helps to memorize this, but you should also be able to figure these out by thinking of the stretches and shifts.

**Ex 5:** Find the amplitude, period, phase shift, and axis of these curves.

(a) $y = -3 \cos(2x + \pi) - 1$  
(b) $y = \frac{1}{2} \sin(\frac{\pi}{2} x)$

**Recognizing the Transforms from a Sine or Cosine**

We just showed how to get a transformed graph from its equation. How do we go the other way and get an equation from a graph? If we want the form $y = a \sin(bx + c) + d$ or $y = a \cos(bx + c) + d$, how do we find $a, b, c, d$? For simplicity, we’re going to keep $a$ and $b$ positive here. (Also, $c$ will be positive the way we do this.)
1. Find the axis and amplitude. The axis is the average of min and max. The amplitude is HALF the distance from min to max.
   - This gives you \( d = (\text{axis}) \) and \( a = (\text{amplitude}) \).

2. Measure the period. Usually, I look for two adjacent crests (max points) or troughs (min points).
   - Since \((\text{period}) = 2\pi/b\), solve for \( b \)... \( b = 2\pi/(\text{period}) \).

3. Phase shift is tricky. There are actually infinitely many possible answers, but we want the most convenient one. Look for the greatest negative “starting point” of the wave (i.e. go left of the origin but as close as possible), and make its \( x \)-coordinate your phase shift.
   - Remember: sin starts at an axis and rises. cos starts at a crest.
   - Since (phase shift) = \(-c/b\), and you already found \( b \), you can get \( c \). If you picked the phase shift correctly, \( c \) will be positive.

**Ex 6:** The graph of a sine function with a positive coefficient is shown on the right.

(a) Find its amplitude, period, and phase shift.
(b) Write the equation in the form \( y = a\sin(bx + c) \).

**NOTE:** If this were a COSINE function, how would that change the phase shift and the equation?

**Tricky Problem Type: Getting the Transformed Wave from Two Points**

**Ex 7:** A function \( f(x) \) is of the form \( f(x) = a + \cos(bx) \), where \( a \) and \( b \) are constants and \( 0 < b < 9 \). If \( f(0) = 2 \) and \( f(\pi/9) = 0.5 \), find \( a \) and \( b \).

**MAIN STEPS:**
1. You have a form given, and you have two points. Plug the points in!
2. When you plug in 0, you get \( a + \cos(0) = 2 \). Since \( \cos(0) = 1 \) (whereas \( \sin(0) = 0 \)), you get \( a = 1 \).
3. When you plug in \( \pi/9 \), you get \( 1 + \cos(b\pi/9) = 0.5 \). Hence, \( \cos(b\pi/9) = -0.5 = -1/2 \).
4. Think of this as a “basic trig equation” where \( \theta = b\pi/9 \). Thus, we want to solve \( \cos(\theta) = -1/2 \).
5. However, we need an interval for \( \theta \). We were given that \( 0 < b < 9 \). Multiply everything by \( \pi/9 \) to find \( 0 < b\pi/9 < \pi \). Therefore, \( \theta \) must be in \((0, \pi)\).

**Ex 8:** Suppose \( g(x) = a + \tan(bx) \), where \(-3.5 < b < 3.5 \). Determine \( a \) and \( b \) if \( g(0) = 3 \) and \( g(\pi/7) = 2 \).

**NOTE:** Here, if \( x \) stands for \( bx/7 \), you should find that \(-\pi/2 < x < \pi/2 \). You’ll find \( \tan(x) \) needs to be negative here, so our \( x \) will be from \(-\pi/2 \) to 0.

**Transforming the Other Four Trigs**

We can do the same types of transforms to the other trig functions, and we make equations like \( y = a\tan(bx + c) + d \), \( y = a\sec(bx + c) + d \), etc. However, the transforms don’t get interpreted quite the same way.

- The phase shift formula is still the same: \(-c/b\). Once again, think about where the graphs “start”. tan starts at the origin, and sec starts at \((0, 1)\). cot and csc have vertical asymptotes at their start.
- The coefficient \( a \) is a vertical stretch: “amplitude” doesn’t make sense here. The ranges of tan and cot are \((−∞, ∞)\). For sec and tan, though, the range is normally broken into two halves: \((−∞, −1] \cup [1, ∞)\) (so there’s a “gap” from −1 to 1). The vertical stretch adjusts the size of the “gap”.
- Remember tan and cot have half the period of the others: their period is \( \pi/|b| \) instead of \( 2\pi/|b| \). In fact, the asymptotes keep repeating every \( \pi/|b| \) in these graphs.

**Ex 9:** Find the ranges and periods of the following curves:

(a) \(-3\tan(5x)\)  
(b) \(4\sec(x/2)\)  
(c) \(9\csc(2x + 3) + 1\)

Try graphing these curves (with a calculator, for instance, in radian mode) to see these changes for yourself!