On Two Nonlinear Biharmonic Evolution Equations: Existence, Uniqueness and Stability

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Abstract

We study the following two nonlinear evolution equations with a fourth order (biharmonic) leading term:

$$-\Delta^2 u - \frac{1}{\epsilon^2}(|u|^2 - 1)u = u_t \qquad \text{in } \Omega \subset \mathbf{R}^2 \text{ or } \mathbf{R}^3$$

and

$$-\Delta^2 u + \frac{1}{\epsilon^2}\nabla \cdot ((|\nabla u|^2 - 1)\nabla u) = u_t \qquad \text{in } \Omega \subset \mathbf{R}^2 \text{ or } \mathbf{R}^3$$

with an initial value and a Dirichlet boundary conditions. We show the existence and uniqueness of the weak solutions of these two equations. For any $t \in [0, +\infty)$, we prove that both solutions are in $L_\infty(0, T, L_2(\Omega)) \cap L_2(0, T, H^2(\Omega))$. We also discuss the asymptotic behavior of the solutions as time goes to infinity. This work lays the ground for our numerical simulations for the above systems in [Lai, Liu and Wenston'03].

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§1. Introduction

In this paper, we are studying the following two biharmonic equations

$$(1.1) \qquad \begin{cases} -\Delta^2 u - \frac{1}{\epsilon^2}(|u|^2 - 1)u = u_t, & \text{in } \Omega \subset \mathbf{R}^2 \text{ or } \mathbf{R}^3 \\ u(x, 0) = u_0(x), & x \in \Omega \\ u(x, t) = u_1(x) & x \in \partial\Omega \\ \frac{\partial u}{\partial n}(x, t) = u_2(x) & x \in \partial\Omega \end{cases}$$

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and

$$(1.2)\qquad \begin{cases} -\Delta^2 u + \frac{1}{\epsilon^2}\nabla\cdot((|\nabla u|^2-1)\nabla u) = u_t, & \text{in } \Omega \subset \mathbf{R}^2 \text{ or } \mathbf{R}^3 \\ u(x,0) = u_0(x), & x\Omega \\ u(x,t) = u_1(x) & x \in \partial\Omega \\ \frac{\partial u}{\partial n}(x,t) = u_2(x) & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^2$ or $\mathbf{R}^3$ and n is the outward normal of the boundary $\partial\Omega$.

Equation (1.1) may be regarded as the gradient flow of the following energy functional:

$$(1.3)\qquad E(u) := \frac{1}{2}\int_\Omega \left(|\Delta u|^2 + \frac{1}{2\epsilon^2}(|u|^2-1)^2\right) dx$$

in the class

$$(1.4)\qquad \mathcal{C} = \{u \in H^2(\Omega),\ u\big|_{\partial\Omega} = u_1, \frac{\partial}{\partial n}u\big|_{\partial\Omega} = u_2\}.$$

Similarly, equation (1.2) is the gradient flow of the energy functional

$$(1.5)\qquad E(u) := \frac{1}{2}\int_\Omega \left(|\Delta u|^2 + \frac{1}{2\epsilon^2}(|\nabla u|^2-1)^2\right) dx$$

in the class (1.4).

Much effort has been devoted to the study of the gradient flow of lower order energy, in particular, the following Ginszburg-Landau type energy, with u in both scalar and vector cases:

$$(1.6)\qquad \min_{\substack{u\in H^1(\Omega)\\ u\big|_{\partial\Omega}=u_1}} \int_\Omega \left(|\nabla u|^2 + \frac{1}{2\epsilon^2}(|u|^2-1)^2\right) dx.$$

The Euler–Lagrange equation is as follows:

$$(1.7)\qquad \begin{cases} -\Delta u + \frac{1}{\epsilon^2}(|u|^2-1)u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = u_1. \end{cases}$$

Such problem has been studied for many years (cf., e.g., [Allen and Cahn'79] and [Bronsard and Kohn'91]). The study of the above energy (1.6) was partially motivated by [Allen and Cahn'79] to understand the motion of free interfaces between different phases. The vector Ginzburg-Landau equation (cf. [Du, Gunzburger and Peterson'92]) is very important in the understanding the vortices in the superconductivities.

Our study is also motivated by the following problem in liquid crystals. Recall that the usual nematic molecule configuration is determined by minimizing the following Oseen–Frank energy:

$$E(\mathbf{n}) = k_1(\mathrm{div}\mathbf{n})^2 + k_2(\mathbf{n}\cdot\mathrm{curl}(\mathbf{n}))^2 + k_3(\mathbf{n}\times\mathrm{curl}(\mathbf{n}))^2 + (k_4+k_2)(\mathrm{trace}(\nabla\mathbf{n})^2 - (\mathrm{div}\mathbf{n})^2)$$

which is equal to $|\nabla \mathbf{n}|^2$ in the case of equal constants. Here, the unit vector $\mathbf{n}$ stands for molecule orientation.

A special type of liquid crystals is called smectic. The molecule posses certain positional order (layer structure). In this case, $\mathbf{n} = \nabla\phi$ where the level sets of ϕ will represent the layer structure in the sample. In order to study this new energy $\int |\nabla\nabla\phi|^2 dx$ under constraint $|\nabla\phi| = 1$, it is natural to introduce the above penalized energy (1.5). We hope that as $\epsilon \to 0$, the minimizer of the above energy, or the solution of the Euler–Lagrange equation (1.2) will be convergent to the corresponding one with the nonlinear constraint $|\nabla\phi| = 1$ (cf. [Kinderlehrer and Liu'96] and [E'97]).

Such a singularly perturbed variational problem also arises in the study of thin film blisters (cf. [Ortiz and Gioia'94]), where the scalar function ϕ will stand for the height of the blistered film and $\int_\Omega |\nabla\nabla\phi|^2$ represents the bending energy. One well-known conjecture is to show that the limit solution is $\phi(x) = \mathrm{dist}(x, \partial\Omega)$ (cf. [Jin'97] and [Kohn'96]).

Thus, for both equations, it is important to study the asymptotic behavior as $\epsilon \to 0$. However, little is known about the variational problem involving the second order derivative term as those in (1.3) and (1.5). One of the main difficulties is the lack of the maximum principle. Our study of the prototype problems (1.1) and (1.2) is in the direction of trying to understand the effectiveness of the second order singular perturbation. Most of our estimates depend on the size of ϵ.

In §2, we first obtain the existence and uniqueness results of the weak solution of (1.1) and then establish the regularity of such weak solutions. After that we let $t \longrightarrow +\infty$ and get the existence of the solution of the steady state case. The uniqueness of the weak solution of the steady state case can be achieved when ϵ is not very small. We also obtain the existence results of the weak solution of (1.2) in §3. If the weak solution is the exact solution, as $t \longrightarrow +\infty$, we show that the time dependent solution converges to the solutions of the steady state case.

We also perform numerical simulations to the equations (1.1) and (1.2) together with their steady state equations. The detail of our numerical experiments and related analysis will be presented in [Lai, Liu and Wenston'02]. There we have applied a special bivariate spline space to numerically solve these equations. We notice many interesting phenomena, for instance, indications of possible nonuniqueness of the stationary solutions and self-similar structures. We include some examples in §4.

§2. The Study of Equation (1.1)

In this section, we study the existence, uniqueness and regularity of the solutions of (1.1). We also discuss the long time behavior and hence the stability property of the solutions.

We start with the "energy law" of the system (1.1). It comes from the natural of being the gradient flow of the system.

Lemma 2.1. *Suppose that $u(x,t)$ is a smooth solution of (1.1). Then the following* *equality holds:*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\left(|\Delta u|^2+\frac{1}{2\epsilon^2}(|u|^2-1)^2\right)dx=-\int_\Omega|u_t|^2dx. \tag{2.1}$$

Proof. Multiplying equation (1.1) by u_t and integrating over the domain Ω, we get

$$\begin{aligned}\int_\Omega u_t^2dx&=-\int_\Omega(\Delta^2u+\frac{1}{\epsilon^2}(|u|^2-1)u)u_tdx\\&=-\frac{1}{2}\frac{d}{dt}\int_\Omega(|\Delta u|^2+\frac{1}{2\epsilon^2}(|u|^2-1)^2)dx.\end{aligned}$$

The second equality employed integration by parts and the fact that the left hand side of (1.1) is the variation of

$$E(u)=\frac{1}{2}\int_\Omega(|\Delta u|^2+\frac{1}{2\epsilon^2}(|u|^2-1)^2)dx \tag{2.2}$$

with respect to u. We also point out that since the boundary values of u does not change with respect to time t, we have $u_t|_{\partial\Omega}=0$. We have thus established the identity. ∎

Lemma 2.2. *Let $u(x,t)$ be a smooth solution of (1.1) with $u_1|_{\partial\Omega}=u_2|_{\partial\Omega}=0$. Then*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|u|^2dx+\int_\Omega(|\Delta u|^2+\frac{1}{\epsilon^2}(|u|^2-1)^2+\frac{1}{\epsilon^2}|u|^2)dx=\frac{1}{\epsilon^2}|\Omega|. \tag{2.3}$$

Moreover, the $L_\infty(0,T,L_2(\Omega))$ and $L_2(0,T,H^2(\Omega))$ norms of u are bounded.

Proof. Equality comes from by multiplying (1.1) by u and integrating by parts. Then we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega|u|^2dx&=-\int_\Omega|\Delta u|^2dx-\frac{1}{\epsilon^2}\int_\Omega(|u|^2-1)|u|^2dx\\&=-\int_\Omega(|\Delta u|^2+\frac{1}{\epsilon^2}(|u|^2-1)^2)dx-\int_\Omega\frac{1}{\epsilon^2}(|u|^2-1)dx\end{aligned}$$

which yields (2.3).

Moreover, integrating (2.3) with respect to $t\in[0,T]$, we get

$$\begin{aligned}&\frac{1}{2}\int_\Omega|u|^2dx+\int_0^T\int_\Omega(|\Delta u|^2+\frac{1}{\epsilon^2}(|u|^2-1)^2+\frac{1}{\epsilon^2}|u|^2)dxdt\\&=\frac{T}{\epsilon^2}|\Omega|+\frac{1}{2}\int_\Omega|u_0|^2dx.\end{aligned} \tag{2.4}$$

It follows that the $L_\infty(0,T,L_2(\Omega)$ and $L_2(0,T,H^2(\Omega)$ norms of u are bounded. This completes the proof. ∎

Remark 1. In case of non-homogeneous boundary conditions (1.8), we let $\phi(x) \in H^2(\Omega)$ solve the following biharmonic problem with Dirichlet boundary conditions

$$(2.5) \qquad \begin{cases} \Delta^2 \phi = 0 & \text{in } \Omega \\ \phi(x) = u_1(x) & \text{on } \partial\Omega \\ \frac{\partial}{\partial n}\phi(x) = u_2(x) & \text{on } \partial\Omega \end{cases}$$

The existence of ϕ is obvious (cf. [Grisvard'85]). Subtracting ϕ from u, we see the new solution called u again satisfies the following

$$(1.1)' \qquad \begin{cases} u_t + \Delta^2 u + \frac{1}{\epsilon^2}(|u|^2 - 1)u = f(u), & \text{in } \Omega \\ u(x,0) = u_1(x), & x \in \Omega \\ u\big|_{\partial\Omega} = 0 \\ \frac{\partial}{\partial n} u\big|_{\partial\Omega} = 0. \end{cases}$$

where $f(u)$ comes from the nonlinear term in (1.1) and in fact

$$f(u) = -\frac{3}{\epsilon^2}\phi u^2 - \frac{3}{\epsilon^2}\phi^2 u - \frac{1}{\epsilon^2}\phi^3 + \frac{1}{\epsilon^2}\phi. \quad \blacksquare$$

Then Lemma 2.2 is also true for the u which satisfies the nonhomogeneous boundary condition. In fact we have

Lemma 2.3. *Let $u(x,t)$ be a smooth solution of $(1.1)'$. Then there exists a constant $C(\epsilon, T)$ independent of u such that*

$$(2.6) \qquad \int_\Omega |u|^2 dx + \int_0^T \int_\Omega (|\Delta u|^2 + \frac{1}{2\epsilon^2}|u|^4) dx dt \le C(\epsilon, T).$$

Proof. Multiplying u both sides of $(1.1)'$, and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \int_\Omega (|\Delta u|^2 + \frac{1}{\epsilon^2}(|u|^2 - 1)u^2) = \int_\Omega f(u) u dx.$$

That is, letting $m = \max_{x \in \Omega} |\phi(x)| < \infty$ by Sobolev Embedding Theorem,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \int_\Omega (|\Delta u|^2 + \frac{1}{\epsilon^2}|u|^4) dx$$

$$\le \frac{(1+3m^2)}{\epsilon^2}\int_\Omega u^2 dx + \frac{3}{\epsilon^2} m \int_\Omega u^3 dx + \frac{(m^3 + m)}{\epsilon^2}\int_\Omega u dx$$

$$\le C_1(\epsilon, m)\int_\Omega |u|^2 dx + \frac{3m}{\epsilon^2}(\int_\Omega |u|^4 dx)^{1/2}(\int_\Omega |u|^2 dx)^{1/2} + |\Omega|.$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2dx + \int_\Omega |\Delta u|^2dx + (\frac{1}{\epsilon^2} - \delta)\int_\Omega |u|^4dx \le C_2(\epsilon, m)\int_\Omega |u|^2dx + |\Omega|$$

for an arbitrary small δ and for a constant C_2 dependent only on ϵ and m.

Then a standard argument implies

$$\frac{d}{dt}\left(e^{-C_2(\epsilon,m)t}\int_\Omega |u|^2dx\right) +$$
$$2e^{-C_2(\epsilon,m)t}\left(\int_\Omega |\Delta u|^2dx + (\frac{1}{\epsilon^2} - \delta)\int_\Omega |u|^4dx\right) \le e^{-C_2(\epsilon,m)t}|\Omega|$$

and hence,

$$\int_\Omega |u|^2dx + 2\int_0^t e^{C_2(\epsilon,m)(t-s)}\int_\Omega \left(|\Delta u|^2 + (\frac{1}{\epsilon^2} - \delta)|u|^4\right)dx$$
$$\le e^{C_2(\epsilon,m)t}\int_\Omega |u_0|^2dx + |\Omega|\int_0^t e^{C_2(\epsilon,m)(t-s)}ds.$$

This completes the proof. ∎

We now introduce the weak formulation for (1.1): find $u \in L_2(0, T, H_0^2(\Omega))$ such that

$$\int_\Omega u_t v dx + \int_\Omega \Delta u \Delta v dx + \frac{1}{\epsilon^2}\int_\Omega (u^2 - 1)uvdx$$
$$= \int_\Omega f(u)vdx$$

for all $v \in H_0^2(\Omega)$.

Let $V_N \in H_0^2(\Omega)$ be a finite dimensional space for each $N \in Z_+$ and for all $u \in H_0^2(\Omega)$ there exists a sequence $u_N \in V_N$ such that

$$u_n \longrightarrow u \text{ in } H_0^2(\Omega) \text{ norm.}$$

We remark here that there are many choice of these spaces V_N's. For instance, since $Lv = \Delta^2 v$ is self-adjoint with zero boundary conditions, we can choose V_N to be the space spanned by the first N eigen functions of L which has an orthonormal basis. For our practice, we shall use the bivariate spline space $v_N = S_{3r}^r(\diamondplus_N)$ where $(\diamondplus_N)$ is the N^{th} uniform refinement of triangulated quadrangulation $\diamondplus$. This will transform $(1.1)'$ into a system of ordinary differential equations in the coefficients of $u \in V_N$ with respect to time variable t. This is the exact spline space we implemented as finite elements in [Lai, Liu and Wenston'02].

The spline space mentioned above is defined as

$$S_{3r}^r(\diamondplus) = \{S \in C^r(\Omega) : \quad s|_t \in \mathbf{P}_{3r}, \forall t \in \diamondplus\},$$

where $\mathbf{P}_{3r}$ is the space of all polynomials of degree $\le 3r$ and $\diamondplus$ is a triangulation of the domain $\Omega \subset \mathbf{R}^2$ which is obtained from a non-degenerate quadrangulation of Ω by adding two diagonals of each quadrilateral in $\diamondsuit$ (cf. [Lai and Schumaker'99]).

It is known that for any finite dimensional ODE system always, a local solution always exists. To prove the global existence of the finite dimensional ODE system, we only need to prove the following *a priori* estimate

Lemma 2.4. *Let $u_N \in V_N$ be the weak solution satisfying*

$$\int_\Omega \frac{\partial}{\partial t}(u_N)\, v dx + \int_\Omega \left[\Delta u_N \Delta v + \frac{1}{\epsilon^2}((u_N)^2 - 1)u_N v\right] dx \tag{2.7}$$
$$= \int_\Omega f(u_N) v dx, \quad \text{for } t \in (0, T_0) \text{ and for all } v \in V_N$$

with $u_N|_{\partial\Omega} = \frac{\partial}{\partial n} V_N|_{\partial\Omega} = 0$ and $u_N(x,0) = u_{0,N}(x)$ which is the projection of $u_0(x)$ in V_N. Then for arbitrary $t \in (0,\infty)$,

$$\int_\Omega |u_N|^2 dx + 2\int_0^t e^{C_2(\epsilon,m)(t-s)} \int_\Omega |\Delta u|^2 dx ds \le e^{C_2(\epsilon,m)t} \int_\Omega |u_{0,N}|^2 dx + C(t,\epsilon,\Omega) \tag{2.8}$$

for a positive constant $C(t,\epsilon,\Omega)$.

Proof: We replace u with u_N in the proof of Lemma 2.3. This completes the proof. ∎

It follows from (2.8) that

$$u_N \in L_\infty(0,T,L_2(\Omega)) \text{ and } \Delta u_N \in L_2(0,T,L_2(\Omega)).$$

These in fact imply that $u_N \in L_2(0,T,H^2(\Omega))$ by the Poincaré inequality. (2.8) also implies the global existence of the solution of (2.7). That is, for any fixed $T > 0$, there exists a smooth solution $u_N \in V_N$ such that (2.7) holds for $v \in V_N$ and all $t \in (0,T)$.

Next, by Rellich's Compactness Theorem (cf.[Adam '75]), there exists a subsequence in $\{u_N\}_{N\in Z_+}$, say $\{u_N\}$ itself, convergent weakly in $L_2(0,T,H^2(\Omega))$ as well as weakly in $L_2(0,T,H^1(\Omega))$ to u. Since the nonlinearity in each term of (2.7) is below the critical power of the Sobolev Embedding Theorem, by passing the limit in the integration of (2.7) with respect to t over $[0,T]$, we know that the limit u is the weak solution of (2.6). Therefore, we have obtained the following

Theorem 2.1(Existence). *for any fixed $T > 0$, there exists a weak solution u of (1.1)' such that*

$$u \in L_\infty(0,T,L_2(\Omega)) \cap L_2(0,T,H^2(\Omega)). \tag{2.9}$$

Remark 3. *Theorem 2.1 implies that there exists a weak solution u for the original system $(1.1)'$.*

Next we study the uniqueness of the weak solution for each $\epsilon > 0$.

Theorem 2.2(Uniqueness). *The weak solution in Theorem 2.1 is in fact unique.*

Proof: Suppose that u_1 and u_2 are two different solutions of $(1.1)'$ and that both u_1 and u_2 satisfy (2.9). Then letting $w = u_1 - u_2$, we have

$$\int_\Omega \frac{d}{dt}(w) v dx + \int_\Omega \Delta w \Delta v dx + \int_\Omega G(u_1,u_2) w v dx = 0$$

for all $v \in H_0^2(\Omega)$, where G is a polynomial in u_1 and u_2 of degree at most 2.

Taking $v = w$, we have

$$\frac{d}{dt}\int_\Omega |w|^2 dx + 2\int_\Omega |\Delta w|^2 dx \le C(\epsilon)\int_\Omega |w|^2 dx \tag{2.10}$$

for a constant $C(\epsilon)$ dependent on the $L_\infty(\Omega)$ norm of u_1 and u_2.

Since $w\big|_{t=0} = 0$, (2.10) implies $\int_\Omega |w|^2 dx \equiv 0$ for $t > 0$. That is, $u_1 \equiv u_2$. This completes the proof. ▪

The above result is not necessarily true for the steady state case. See Theorem 2.4 for a partial answer.

For a fixed $\epsilon > 0$, and when the boundary $\partial\Omega$ of the domain is smooth, the weak solution u we get in Theorem 2.1 is in fact a classical C^∞ smooth solution. This is the consequence of (2.9) and the regularity theory of the biharmonic heat equation (cf. [Grisvard'85]).

Furthermore, let us discuss the asymptotic behavior of $(1.1)'$ as $t \to +\infty$. We have

Theorem 2.3. *Suppose that $u_0 \in H^2(\Omega)$. The solution of $(1.1)'$ subsequently converges to the solution of the steady state problem as $t \to +\infty$. That is, for almost all sequence $\{t_j\}$ with $t_j \longrightarrow +\infty$, there exists a subsequence convergent to a weak solution of the steady state problem.*

Proof: By Lemma 2.1, we have

$$\begin{aligned}\int_0^T \int_\Omega |u_t|^2 dx dt &= -\int_\Omega |\Delta u(x,T)|^2 + \frac{2}{\epsilon^2}(|u(x,T)|^2 - 1)^2) dx \\ &\quad + \int_\Omega (|\Delta u(x,0)|^2 + \frac{2}{\epsilon^2}(|u_0|^2 - 1)^2) dx \\ &\le \int_\Omega (|\Delta u(x,0)|^2 + \frac{2}{\epsilon^2}(|u_0|^2 - 1)^2) dx\end{aligned}$$

Since the solution u is smooth, we have $u_t \in L_2(0, +\infty, L_2(\Omega))$. This implies

$$\int_\Omega |u_t(x,t_j)|^2 dx \to 0$$

for almost all sequence $\{t_j\}_{j\in Z_+} \subset (0, +\infty)$ with $t_j \to +\infty$. Moreover, by Fatou's Lemma, $u_t(x, t_j)$ converges weakly to zero in $L_2(\Omega)$.

Furthermore, Lemma 2.1 also implies

$$\int_\Omega (|\Delta u|^2 + \frac{1}{\epsilon^2}(|u|^2 - 1)^2) dx \le \int_\Omega (|\Delta u_0|^2 + \frac{1}{\epsilon^2}(|u_0|^2 - 1)^2) dx$$

for any $t \in (0, +\infty)$. That is, $\Delta u \in L_\infty(0, +\infty, L_2(\Omega))$. Since $\displaystyle\int_\Omega |u|^4 dx = \int_\Omega (|u|^2 - 1 + 1)^2 dx \le 2\int_\Omega (|u|^2 - 1)^2 dx + 2|\Omega|$, we have $\int_\Omega |u|^4 dx \le C(\epsilon) < +\infty$ for any $t \in (0, +\infty)$. Thus, $u \in L_\infty(0, +\infty, L^4(\Omega))$.

It now follows that $u \in L_\infty(0, +\infty, H^2(\Omega))$. There exists a subsequence $\{u(x,t_j), j \in Z\}$ convergent weakly in $H^2(\Omega)$. For simplicity, let us say $u(x,t_j) \to u^*(x)$ weakly in $H^2(\Omega)$. Moreover, $u(x,t_j) \to u^*(x)$ in $L_p(\Omega)$, $\forall p > 2$.

By passing $t_j \to +\infty$ in the following equation,

$$\int_\Omega \frac{\partial}{\partial t} u(x,t_j)v(x)dx + \int_\Omega \left[\Delta u(x,t_j)\Delta v(x) + \frac{1}{\epsilon^2}(|u(x,t_j)|^2 - 1)u(x,t_j)v(x)\right] dx$$
$$= \int_\Omega f(u(x,t_j))v(x)dx,$$

we have obtained

$$\int_\Omega \left(\Delta u^*(x)\Delta v + \frac{1}{\epsilon^2}(|u^*|^2 - 1)u^* v\right) dx$$
$$= \int_\Omega f(u^*)v(x)dx$$

or u^* is a solution of the steady state problem. This completes the proof. ∎

We now study the properties of the weak solution u of the steady state problems satisfying

$$\int_\Omega \left[\Delta u\ \Delta v dx + \frac{1}{\epsilon^2}(|u|^2 - 1)uv\right] dx = \int_\Omega f(u)v dx \tag{2.11}$$

for $v \in H_0^2(\Omega)$. We have

Lemma 2.5. *Let u be a weak solution satisfying (2.11). Then, for a positive constant $C(\epsilon)$,*

$$\int_\Omega |\Delta u|^2 dx \le C(\epsilon)|\Omega| \tag{2.12}$$

Proof: Let $v = u$ in (2.11) and recall $m = \max_{x\in\Omega} |\phi| < \infty$, we have

$$\begin{aligned}
\int_\Omega |\Delta u|^2 dx + \frac{1}{\epsilon^2}\int_\Omega |u|^4 dx
&\le \frac{1+3m^2}{\epsilon^2}\int_\Omega |u|^2 dx + \frac{3m}{\epsilon^2}\int_\Omega |u|^3 dx + \frac{m(1+m^2)}{\epsilon^2}\int_\Omega |u| dx \\
&\le \frac{1+3m^2}{\epsilon^2}\int_\Omega |u|^2 dx + \frac{3m}{\epsilon^2}\left(\int_\Omega |u|^4 dx\right)^{\frac{1}{2}}\left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}} \\
&\quad + \frac{m(1+m^2)}{\epsilon^2}\left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}}\sqrt{|\Omega|} \\
&\le \frac{C_1}{\epsilon^2}\int_\Omega |u|^2 dx + \frac{1}{2\epsilon^2}\int_\Omega |u|^4 dx + \frac{C_2}{\epsilon^2}|\Omega|
\end{aligned}$$

for some constants C_1 and C_2. Thus

$$\begin{aligned}\int_\Omega |\Delta u|^2 dx + \frac{1}{2\epsilon^2}\int_\Omega |u|^4 dx &\leq \frac{C_1}{\epsilon^2}\int_\Omega |u|^2 dx + \frac{C_2}{\epsilon^2}|\Omega| \\ &\leq \frac{C_1}{\epsilon^2}\left(\int_\Omega |u|^4 dx\right)^{1/2}\sqrt{|\Omega|} + C_2|\Omega| \\ &\leq \frac{1}{2\epsilon^2}\int_\Omega |u|^4 dx + \frac{C_1^2}{2\epsilon^2}|\Omega| + \frac{C_2}{\epsilon^2}|\Omega|.\end{aligned}$$

The inequality (2.12) thus follows. ∎

Regarding the uniqueness of the weak solution u of the steady state problem, we have

Theorem 2.4. *If ϵ is not very small, i.e., $\epsilon > K^2$, then the weak solution u of* (2.11) *is unique, where K is the Poincaré constant dependent on Ω.*

Proof: Suppose that there exists two solutions u_1 and u_2 in $H_0^2(\Omega)$. Let

$$a_2(\phi,\psi) = \int_\Omega \Delta\phi\Delta\psi dx \text{ and } b(\theta,\phi,\psi) = \frac{1}{\epsilon^2}\int_\Omega (|\theta|^2 - 1)\phi\psi dx.$$

Then we have

$$\begin{aligned}&a_2(u_1 - u_2, \psi) + b(u_1, u_1, \psi) - b(u_2, u_2, \psi) \\ &= \frac{-3}{\epsilon^2}\int_\Omega \phi(u_1^2 - u_2^2)\psi dx + \frac{-3}{\epsilon^2}\int_\Omega \phi^2(u_1 - u_2)\psi dx.\end{aligned}$$

Let $\psi = u_1 - u_2 \in H_0^2(\Omega)$. Then

$$\begin{aligned}&\epsilon^2(a_2(u_1 - u_2, \psi) + b(u_1, u_1\psi) - b(u_2, u_2, \psi)) \\ =&\epsilon^2\int_\Omega |\Delta(u_1 - u_2)|^2 dx + \frac{1}{4}\int_\Omega |u_1 - u_2|^4 dx + \frac{3}{4}\int_\Omega |u_1^2 - u_2^2|^2 dx - \int_\Omega |u_1 - u_2|^2 dx\end{aligned}$$

and

$$-3\int_\Omega \phi(u_1^2 - u_2^2)\psi dx - 3\int_\Omega \phi^2(u_1 - u_2)\psi dx \leq \frac{3}{4}\int_\Omega |u_1^2 - u_2^2|^2.$$

Combining the above equality and inequality, we have

$$\begin{aligned}&\epsilon^2\int_\Omega |\Delta(u_1 - u_2)|^2 dx + \frac{1}{4}\int_\Omega |u_1 - u_2|^4 dx \\ &\leq \int_\Omega |u_1 - u_2|^2 dx \leq K^4\int_\Omega |\Delta(u_1 - u_2)|^2 dx\end{aligned}$$

where we have used the Poincaré inequality twice. Thus, if $\epsilon > K^2$, then $u_1 \equiv u_2$. This completes the proof. ∎

We will end the section by looking at the self-dual solutions of the equation (1.1) in the 1–D stationary case. For simplicity, we set $\epsilon = 1$. That is, since $u'' = u^2 - 1$, we have

$$\frac{1}{2}(u')^2 = \frac{1}{3}u^3 - u + C.$$

Clearly, the above solution can not be an odd function. If the boundary conditions are 1 and -1, this hints that the sharp interface is unlikely to appear as $\epsilon \longrightarrow 0$. This is different from the solution of the self-dual of the energy of (1.6) in the 1-D case. Our numerical simulation of the stationary (1.1) in the 2-D boundary value problem also is consistent with the above observation.

§3. The Study of Equation (1.2)

In this section, we study the existence, uniqueness, and regularity of the solutions of nonlinear biharmonic equation (1.2). The higher order nonlinearity of (1.2) than (1.1) will play a crucial role in the analysis as well as in our numerical simulation (cf. [Lai, Liu and Wenston'02]).

Again let $\varphi \in H^2(\Omega)$ be the weak solution of the following biharmonic equation

$$\begin{cases} \Delta^2\varphi = 0 & x \in \Omega \\ \varphi(x) = u_1(x) & x \in \partial\Omega \\ \frac{\partial}{\partial n}\varphi(x) = u_2(x) & x \in \partial\Omega. \end{cases}$$

Then $\hat{u} = u - \varphi$ satisfies the following

$$\begin{aligned} &-\Delta^2\hat{u} + \frac{1}{\varepsilon^2}\nabla\cdot((|\nabla\hat{u}|^2 - 1)\nabla\hat{u}) \\ &+ \frac{1}{\varepsilon^2}\nabla\cdot((|\nabla\hat{u}|^2 - 1)\nabla\varphi) + \frac{2}{\varepsilon}\nabla\cdot((\nabla\hat{u}\cdot\nabla\varphi)\nabla\hat{u}) \\ &+ \frac{2}{\varepsilon^2}\nabla\cdot((\nabla\hat{u}\cdot\nabla\varphi)\nabla\varphi) + \frac{1}{\varepsilon^2}\nabla\cdot(|\nabla\varphi|^2\nabla\hat{u}) + \frac{1}{\varepsilon^2}\nabla\cdot(|\nabla\varphi|^2\nabla\varphi) \\ &= \hat{u}_t \end{aligned}$$

For convenience, write $u = \hat{u}$ and let

$$\begin{aligned} f(u) =& \frac{1}{\varepsilon^2}\nabla\cdot(|\nabla u|^2\nabla\varphi) + \frac{2}{\varepsilon^2}\nabla\cdot(\nabla u\cdot\nabla\varphi\nabla u) \\ &+ \frac{2}{\varepsilon^2}\nabla\cdot(\nabla u\cdot\nabla\varphi\nabla\varphi) + \frac{1}{\varepsilon^2}\nabla\cdot(|\nabla\varphi|^2\nabla u) \\ &- \frac{1}{\varepsilon^2}\Delta\varphi + \frac{1}{\varepsilon^2}\nabla\cdot(|\nabla\varphi|^2\nabla\varphi). \end{aligned}$$

Then we may rewrite the above nonlinear biharmonic equation in the following simplified form

$$\text{(3.1)} \qquad \begin{cases} u_t + \Delta^2 u - \frac{1}{\varepsilon^2}\nabla\cdot((|\nabla u|^2 - 1)\nabla u) & = f(u) \\ u(x,0) = u_0(x) - \varphi(x) & x \in \Omega \\ u(x,t) = 0, \frac{\partial}{\partial n}u(x,t) = 0 & \text{on } \partial\Omega. \end{cases}$$

We now introduce the weak formulation: Find $u \in L_2(0,T,H_0^2(\Omega))$ s.t. $u(x,0) = u_0(x) - \varphi(x)$ and

$$\text{(3.2)} \qquad \int_\Omega u_t v dx = -\int_\Omega \left[\Delta u\Delta v + \frac{1}{\varepsilon^2}(|\nabla u|^2 - 1)\nabla u\nabla v\right] dx + \int_\Omega f(u) v dx$$

for all $v \in H_0^2(\Omega)$.

Suppose that Ω has smooth boundary $\partial\Omega$ and $u_1 \in H^{3/2+\sigma}(\partial\Omega)$, $u_2 \in H^{1/2+\sigma}(\partial\Omega)$ with $\sigma > 0$. Then

$$m = \max_{x\in\Omega} |\nabla\varphi(x)| < +\infty$$

(cf. [Grisvard'85]). We are now ready to prove the following L^2 estimate. Notice the differences between Lemma 2.2 and the following Lemma 3.1 due to the nonlinearity of (1.2).

Lemma 3.1. *Suppose that u is a weak solution of (3.2). Then there exists a constant $C(\varepsilon, m) > 0$ such that*

$$\int_\Omega |u|^2 dx + \int_0^t e^{C(\varepsilon,m)(t-s)} \int_\Omega |\Delta u|^2 dx ds \le e^{C(\varepsilon,m)t}|\Omega| + \int_\Omega |u_0|^2 dx.$$

Proof: Let $v = u$ in (3.2). We have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx = &-\int_\Omega \left(|\Delta u|^2 + \frac{1}{\varepsilon^2}(|\nabla u|^2 - 1)|\nabla u|^2\right) dx \\ &+ \int_\Omega f(u) u dx.\end{aligned}$$

That is,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \int_\Omega |\Delta u|^2 dx + \frac{1}{\varepsilon^2}\int_\Omega |\nabla u|^4 dx \\ = \int_\Omega f(u) u dx + \frac{1}{\varepsilon^2}\int_\Omega |\nabla u|^2 dx.\end{aligned}$$

Note that

$$\begin{aligned}\int_\Omega f(u) u dx = &-\frac{3}{\varepsilon^2}\int_\Omega |\nabla u|^2 \nabla\varphi\cdot\nabla u dx - \frac{2}{\varepsilon^2}\int_\Omega |\nabla u\cdot\nabla\varphi|^2 dx \\ &- \frac{1}{\varepsilon^2}\int_\Omega |\nabla\varphi|^2|\nabla u|^2 dx + \frac{1}{\varepsilon^2}\int_\Omega \nabla\varphi\nabla u dx \\ &- \frac{1}{\varepsilon^2}\int_\Omega |\nabla\varphi|^2\nabla\varphi\nabla u dx \\ \le &\frac{3m}{\varepsilon^2}\int_\Omega |\nabla u|^2|\nabla u| dx - \frac{2}{\varepsilon^2}\int_\Omega |\nabla u\cdot\nabla\varphi|^2 dx \\ &- \frac{1}{\varepsilon^2}\int_\Omega |\nabla\varphi|^2|\nabla u|^2 dx + \frac{m}{\varepsilon^2}\int_\Omega |\nabla u| dx \\ &+ \frac{m^3}{\varepsilon^2}\int_\Omega |\nabla u| dx.\end{aligned}$$

We have

$$\begin{aligned}\frac{3m}{\varepsilon^2}\int_\Omega |\nabla u|^2|\nabla u| dx &\le \frac{3m}{\varepsilon^2}\left(\int_\Omega |\nabla u|^4 dx\right)^{1/2}\left(\int_\Omega |\nabla u|^2 dx\right)^{1/2} \\ &\le \frac{1}{\varepsilon^2}\int_\Omega |\nabla u|^4 dx + \frac{9m^2}{4\epsilon^2}\int_\Omega |\nabla u|^2 dx\end{aligned}$$

and

$$\frac{m+m^3}{\varepsilon^2}\int_\Omega |\nabla u| dx \le \frac{m+m^3}{\varepsilon^2}\sqrt{|\Omega|}\left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}$$
$$\le \frac{m+m^3}{2\varepsilon^2}|\Omega| + \frac{m+m^3}{2\varepsilon^2}\int_\Omega |\nabla u|^2 dx.$$

Thus, we put the above inequalities together to have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \int_\Omega |\Delta u|^2 dx + \frac{1}{\varepsilon^2}\int_\Omega |\nabla\varphi|^2|\nabla u|^2 dx$$
$$+\frac{2}{\varepsilon^2}\int_\Omega |\nabla u\cdot\nabla\varphi|^2 dx$$
$$\le \frac{m+m^3}{2\varepsilon^2}|\Omega| + \left(\frac{9m^2}{4\epsilon^2}+\frac{m+m^3}{2\varepsilon^2}\right)\int_\Omega |\nabla u|^2 dx.$$

Note that

$$\int_\Omega |\nabla u|^2 dx \le C_1\int_\Omega |u|^2 dx + C_2\int_\Omega |\Delta u|^2 dx.$$

Thus, we have

$$\frac{d}{dt}\int_\Omega |u|^2 dx + \int_\Omega |\Delta u|^2 dx \le C(\varepsilon, m)|\Omega| + C(\varepsilon, m)\int_\Omega |u|^2 dx.$$

A standard argument now completes the proof of Lemma 3.1. ∎

Let $V_N = S^r_{3r}(\diamondplus)_N) \cap H^2_0(\Omega)$ be the finite dimensional subspace of $H^2_0(\Omega)$ introduced before. Let $u_N \in V_N$ be a weak solution satisfying

$$\int_\Omega \frac{\partial}{\partial t}(u_N)v dx = -\int_\Omega (\Delta u_N\Delta v + \frac{1}{\varepsilon^2}(|\Delta u_N|^2-1)\nabla u_N\nabla v)dx + \int_\Omega f(u_N)v dx \tag{3.3}$$

for all $v\in V_N$ with initial value $u_N(x,0) = u_{0,N}(x) - \varphi_N(x)$ which is a spline approximation of $u_0-\varphi$, e.g., $u_{0N}(x)-\varphi_N(x) = Q_N(u_0-\varphi)$ as in [Lai and Schumaker'98]. The equation (3.3) is a finite dimensional ODE system. We know that u_N exists for $t\in[0,T_N)$ with $T_N\le T$.

We need to show $T_N \ge T$. Otherwise, $T_N < T$ implies that u_N will blow up when $t\to T_N$. However, the proof of Lemma 3.1 shows that

Lemma 3.2. *Let $u_N \in V_N$ be a solution of (3.3) for $t\in[0,T_N)$. Then*

$$\int_\Omega |u_N|^2 dx + \int_0^t e^{C(\varepsilon,m)(t-s)}\int_\Omega |\Delta u_N|^2 dx ds \le e^{C(\varepsilon,m)t}|\Omega| + \int_\Omega |u_{0,N}-\varphi_N|^2 dx. \tag{3.4}$$

Since $u_{0,N}-\varphi \to u_0-\varphi$ in $L_2(\Omega)$, we have

$$u_N \in L_\infty(0,T,L_2(\Omega)) \quad\text{and}\quad \Delta u_N \in L_2(0,T,L_2(\Omega)).$$

These facts imply $u_N \in L_2(0,T,H^2(\Omega))$.

We know, by Rellich's compactness theorem, there exists a subsequence in $\{u_N\}_{N\in Z_+}$, say $\{u_N\}$ convergent weakly in $H^2(\Omega)$ and strongly in $W^{1,p}(\Omega)$ to u for $p<+\infty$.

Note that

$$\begin{aligned}
&\left|\int_\Omega |\nabla u_N|^2\nabla u_N\nabla v dx-\int_\Omega |\nabla u|^2\nabla u\nabla v dx\right|\\
&\leq \left|\int_\Omega |\nabla u_N|^2(\nabla u_N-\nabla u)\nabla v dx\right|+\left|\int_\Omega (|\nabla u_N|^2-|\nabla u|^2)\nabla u\cdot\nabla v dx\right|\\
&\leq \left(\int_\Omega |\nabla u_N|^4dx\right)^{1/2}\left(\int_\Omega |\nabla u_N-\nabla u|^4dx\right)^{1/4}\left(\int_\Omega |\nabla v|^4dx\right)^{1/4}\\
&\quad+\left(\int_\Omega |\nabla u_N-\nabla u|^2dx\right)^{1/2}\left(\int_\Omega |\nabla u_N+\nabla u|^2|\nabla u\cdot\nabla v|^2dx\right)^{1/2}\\
&\to 0
\end{aligned}$$

as $N\to+\infty$ for each $t\in[0,T)$.

We next claim that $\int_\Omega f(u_N)vdx \to \int_\Omega f(u)vdx$.

First of all, we have

$$\begin{aligned}
&\left|\int_\Omega |\nabla u_N|^2\nabla\varphi\nabla v dx-\int_\Omega |\nabla u|^2\nabla\varphi\nabla u dx\right|\\
&\leq \left(\int_\Omega |\nabla u_N-\nabla u|^2dx\right)^{1/2}\left(\int_\Omega |\nabla u_N+\nabla u|^2|\nabla\varphi\nabla u|^2dx\right)^{1/2}\\
&\leq \left(\int_\Omega |\nabla u_N-\nabla u|^2dx\right)^{1/2}\left(\int_\Omega |\nabla u_N+\nabla u|^4\right)^{1/4}\left(\int_\Omega |\nabla\varphi\nabla v|^4dx\right)^{1/4}\\
&\leq \left(\int_\Omega |\nabla u_N-\nabla u|^2dx\right)^{1/2}\left(\left(\int_\Omega |\nabla u_N|^4dx\right)^{1/4}+\left(\int_\Omega |\nabla u|^4dx\right)^{1/4}\right)\\
&\quad\times\left(\int_\Omega |\nabla\varphi|^8dx\right)^{1/8}\left(\int_\Omega |\nabla v|^8dx\right)^{1/8}\\
&\longrightarrow 0,\ \text{as } N\to\infty.
\end{aligned}$$

Secondly, we have

$$\begin{aligned}
&\left|\int_\Omega \nabla u_N\cdot\nabla\varphi\nabla u_N\cdot\nabla v dx-\int_\Omega \nabla u\cdot\nabla\varphi\nabla u\cdot\nabla v dx\right|\\
&\leq \left|\int_\Omega \nabla u_N\cdot\nabla\varphi(\nabla u_N-\nabla u)\cdot\nabla v dx\right|+\left|\int_\Omega (\nabla u_N-\nabla u)\cdot\nabla\varphi\nabla u\cdot\nabla v dx\right|\\
&\leq 2\left(\int_\Omega |\nabla u_N-\nabla u|^2dx\right)^{1/2}\left(\int_\Omega |\nabla u_N|^4dx\right)^{1/4}\left(\int_\Omega |\nabla\varphi|^8dx\right)^{1/8}\times\\
&\qquad\left(\int_\Omega |\nabla v|^8dx\right)^{1/8}\\
&\longrightarrow 0,\ \text{as } N\to+\infty.
\end{aligned}$$

Thus, we have the claim. This completes the proof of the following

Theorem 3.1 (Existence). *For any fixed $T > 0$, there exists a weak solution u satisfying (3.2). By Lemma 3.1, $u \in L_\infty(0,T,L_2(\Omega) \cap L_2(0,T,H^2(\Omega))$. Hence, there exists a weak solution u for the original system (1.2).*

Next we study the uniqueness of the weak solution for each $\varepsilon > 0$. We have

Theorem 3.2(Uniqueness). *The weak solution in Theorem 3.1 is in fact unique.*

Proof: Suppose that u_1 and u_2 are two different solutions of (1.2). Letting $w = u_1 - u_2$, we have

$$\begin{aligned}(3.5)\qquad &\int_\Omega \frac{\partial}{\partial t}(w)v dx + \int_\Omega \Delta w \Delta v dx + \frac{1}{\varepsilon^2}\int_\Omega |\nabla u_1|^2 \nabla u_1 \nabla v dx \\ &- \frac{1}{\varepsilon^2}\int_\Omega |\nabla u_2|^2 \nabla u_2 \nabla v dx - \frac{1}{\varepsilon^2}\int_\Omega \nabla w \nabla v dx = 0\end{aligned}$$

for all $v \in H_0^2(\Omega)$. That is, we have

$$\begin{aligned}&\int_\Omega \frac{\partial}{\partial t}(w)v dx + \int_\Omega \Delta w \Delta v dx \\ &+ \frac{1}{\varepsilon^2}\int_\Omega |\nabla u_1|^2 \nabla w \nabla v dx + \frac{1}{\varepsilon^2}\int_\Omega \nabla u_2 \cdot \nabla w \nabla u_2 \cdot \nabla v dx \\ =&\frac{1}{\varepsilon^2}\int_\Omega \nabla w \nabla v dx - \frac{1}{\varepsilon^2}\int_\Omega \nabla w \cdot \nabla u_1 \nabla u_2 \cdot \nabla v dx.\end{aligned}$$

Let $v = w$. We have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int_\Omega w^2 dx\right) + \int_\Omega |\Delta w|^2 dx + \frac{1}{\varepsilon^2}\int_\Omega |\nabla u_1|^2 |\nabla w|^2 dx + \frac{1}{\varepsilon^2}\int_\Omega |\nabla u_2 \cdot \nabla w|^2 dx \\ &= \frac{1}{\varepsilon^2}\int_\Omega |\nabla w|^2 dx - \frac{1}{\varepsilon^2}\int_\Omega (\nabla w \cdot \nabla u_1 \nabla w \cdot \nabla u_2) dx.\end{aligned}$$

Similarly, letting $\widetilde{w} = u_2 - u_1$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \widetilde{w}^2 dx\right) + \int_\Omega |\Delta \widetilde{w}|^2 dx + \frac{1}{\varepsilon^2}\int_\Omega |\nabla u_2|^2 |\nabla \widetilde{w}|^2 dx + \frac{1}{\varepsilon^2}\int_\Omega |\nabla u_1 \cdot \nabla \widetilde{w}|^2 dx \\ &= \frac{1}{\varepsilon^2}\int_\Omega |\nabla \widetilde{w}|^2 dx - \frac{1}{\varepsilon^2}\int_\Omega (\nabla \widetilde{w} \cdot \nabla u_1 \nabla \widetilde{w} \cdot \nabla u_2) dx.\end{aligned}$$

Since $|w| = |\widetilde{w}|$, we add the above two equations together to get

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega w^2 dx\right) + 2\int_\Omega |\Delta w|^2 dx + \frac{1}{\varepsilon^2}\int_\Omega |\nabla u_1|^2 |\nabla w|^2 dx + \\ &\frac{1}{\varepsilon^2}\int_\Omega |\nabla u_2|^2 |\nabla w|^2 dx + \frac{1}{\varepsilon^2}\int_\Omega |\nabla u_1 \cdot \nabla w|^2 dx + \frac{1}{\varepsilon^2}\int_\Omega |\nabla u_2 \cdot \nabla w|^2 dx \\ =&\frac{2}{\varepsilon^2}\int_\Omega |\nabla w|^2 dx - \frac{2}{\varepsilon^2}\int_\Omega (\nabla w \cdot \nabla u_1 \nabla w \cdot \nabla u_2) dx \\ \le&\frac{2}{\varepsilon^2}\int_\Omega |\nabla w|^2 dx + \frac{1}{\varepsilon^2}\left(\int_\Omega (\nabla w \cdot \nabla u_1)^2 dx + \int_\Omega (\nabla w \cdot \nabla u_2)^2 dx\right).\end{aligned}$$

That is, we have

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega w^2 dx\right) + 2\int_\Omega |\Delta w|^2 dx + \\ &\quad \frac{1}{\varepsilon^2}\int_\Omega |\nabla u_1|^2 |\nabla w|^2 dx + \frac{1}{\varepsilon^2}\int_\Omega |\nabla u_2|^2 |\nabla w|^2 dx \\ \leq &\frac{2}{\varepsilon^2}\int_\Omega |\nabla w|^2 dx \\ \leq &C(\epsilon)\int_\Omega |w|^2 dx + \int_\Omega |\Delta w|^2 dx \end{aligned}$$

for a constant $C(\epsilon)$ dependent only on ϵ. Thus, we multiply $e^{-C(\epsilon)t}$ both sides of the inequality above to get

$$\frac{d}{dt}\left(e^{-C(\epsilon)t}\int_\Omega |w|^2 dx\right) \leq 0.$$

Since $\int_\Omega |w|^2 dx = 0$ when $t = 0$ we have

$$e^{-C(\epsilon)t}\int_\Omega |w|^2 dx \leq 0$$

for all $t \in [0, T]$. Hence $\int_\Omega |w|^2 dx = 0$ for all $t \in [0, T]$. That is, $u_1 \equiv u_2$. We have thus complete the proof. ∎

Furthermore, let us discuss the asymptotic behavior of (1.2) as $t \to +\infty$. We have

Lemma 3.3. *Let $u(x, t)$ be a smooth solution of (1.2). Then the following equality holds*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (|\Delta u|^2 + \frac{1}{2\varepsilon^2}(|\nabla u|^2 - 1)^2) dx = -\int_\Omega |u_t|^2 dx \tag{3.6}$$

The proof is the same as that of Lemma 2.1. We may omit the details. With the above lemma, we can prove the following

Theorem 3.3. *Suppose that $u_0 \in H^2(\Omega)$. Then the smooth solution of (1.2) subsequently converges to the solution of the steady state problem on $t \to +\infty$.*

Proof: By Lemma 3.3, we have

$$\int_0^T \int_\Omega |u_t|^2 dx dt \leq \int_\Omega \left(|\Delta u_0|^2 + \frac{2}{\varepsilon^2}(|\nabla u_0|^2 - 1)^2\right) dx$$

which is true for any $t \in [0, +\infty)$. This implies $u_t \in L_2(0, +\infty, L_2(\Omega))$. That is,

$$\int_\Omega |u_t(x, t_j)|^2 dx \to 0$$

as $t_j \to +\infty$. This implies $u_t(x, t_j)$ converges to zero weakly in $L_2(\Omega)$.

Another application of Lemma 3.3 yields

$$\int_\Omega (|\Delta u(x,t)|^2 + \frac{1}{\varepsilon^2}(|\nabla u(x,t)|^2-1)^2)dx \le \int_\Omega (|\Delta u_0|^2 + \frac{1}{\varepsilon^2}(|\nabla u_0|^2-1)^2)dx$$

for any $t \in [0,+\infty)$. That is, $\Delta u \in L_\infty(0,+\infty, L_2(\Omega))$. It also follows that $\nabla u \in L_\infty(0,+\infty, L_4(\Omega))$. By Poincaré's inequality we have

$$\int_\Omega |u|^2 dx \le K\int_\Omega |\nabla u|^2 dx \le K|\Omega|^{1/2}\left(\int_\Omega |\nabla u|^4 dx\right)^{1/2}$$

which implies $u \in L_\infty(0,+\infty, L_2(\Omega))$. Together with the facts $\Delta u \in L_\infty(0,+\infty, L_2(\Omega))$ and $\nabla u \in L_\infty(0,+\infty, L_2(\Omega))$ we know that $u \in L_\infty(0,+\infty, H^2(\Omega))$.

The boundedness of $\{u(x,t_j)\}_{j\in Z_+}$ in $H^2(\Omega)$ implies that there exists a subsequence in $\{u(x,t_j)\}_{j\in Z_+}$, say $\{u(x,t_j)\}_{j\in Z_+}$ converges to $u^* \in H^2(\Omega)$ weakly. By Rellich's compactness theorem, $\{\nabla u(x,t_j)\}_{j\in Z_+}$ converges to ∇u^* in $L_p(\Omega)$, $p \ge 2$. Recall the fact that $u_j(x,t_j)$ converges to zero weakly in $L_2(\Omega)$. By passing $t_j \to +\infty$ in the following

$$\int_\Omega u_t(x,t_j)v dx + \int_\Omega \left[\Delta u(x,t_j)\Delta v + \frac{1}{\varepsilon^2}(|\nabla u(x,t_j)|^2-1)\nabla u(x,t_j)\nabla v\right) dx$$
$$= \int_\Omega f(u(x,t_j))v dx,$$

we obtain

$$\int_\Omega \left[\Delta u^*\Delta v + \frac{1}{\varepsilon^2}\left(|\nabla u^*|^2-1\right)\nabla u^* v\right] dx = \int_\Omega f(u^*)v dx.$$

That is u^* is a solution of the steady state problem. This completes the proof. ∎

We notice that in general the weak solution of (1.2) may not be smooth enough to make a sense of (3.6) in Lemma 3.3.

We next study the properties of the weak solution u of the steady state problems satisfying

$$\int_\Omega \left[\Delta u \ \Delta v dx + \frac{1}{\epsilon^2}(|\nabla u|^2-1)\nabla u\nabla v\right] dx = \int_\Omega f(u)v dx \tag{3.7}$$

for $v \in H_0^2(\Omega)$. We have the following

Lemma 3.4. *Let u be a weak solution satisfying (3.7). Then, for a positive constant $C(\epsilon)$,*

$$\int_\Omega |\Delta u|^2 dx \le C(\epsilon)|\Omega|. \tag{3.8}$$

The proof is similar to that of Lemma 2.5. We omit the detail. Regarding the uniqueness of the weak solution u of the steady state problem, we have

Theorem 3.4. *If ϵ is not very small, i.e., $\epsilon > K^2$ then the weak solution u of (3.7) is unique, where K is the Poincaré constant which is dependent on Ω.*

The proof is again similar to that of Theorem 2.4. We leave the detail to the interest reader.

§4. Some Numerical Examples

We want to end the paper by presenting some numerical examples in this section. The examples here are the stationary solutions of equations (1.1) and (1.2). The solutions indicate many interesting nonlinear effects.

Example 1. *We simulate the solution of*

$$\begin{cases} \Delta^2 u + \frac{1}{\epsilon^2}(|u|^2 - 1)u = 0 & \\ u|_{2\Omega} = 1 & \text{on the upper half domain} \\ u|_{2\Omega} = -1 & \text{on the lower half domain} \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0 & \end{cases} \tag{4.1}$$

with $\Omega = [0, 1/2] \times [0, 1]$. The following graphs are numerical solution for $\epsilon = \sqrt{0.001}$ and $\epsilon = \sqrt{0.0001}$.

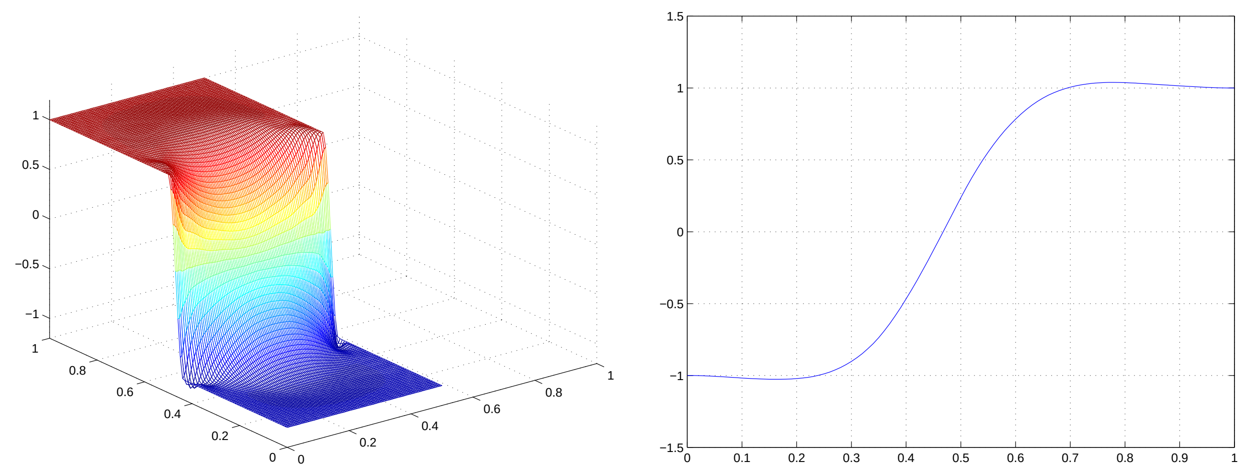

Fig. 4.1. Numerical solution of equation (4.1) and the middle cross section with $\epsilon = \sqrt{0.001}$

We observed that the surfaces pop up in the middle of Ω which are not like the solution of (1.7) which is subject to the maximum principle. We also observe that the portion of the surfaces transiting from 1 to -1 does not become narrow rapidly as $\epsilon \to 0$. This is another indication of the possible absence of the sharp interface, even with the help of the prescribed boundary conditions in (4.1).

Example 2. *We simulate the solution of*

$$\begin{cases} \Delta^2 u + \frac{1}{\epsilon^2}\nabla \cdot (|\nabla u|^2 - 1)\nabla u = 0, \\ u|_{\partial\Omega} = 0 \\ \frac{\partial}{\partial n} u\big|_{\partial\Omega} = 1 \end{cases} \tag{4.2}$$

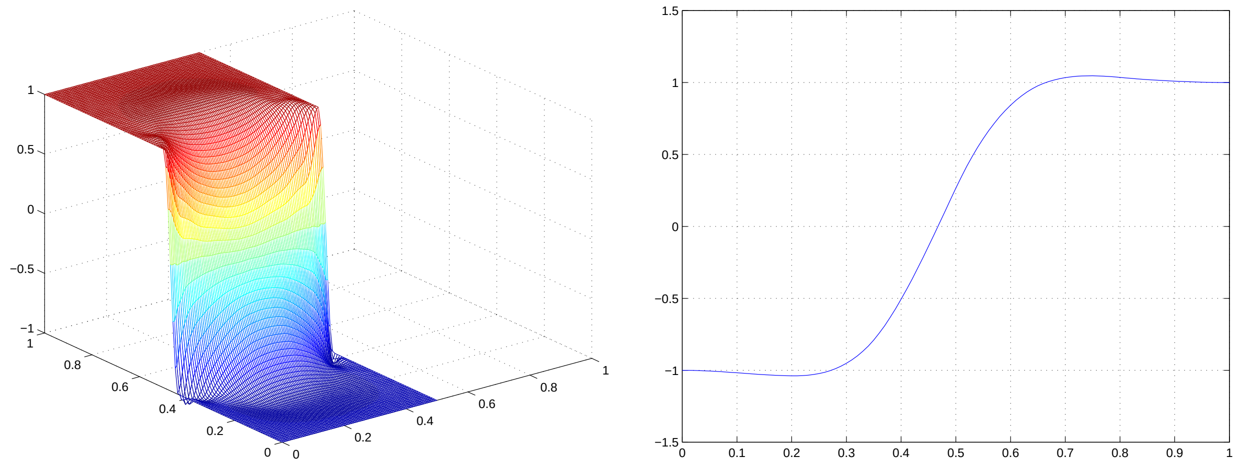

Fig. 4.2. Numerical solution of equation (4.1) and the middle cross section with $\epsilon = \sqrt{0.0001}$

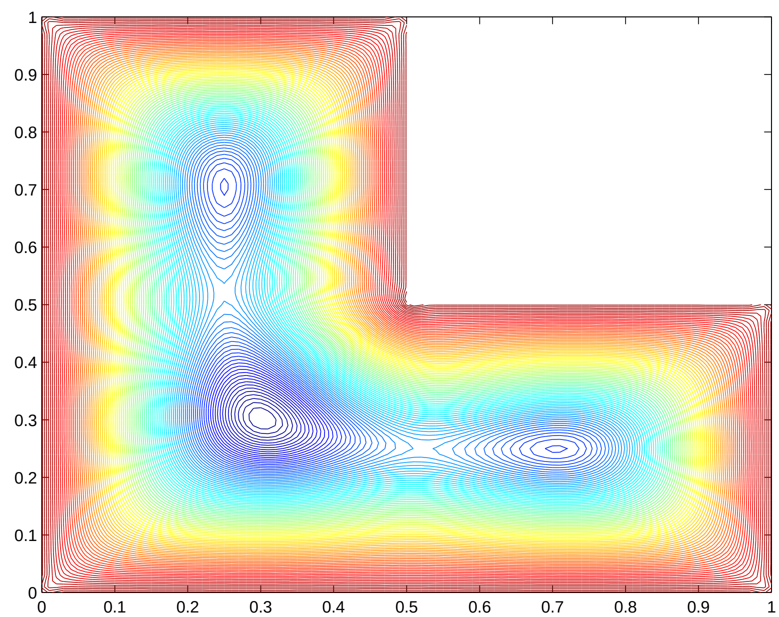

Fig. 4.3. The contour plot of the numerical solutions of equation (4.2) with $\epsilon = \sqrt{0.001}$

with Ω as shown in the figures below. The following graphs are numerical solution for $\epsilon = \sqrt{0.001}$ and $\epsilon = \sqrt{0.0001}$.

In the case of the L-shape domain (Fig 4.3 and Fig 4.4), the numerical solution indicates that as ϵ approaches zero, the solution approaches to the distance function to the domain boundary. We want to point out this result does not contradict to the results in [Carme Calderer, Liu and Voss'97] since all the boundaries of the domain are straight lines. In particular, the level set plot illustrates the defect lines coming out of the 5 corners and bisect the angles. Moreover, the center defect line consists of 2 straight line with 2 parabola segments. This is consistent with the special solution constructed in [Carme Calderer, Liu and Voss'98].

References

[1] S. Allen and J. Cahn, A microscopic theory for the antiphase boundary motion and

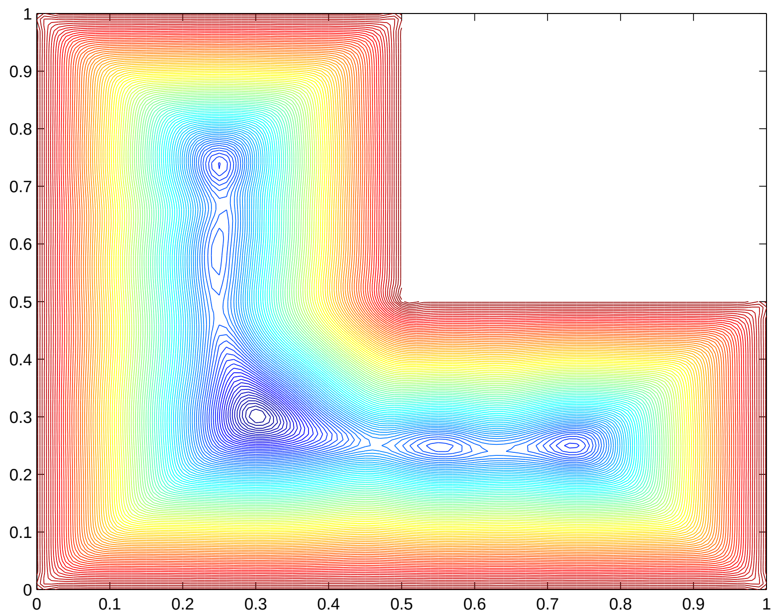

Fig. 4.4. The contour plot of the numerical solutions of equation (4.2) with $\epsilon = \sqrt{0.0001}$

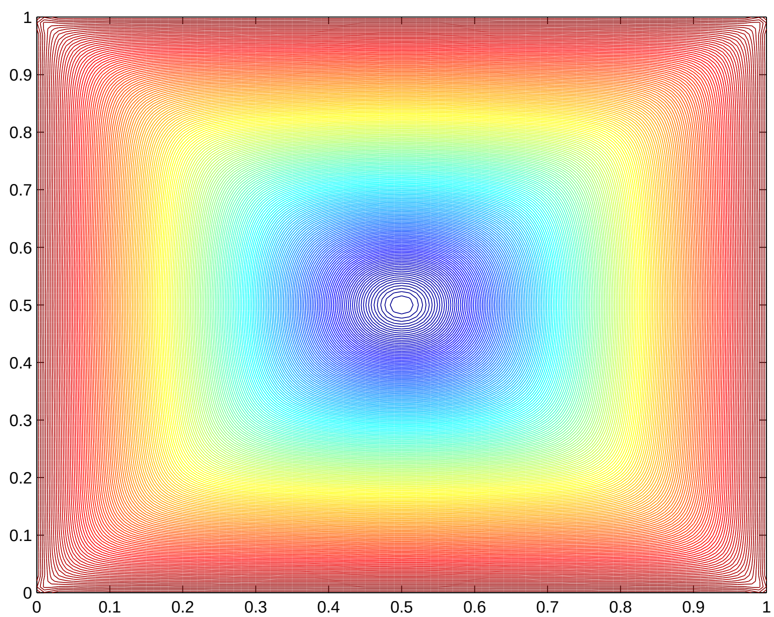

Fig. 4.5. The contour plot of the numerical solutions of equation (4.2) with $\epsilon = \sqrt{0.001}$

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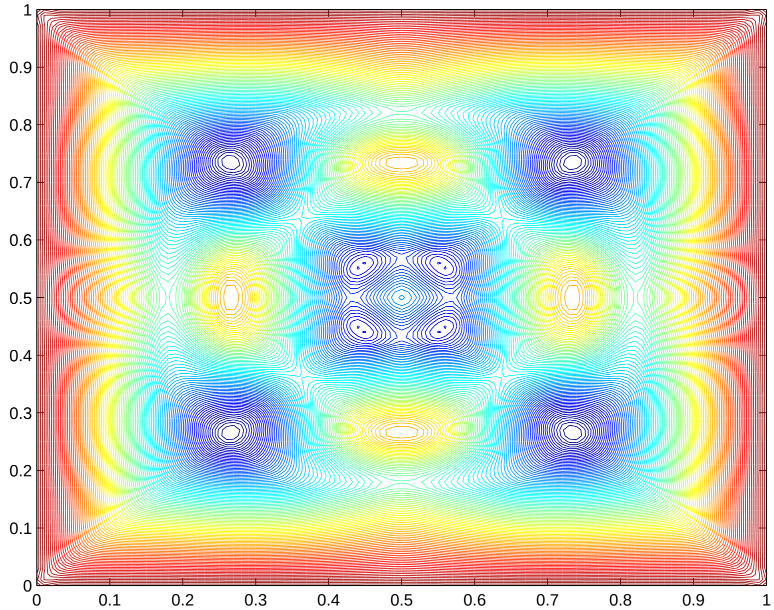

Fig. 4.6. The contour plot of the numerical solutions of equation (4.2) with $\epsilon = \sqrt{0.0001}$

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