Spherical Splines

for

Scattered Data Fitting

by

Victoria Baramidze

(Under the direction of Ming Jun Lai)

Abstract

We study properties of spherical Bernstein-Bézier splines. Algorithms for practical implementation of the global splines are presented for a homogeneous case as well as a non-homogeneous. Error bounds are derived for the global splines in terms of Sobolev type spherical semi-norms. Multiple star technique is studied for the minimal energy interpolation problem. Numerical summary supporting theoretical considerations is provided.

Index words: Spherical splines, Interpolation-on-the-sphere

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Chapter 1

Introduction

Suppose we are given a set $\mathcal{V}$ of locations on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ along with real numbers $f(v), v \in \mathcal{V}$ associated with these locations. We seek a smooth function defined on $\mathbb{S}^2$ interpolating or approximating these data. This constitutes a scattered data interpolation/approximation problem on the unit sphere. In this dissertation we solve the problem using spherical splines: piecewise spherical polynomials.

Data fitting problems have applications in geodesy, geometric surface design, food science, etc. Let us describe some examples.

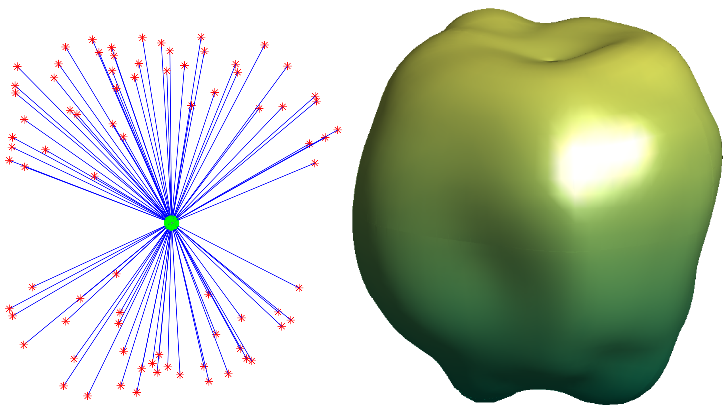

Figure 1.1: Cubic minimal energy spline interpolating apple data.

Example 1. In this example we measure Cartesian coordinates of points on the surface of an apple with the center of the coordinate system approximately corresponding to the center mass of the apple. Normalized coordinate triples produce scattered locations on the unit sphere. Distances from the origin to the points on the surface are the function values corresponding to the locations. In Fig. 1.1 (left) we show sampled points and directions in which they project onto the sphere. On the right we display a spherical spline interpolating the point cloud, which is a surface of an apple.

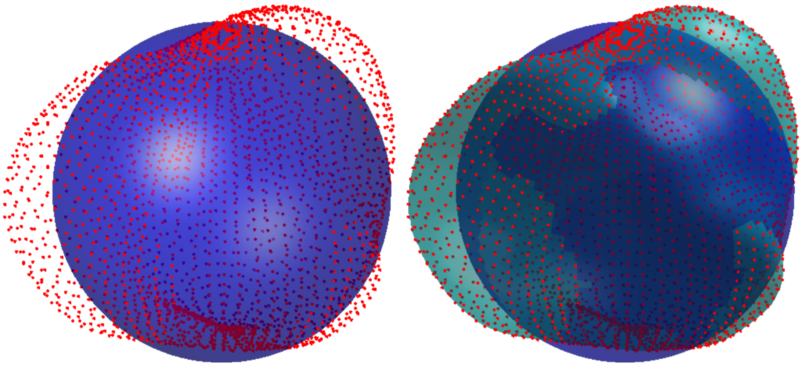

Figure 1.2: Minimal energy cubic spline interpolating data on $\mathbb{S}^2$.

Example 2. Consider Dirichlet's interior problem for the Laplace's equation on the unit ball

$$\begin{cases} \Delta V = 0 \text{ inside } \mathbb{S}^2 \\ V|_{\mathbb{S}^2} = f. \end{cases}$$

Given an arbitrary function f on $\mathbb{S}^2$ we need to determine a function V harmonic inside $\mathbb{S}^2$ and assuming the values of f on $\mathbb{S}^2$. The exact solution V to the interior problem $|u| < 1$ can be presented in terms of Poisson integral (cf. [Evans'98])

$$V(u) = \frac{1 - |u|^2}{4\pi} \int_{\mathbb{S}^2} \frac{f(v)}{|u - v|^3} d\sigma(v). \tag{1.1}$$

In practice the boundary function f may be available only as a discrete finite set of measurements at some scattered locations on $\mathbb{S}^2$. Having a spline approximation s of f (see Fig. 1.2) we can obtain

$$V_s(u) = \frac{1 - |u|^2}{4\pi} \int_{\mathbb{S}^2} \frac{s(v)}{|u - v|^3} d\sigma(v) \tag{1.2}$$

the approximation of V at any point interior to $\mathbb{S}^2$. By the maximum principle the error of this approximation is bounded by the error in the approximation of f by s:

$$\|V - V_s\|_{\infty,\text{ inside }\mathbb{S}^2} \leq \|f - s\|_{\infty,\mathbb{S}^2}$$

it is therefore essential to find a good approximation of f.

Example 3. In this example we are modeling a part of an aircraft from a point cloud in $\mathbb{R}^3$. Translate the cloud so that its center is located at the origin. Then directions from the origin to each point give us locations on the unit sphere, and distances between the origin and each point in the cloud give us the corresponding experimental function values. Note that the data locations do not have tensor structure, and therefore this fitting problem cannot be solved by tensor product splines. We solve the interpolation problem on a spherical triangulation using minimal energy method and display the spherical spline together with the point cloud in Fig. 1.3.

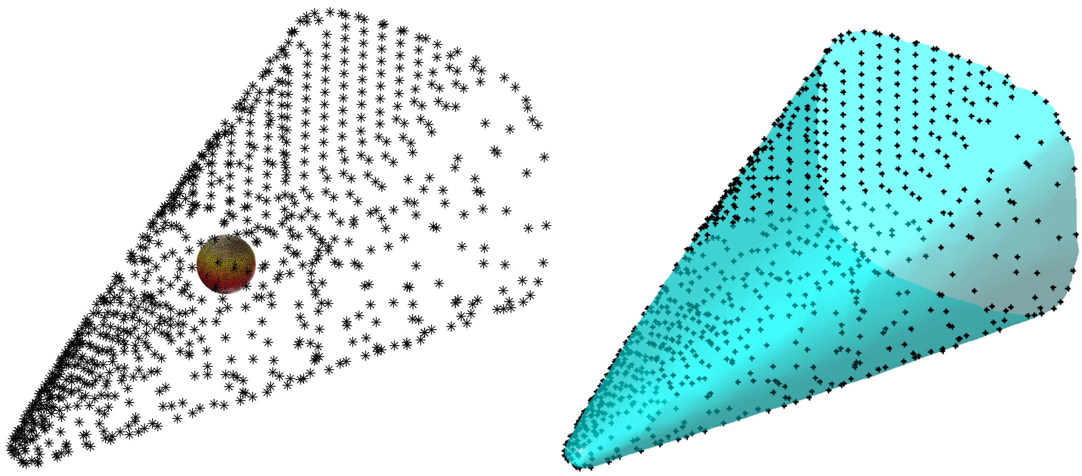

Figure 1.3: Minimal energy cubic interpolant of the point cloud

Example 4. Consider a blueberry exposed to convective heat [11]. A food scientist is interested in modeling the berry as it dries. The process can be described by heat equations with convective boundary conditions. Moisture and temperature on the surface are sampled at several locations and can be approximated everywhere on the surface by spherical splines. Consider a general homogeneous heat equation

$$u_t(v,t) - \Delta u(v,t) = 0$$

on the unit ball in $\mathbb{R}^3$ and time interval $(0,T]$ with an initial condition

$$u(v,0) = u_0(v), v \in B_1(0) \in \mathbb{R}^3$$

and a boundary condition

$$u(v,t) = g(v), v \in \mathbb{S}^2.$$

Using the Euler time discretization method [5] we subdivide the time interval $(0,T]$ into n subintervals of equal length $h = T/n$ with the right-hand side endpoints $t_k := kh, k = 1, ..., n$. Denote $u_k(v) := u(v,t_k)$. The time derivative is approximated by a forward difference

$$u_t(v,t_k) \approx \frac{u_k(v) - u_{k-1}(v)}{h}$$

therefore we obtain an iterational scheme

$$-h\Delta u_k(v) + u_k(v) = u_{k-1}(v).$$

This is a second order elliptic nonhomogeneous equation on the unit ball where $u_{k-1}(v)$ is known at the k-th step and $u_k(v)$ is to be solved for. The time discretization of the boundary conditions leads to

$$u_k(v)|_{\mathbb{S}^2} = g(v), k = 1, .., n.$$

Therefore at every time step we need to solve the problem

$$-\Delta U(v) + w^2 U(v) = F(v), v \in B_1(0)$$

subject to the boundary conditions

$$U(v)|_{\mathbb{S}^2} = G(v).$$

The function $F(v)$ is known everywhere on $B_1(0)$. The boundary conditions $G(v)$ in the berry problem are given as a set of scattered points coupled with heat/moisture values. These values can be approximated by spherical splines.

There are various ways to approximate scattered data on the sphere available in the literature, for example, [13]. Let us review two approaches for solving this problem. One approach is using splines on the plane and another spherical harmonics. The first step in approximating functions is to identify an approximating space well suited for the problem at hand.

Since the data fitting problem is well studied on the plane it is natural to attempt to map the sphere onto a planar rectangular region $[0, 2\pi] \times [0, \pi]$ in terms of spherical coordinates (θ, ϕ) and use well-known planar techniques to solve the problem. Note that we would like to use polynomial functions since their approximating properties are powerful and they are easy to compute along with their derivatives and integrals. It is not difficult to see what type of difficulties arise in this case. Functions on the sphere are naturally periodic, and polynomials in (θ, ϕ) are not. We need to pose extra conditions along the boundaries $\theta = 0, \theta = 2\pi$ to ensure continuity and differentiability of the resulting functions. The poles also pose a threat to the smoothness of the solution, as each of the poles on the sphere is mapped to entire line segment in (θ, ϕ) plane. Therefore the solution must be constant along the lines $\phi = 0$ and $\phi = \pi$.

A classical approach to the problem is to use spherical harmonics. It is obvious that for computational purposes one has to cut off the tail of the series and to have a finite number of coefficients to compute. The spherical harmonics are naturally periodic on the sphere and have powerful approximation properties. One disadvantage of using the spherical harmonics arises when one has to code applications involving them. The trigonometric polynomial representation of spherical harmonics requires a change from Cartesian coordinates to spherical coordinates. The formulas are not very convenient for coding: implicit formulas depend on the degree in such a way that it cannot be passed to a program as a parameter. Another disadvantage of using spherical harmonics for fitting problems is their global support; even a small change in a value of a coefficient will affect the resulting function everywhere else on the sphere. Also oscillatory behavior of the harmonics as their degree increases may not be desirable for applications.

A relatively new approach is to use spherical splines. A spherical spline is a piecewise polynomial function defined on a spherical triangulation. Let τ be a spherical triangle with vertices v_1, v_2, v_3. A homogeneous spherical Bernstein-Bézier (BB) polynomial p is defined on τ by

$$p = \sum_{i+j+k} c_{ijk} B^d_{ijk},$$

where

$$B^d_{ijk}(v) = \frac{d!}{i!j!k!} b_1(v)^i b_2(v)^j b_3(v)^k, i + j + k = d,$$

are called BB basis functions of degree d and b_i's satisfying

$$\sum_{i=1}^{3} b_i(v) v_i = v$$

are called spherical barycentric coordinates of v with respect to τ. To construct spherical splines we operate with BB-polynomials restricted to the domain of their definition τ, i.e. a spherical spline is a smooth piecewise BB-polynomial. A collection of all spherical splines of degree d and smoothness r defined on a triangulation Δ is denoted by $S^r_d(\Delta)$ and is called a spline space of degree d and smoothness r.

The BB spherical splines are analogous to the well known bivariate Bernstein-Bézier splines defined on planar triangulations. The bivariate BB splines are very convenient in solving data fitting problems. They have many attractive properties, such as derivative and integral representations, evaluation and subdivision algorithms, easy implementable smoothness conditions. The BB planar splines are wildly used in numerical methods for solving PDE's as well. Many of these properties are carried over to the spherical splines, which makes them a great tool in solving fitting problems on the sphere.

Let us now describe methods using BB spherical splines for solving data fitting problem. Note that with the polynomial definition above to solve the interpolation problem means to find appropriate values for the coefficients $c_{ijk}, i + j + k = d$ for every triangle. Local methods, such as Finite Element, assign values to the coefficients independently of the values on any other triangle. These methods are very fast and relatively simple. However they require more information than is often available. For example to construct a quintic C^1 macro-element for every triangle we need to know the function values at the vertices of τ, certain first and second order directional derivatives at the vertices, and first order directional derivatives at the midpoints of edges, totaling 21 data for every single triangle. These derivative values are estimated locally from the data available. Naturally, the approximation power of such a method depends on the accuracy of the derivative estimates, which may not be very high from the global point of view. Another disadvantage of using macro-elements is that they are degree dependent, i.e. for every degree and smoothness one has to construct and therefore program a new macro-element [14].

Alternatively there are global techniques for solving the interpolation/fitting problem. The three widely used methods [3] are:

Minimal energy interpolation,

Discrete least squares approximation,

Penalized least squares approximation.

These methods require simultaneous involvement of all coefficients of the spline. For the minimal energy interpolation problem we are given a set of locations $\mathcal{V} \in \mathbb{S}^2$ and corresponding values $f(v), v \in \mathcal{V}$. We need to find a function $s \in S_d^r(\Delta)$ satisfying $s(v) = f(v), v \in \mathcal{V}$. To find a unique interpolating spline we minimize an energy functional

$$\mathcal{E}_\delta(f) = \int_{\mathbb{S}^2} \sum_{|\alpha|=2} (D^\alpha f_\delta)^2 \tag{1.3}$$

over the set

$$\Gamma_f := \{s \in S_d^r(\Delta) : s(v) = f(v), \forall v \in \mathcal{V}\}.$$

Here f_δ is a homogeneous extension of f to $\mathbb{R}^3 \backslash \{0\}$ to be defined later.

In [3] the authors discussed several choices for the functional $\mathcal{E}$ above. We defined (1.3) by analogy with the bivariate thin plate energy functional. The necessary adjustment is to take a homogeneous extension of f to $\mathbb{R}^3 \backslash \{0\}$ of degree δ prior to differentiation. The value of δ is taken to be 1 if d is odd and 0 if d is even. This choice allows reproduction of linear homogeneous functions in odd degree spline spaces, and reproduction of constants in even degree spaces.

In case of discrete least squares splines a given data set is extremely large, e.g., $n \geq 10,000$ and highly redundant. Let $\mathcal{L}(s)$ be the discrete squares functional

$$\mathcal{L}(s) = \sum_{v \in \mathcal{V}} (s(v) - f(v))^2.$$

The discrete least squares spherical spline $S_f \in S_d^r(\Delta)$ is the function which minimizes the quantity $\mathcal{L}(s), s \in S_d^r(\Delta)$, i.e.

$$\mathcal{L}(S_f) = \min\{\mathcal{L}(s) : s \in S_d^r(\Delta)\}.$$

Penalized least squares fitting is used when data locations are not uniformly spaced over $\mathbb{S}^2$ and interpolation is not required. Let Δ be a regular triangulation of the unit sphere $\mathbb{S}^2$ whose vertices form a subset of $\mathcal{V}$. We seek a spline solution $S_f \in S_d^r(\Delta)$ satisfying

$$\mathcal{P}_\lambda(S_f) = \min\{P_\lambda(s) : s \in S_d^r(\Delta)\}$$

where λ is a positive weight and

$$\mathcal{P}_\lambda(s) := \mathcal{L}(s) + \lambda \mathcal{E}_\delta(s).$$

Here $\mathcal{L}$ is the least squares functional and $\mathcal{E}_\delta$ is the energy functional defined in (1.3).

For the spherical fitting problems these methods were first introduced in [3]. To obtain the unknown spline coefficients $\mathbf{c} := \{c^\tau_{ijk}, i+j+k=d, \tau \in \Delta\}$ in every case we have to solve a linear system of equations in the form

$$\begin{bmatrix} A & L^T \\ L & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \eta \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix},$$

where η is a vector of Lagrange multiplier coefficients. In [3] the global methods above were tested in cubic spaces. Authors note that high running time is devoted to solving associated linear system. These were solved by public domain matrix solvers, for example Y12M package. For the global methods the size of the linear system is the main determinator of the amount of space and time needed to run the program, and they were not able to run substantially larger problems. To deal with these restrictions we have two suggestions. One: the linear system can be solved using an iterational algorithm [6]

$$\begin{bmatrix} A & L^T \\ L & -\epsilon I \end{bmatrix} \begin{bmatrix} \mathbf{c}^{(\ell+1)} \\ \lambda^{(\ell+1)} \end{bmatrix} = \begin{bmatrix} F \\ G - \epsilon\lambda^{(\ell)} \end{bmatrix},$$

for $\ell = 0, 1, 2, \cdots$, where $\epsilon > 0$ is a fixed number, e.g. $\epsilon = 10^{-4}$, $\lambda^{(\ell)}$ is iterative solution of a Lagrange multiplier with $\lambda^0 = 0$ and I is the identity matrix. The above matrix iterative steps can in fact be rewritten as follows:

$$(A + \frac{1}{\epsilon}L^T L)\mathbf{c}^{(\ell+1)} = AF\mathbf{c}^{(\ell)} + \frac{1}{\epsilon}L^T G$$

with $\mathbf{c}^{(0)} = 0$.

It is proved in [6] that the matrix $A + \frac{1}{\epsilon}L^T L$ is always invertible for any $\epsilon > 0$ if A is symmetric and positive definite with respect to L in the sense that $A\mathbf{c} = 0$ and $L\mathbf{c} = 0$ imply that $\mathbf{c} = 0$. Also, under the assumption that A is symmetric and positive definite with respect to L, the vectors $\mathbf{c}^{(\ell)}$ converge to the solution $\mathbf{c}$: there exists a constant C such that

$$\|\mathbf{c}^{(\ell+1)} - \mathbf{c}\| \leq C\epsilon \|\mathbf{c}^{(\ell)} - \mathbf{c}\|$$

for all ℓ [6]. One of the advantages of this iterative method over the least square approach is that it involves only the inverse of the matrix $A + \frac{1}{\epsilon} L^T L$ instead of the SVD of the entire coefficient matrix of the singular linear system which is apparently more expensive.

Two: we apply the minimal energy interpolation method on certain subdomains of $\mathbb{S}^2$. Having data locations $\mathcal{V}$ construct a triangulation Δ. Divide Δ into clusters $\Omega_i, i = 1, .., n$. Enlarge each cluster by attaching adjacent triangles to create $\Omega_{i,k}$. Find the minimal energy interpolant $s_{f,i,k}$ on each $\Omega_{i,k}$, then use its restriction to Ω_i for the overall solution. We show that as the number k of triangle rings around Ω_i increases, the subdomain spline converges to the global minimal energy interpolant.

We remark, that one of disadvantages of using homogeneous spherical splines is that spline spaces of even and odd degrees have only the zero function in common due to homogeneity of the basis. More explicitly, $S^r_d(\Delta)$ with d odd does not contain constant functions, and $S^r_d(\Delta)$, d even, does not contain linear functions. It is well-known that on planar triangulations minimal energy splines with functionals involving second-order differential operators are capable of reproducing constants and linear functions. This is however not the case for homogeneous spherical splines, since $S^r_d(\Delta)$, d odd, simply does not contain constant polynomials. Therefore only homogeneous linear functions can be reproduced. Similarly, if d is even, we can only reproduce constants. Analogous problems hold in the case of discrete and penalized least square approximations.

It turns out that non-homogeneous splines can be constructed easily from the homogeneous splines, since on the unit sphere [19] the space $\mathcal{P}_d$ of polynomials of degree d has a direct sum decomposition

$$\mathcal{H}_d \oplus \mathcal{H}_{d-1} = \mathcal{P}_d$$

into two spaces $\mathcal{H}_d$ and $\mathcal{H}_{d-1}$ of homogeneous polynomials. Nonhomogeneous spline spaces allow reproduction of nonhomogeneous polynomials.

The dissertation is organized as follows. In Chapter 2 we introduce definitions and concepts related to spherical triangulations. We define spherical BB polynomials and thoroughly study their properties. We define a spherical analog of Sobolev spaces and associated semi-norms. A brief description of local bases and important results concerning quasi-interpolants are presented as well. These considerations set the foundation for error bounds derived in Chapter 3. The results of this chapter are applied to demonstrate the approximation power of the multiple star technique in Chapter 4. Chapter 5 is devoted to practical issues related to implementation of the global methods and multiple star technique. We conclude by presenting numerical experiments in Chapter 6.

Chapter 2

Preliminaries

2.1 Spherical triangulations

In this section we introduce basic notation and definitions used throughout the dissertation. Let $\mathbb{S}^2$ denote the unit sphere in $\mathbb{R}^3$. Given two points u, v on $\mathbb{S}^2$ that are not antipodal the shortest curve connecting them is an arc $\widehat{uv}$ of the the great circle through them. Given three points v_1, v_2 and v_3 on $\mathbb{S}^2$ such that the vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ form a basis for $\mathbb{R}^3$, a spherical triangle τ is a domain bounded by the arcs $\widehat{v_1v_2}$, $\widehat{v_2v_3}$ and $\widehat{v_3v_1}$, which are called edges of the spherical triangle τ. The points v_1, v_2 and v_3 are called vertices of τ.

Given a set $\mathcal{V}$ of points on $\mathbb{S}^2$ we can form a triangulation Δ: a collection of spherical triangles. We will assume that the triangulation Δ is regular in the sense that any two triangles do not intersect each other or else share either a common vertex or a common edge and every edge of Δ is shared by exactly two triangles.

Under the assumption that Δ is regular we can state the following properties of Δ.

- For Δ to exist the cardinality of $\mathcal{V}$ must be at least 4.
- The number of vertices $\#\mathcal{V}$ and the number of triangles $\#T$ are related as $\#T = 2(\#\mathcal{V} - 2)$.
- The number $\#E$ of edges of Δ is related to the number of triangles as $\#E = 3\#T/2$.

As for any spline space we need a notion of the size of a spherical partition.

Given a spherical triangle τ let $|\tau|$ denote the diameter of the smallest spherical cap containing τ and let ρ_τ denote the diameter of the largest spherical cap contained in τ. Then

$$|\Delta| = \max\{|\tau|, \quad \tau \in \Delta\}$$

$$\rho_\Delta = \min\{\rho_\tau, \quad \tau \in \Delta\}$$

are correspondingly the diameter of the largest triangle in Δ and the diameter of the smallest spherical cap inscribed in Δ.

Definition 2.1. Let β be a positive real number. A triangulation Δ is said to be β-quasi-uniform provided that

$$\frac{|\Delta|}{\rho_\Delta} \leq \beta.$$

It is well-known that in the planar case, the smallest angle of a quasi-uniform triangulation is bounded below by $1/\beta$ [20]. We make use of a concept of a natural radial projection developed in [23] to relate properties of planar quasi-uniform triangulations to the spherical ones. It will be clear from our construction that we need to bound triangulation size. In order for the results of [23] to be applicable we choose this bound to be 1.

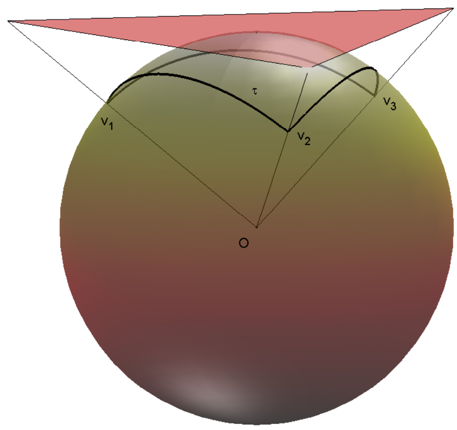

Figure 2.1: Radial projection.

Fix a spherical triangle τ with $|\tau| \le 1$. Define r_τ to be the center of a spherical cap of smallest possible radius containing τ, and let $\mathbf{T}_\tau$ be the tangent plane touching $\mathbb{S}^2$ at r_τ . We define the radial projection from $\mathbf{T}_\tau$ into $\mathbb{S}^2$ by

$$w := R_\tau \bar{w} := \frac{\bar{w}}{|\bar{w}|} \in \mathbb{S}^2, \bar{w} \in \mathbf{T}_\tau.$$

Since R_τ is one-to-one, R_τ^{-1} is well-defined. Let $\bar{\tau}$ be the image of τ under R_τ^{-1}. Let $\rho_{\bar{\tau}}$ and $|\bar{\tau}|$ be diameters of the inscribed and outscribed circles of $\bar{\tau}$ correspondingly. It is not too difficult to check that

$$\begin{aligned} |\tau| \le |\bar{\tau}| \le K_1|\tau|, \\ K_2^{-1}\rho_\tau \le \rho_{\bar{\tau}} \le K_2\rho_\tau, \end{aligned} \tag{2.1}$$

for some positive constants K_1 and K_2 (cf. [23]). However we make use of the following

LEMMA 2.1. Let τ be a spherical triangle with $|\tau| \le 1$. Let $\bar{\tau}$ denote the image of τ under the map R_τ^{-1}. Then

$$2\tan\frac{|\tau|}{2} = |\bar{\tau}| \tag{2.2}$$

and

$$2\tan\frac{\rho_\tau}{2} \le \rho_{\bar{\tau}}. \tag{2.3}$$

PROOF. By the definition of R_τ the center of the smallest spherical cap containing τ is the center of the circle outscribing $\bar{\tau}$. Let $\bar{v}$ be one of the vertices of $\bar{\tau}$. The center of the unit sphere O, $\bar{v}$ and r_τ form a right triangle with the leg Or_τ of length 1, the leg $\bar{v}r_\tau$ having length $\frac{|\bar{\tau}|}{2}$ and the angle $\angle\bar{v}Or_\tau$ having radian measurement $\frac{|\tau|}{2}$. Then (2.2) follows immediately.

The largest spherical cap σ contained in τ is mapped onto an ellipse ϵ in the plane $\mathbf{T}_\tau$ which is contained in $\bar{\tau}$. The largest circle $\bar{\sigma}$ contained in $\bar{\tau}$ has a radius $\frac{\rho_{\bar{\tau}}}{2}$ greater than or equal to r_ϵ - the radius of the largest circle contained in the ellipse. Let o be the center of σ and v be any point on the boundary $\delta\sigma$ of the cap. Let $\bar{o}$

and $\bar{v}$ be the images of o and v under R_τ^{-1} correspondingly. Then r_ϵ can defined by $r_\epsilon := \min_{v\in\delta\sigma}\{|\bar{o}-\bar{v}|\}$. Note now that

$$|\bar{o}-\bar{v}| \geq \tan|o-v|, \forall v \in \delta\sigma.$$

Therefore

$$\frac{\rho_{\bar{\tau}}}{2} \geq r_\epsilon \geq \tan\frac{\rho_\tau}{2}$$

and we have (2.3).

Note that since great circles are mapped into straight lines under the inverse of the radial projection R_τ, any cluster of spherical triangles ω with $|\omega| \leq 1$ is mapped into a planar triangulation $\bar{\omega}$.

LEMMA 2.2. Let Δ be a β-quasi-uniform triangulation of the unit sphere with $|\Delta| \leq 1$. Let Θ_Δ denote the smallest angle of Δ. There exists a constant A_1 such that

$$\Theta_\Delta \geq \frac{1}{A_1\beta}. \tag{2.4}$$

PROOF. Fix a spherical triangle $\tau \in \Delta$ and construct the radial projection R_τ. By Lemma 2.1 we have

$$\frac{|\bar{\tau}|}{\rho_{\bar{\tau}}} \leq \frac{\tan\frac{|\tau|}{2}}{\tan\frac{\rho_\tau}{2}} \leq 2\tan\frac{1}{2}\,\beta.$$

Since $\bar{\tau}$ is a planar triangle, its every angle is bounded below by $\frac{1}{A_1\beta}$ with $A_1 := 2\tan\frac{1}{2}$. Since the corresponding spherical angles are even greater (2.4) follows.

We will need another lemma comparing areas A_τ of spherical triangles to the size parameters $|\Delta|$ and ρ_Δ characterizing spherical triangulations.

LEMMA 2.3. For every spherical triangle $\tau \in \Delta$ with $|\Delta| \leq 1$

$$\frac{\pi\rho_\Delta^2}{5} \leq A_\tau \leq \frac{\pi|\Delta|^2}{4}. \tag{2.5}$$

PROOF. The area A_τ of a spherical triangle is bounded above by the area of the smallest spherical cap containing τ. The diameter of this cap is $|\tau|$. Without loss of generality we assume that the center of this cap is located at the north pole. Then

$$A_\tau \le \int_0^{2\pi} \int_0^{|\tau|/2} \sin \eta d\eta d\theta = 2\pi(1 - \cos(|\tau|/2)) \le \pi \frac{|\Delta|^2}{4}.$$

Similarly, A_τ is bounded below by the area of the largest spherical cap contained in τ, which by the definition has a diameter ρ_τ. Therefore

$$A_\tau \ge 2\pi(1 - \cos(\rho_\tau/2)) \ge \frac{\pi \rho_\Delta^2}{5}.$$

Another result that we need concerning β-quasi-uniform triangulations is a bound on the number of triangles n_k in the k-th disk around τ. We denote the union of all triangles in Δ that share the vertex v by $\mathrm{star}^1(v)$. Define recursively

$$\mathrm{star}^\ell(v) := \cup\{\mathrm{star}^1(w) : w \text{ is a vertex of } \mathrm{star}^{\ell-1}(v)\}, \ell > 1,$$

and

$$\mathrm{star}^\ell(\tau) := \cup\{\mathrm{star}^\ell(w) : w \text{ is a vertex of } \tau\}, \ell > 1.$$

LEMMA 2.4. Suppose Δ is a β-quasi-uniform triangulation such that $|\Delta| \le 1$. Then for any triangle $\tau \in \Delta$ and any $k \ge 0$ the number n_k of triangles in $\mathrm{star}^k(\tau)$ is

$$n_k \le \frac{5\beta^2}{4}(2k+1)^2, \tag{2.6}$$

and

$$n_k \ge \frac{2}{\pi\beta^2}(2k+1)^2. \tag{2.7}$$

PROOF. Note that $\mathrm{star}^k(\tau)$ is contained in a spherical cap of radius $R = (2k+1)\frac{|\Delta|}{2}$ and area $A_R = 2\pi(1 - \cos R)$. By Lemma 2.3 we have

$$\frac{\pi \rho_\Delta^2}{5} \le A_\tau.$$

Then

$$n_k \frac{\pi \rho_\Delta^2}{5} \le A_R = 2\pi(1 - \cos R) \le \pi R^2.$$

Therefore

$$n_k \leq \frac{5\beta^2(2k+1)^2}{4}.$$

On the other hand $\mathrm{star}^k(\tau)$ contains a spherical cap of radius $r = (2k+1)\frac{\rho_\Delta}{2}$ and area $A_r = 2\pi(1-\cos r)$. Then by Lemma 2.3

$$2r^2 \leq 2\pi(1-\cos r) = A_r \leq n_k \frac{\pi|\Delta|^2}{4},$$

therefore

$$n_k \geq \frac{2(2k+1)^2}{\pi\beta^2}.$$

2.2 Spherical barycentric coordinates

In this section we define an analog of planar barycentric coordinates on the sphere and analyze some of their properties. We start by introducing a special set of coordinates in $\mathbb{R}^3$ which will be used later to construct barycentric coordinates on the sphere.

Definition 2.2. Let $V := \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be a basis for $\mathbb{R}^3$. We call

$$\mathcal{T} := \{v \in \mathbb{R}^3 : \mathbf{v} = b_1\mathbf{v_1} + b_2\mathbf{v_2} + b_3\mathbf{v_3},\ b_i \geq 0\} \tag{2.8}$$

the trihedron generated by V. Each $\mathbf{v} \in \mathbb{R}^\mathbf{3}$ can be written in the form

$$\mathbf{v} = b_1\mathbf{v_1} + b_2\mathbf{v_2} + b_3\mathbf{v_3}. \tag{2.9}$$

We call b_1, b_2, b_3 the trihedral coordinates of v with respect to V. Equation (2.9) defining the trihedral coordinates can be written as a system of three equations for b_i's:

$$\begin{bmatrix} v_1^x & v_2^x & v_3^x \\ v_1^y & v_2^y & v_3^y \\ v_1^z & v_2^z & v_3^z \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} v^x \\ v^y \\ v^z \end{bmatrix},$$

where v^x denotes the x-coordinate of $\mathbf{v}$, etc. The matrix above is nonsingular since $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent. Using Cramer's rule we immediately have

$$b_1 = \frac{\det(v, v_2, v_3)}{\det(v_1, v_2, v_3)}, \; b_2 = \frac{\det(v_1, v, v_3)}{\det(v_1, v_2, v_3)}, \; b_3 = \frac{\det(v_1, v_2, v)}{\det(v_1, v_2, v_3)}, \tag{2.10}$$

where

$$\det(v_1, v_2, v_3) = \det \begin{bmatrix} v_1^x & v_2^x & v_3^x \\ v_1^y & v_2^y & v_3^y \\ v_1^z & v_2^z & v_3^z \end{bmatrix}$$

and so forth. Equations above show that the b_i's are ratios of volumes of tetrahedra.

The concept of homogeneity plays an important role in the construction of spline functions we are going to use. Let us present a formal definition and relate it to trihedral coordinates.

Definition 2.3. A trivariate function F is said to be homogeneous of degree n provided that for every real number $\alpha \neq 0$,

$$F(\alpha v) = \alpha^n F(v), \quad v \in \mathbb{R}^3 \backslash \{0\}. \tag{2.11}$$

Clearly, for all $\alpha \in \mathbb{R}$, $b_i(\alpha v) = \alpha b_i(v)$, $i = 1, 2, 3$, which implies that the b_i's are homogeneous linear functions of v of degree of homogeneity 1.

We summaries some additional properties of trihedral coordinates in the following

Lemma 2.5.

1) $\{b_i(v), i = 1, 2, 3\}$ is a linearly independent set,

2) If $\mathcal{L}$ is the space of trivariate linear homogeneous polynomials, then $\mathcal{L} = \text{span}\{b_1, b_2, b_3\}$,

3) $b_i(v_j) = \delta_{ij}, i, j = 1, 2, 3$,

4) $b_i(v) > 0$ for all v in the interior of trihedron $\mathcal{T}$,

Proof.

1) Suppose there are scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1 b_1(v) + \alpha_2 b_2(v) + \alpha_3 b_3(v) = 0, \forall v \in \mathbb{R}^3. \tag{2.12}$$

Define $\mathbf{v_0} := \alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2} + \alpha_3 \mathbf{v_3}$. By uniqueness of trihedral coordinates, we must have

$$\alpha_i = b_i(v_0), i = 1, 2, 3.$$

Then (2.12) implies

$$\sum_{i=1}^{3} \alpha_i^2 = 0,$$

and thus $\alpha_i = 0, i = 1, 2, 3$.

2) Since b_i's are homogeneous linear functions, clearly

$$\text{span}\{b_1, b_2, b_3\} \subset \mathcal{L}$$

Let $P(x, y, z) = ax + by + cz + d \in \mathcal{L}$. Since $P(x, y, z)$ is linearly homogeneous $P(\alpha x, \alpha y, \alpha z) = \alpha P(x, y, z), \forall \alpha \in \mathbb{R}$. Choose $\alpha \neq 1$. Then we must have

$$\alpha(ax + by + cz) + d = \alpha(ax + by + cz) + \alpha d$$

and thus $d = 0$. Then $P(x, y, z) = ax + by + cz$, and $\mathcal{L} = \text{span}\{x, y, z\}$. Since x, y, z are linearly independent $\dim(\mathcal{L}) = 3$. Since b_1, b_2, b_3 are linearly independent and $\dim(\text{span}\{b_1, b_2, b_3\}) = 3$, $\mathcal{L} = \text{span}\{b_1, b_2, b_3\}$.

3) Consider for some $v_j, j = 1, 2, 3$,

$$\mathbf{v_j} = \sum_{i=1}^{3} b_i(v_j)\mathbf{v_i}.$$

Then

$$(b_j(v_j) - 1)\mathbf{v_j} + \sum_{i=1, i\neq j}^{3} b_i(v_j)\mathbf{v_i} = 0.$$

Since $\mathbf{v_i}$'s are linearly independent we must have

$$b_j(v_j) = 1$$

$$b_i(v_j) = 0, i \neq j.$$

4) If $b_i(v) = 0$ for some i, then $\mathbf{v} = \sum_{j=1,j\neq i}^{3} b_j(v)\mathbf{v_j}$. Hence $\mathbf{v} \in \text{span}\{b_j, j \neq i\}$, thus v is not in the interior of $\mathcal{T}$. Thus if v is in the interior of $\mathcal{T}$ we must have $b_i(v) \neq 0$ for all i. By the definition of $\mathcal{T}$ $b_i(v) > 0$ for $i = 1, 2, 3$, and all v in the interior of $\mathcal{T}$.

THEOREM 2.6. Let R be any nonsingular matrix. Then

$$b_i^R(Rv) = b_i(v), \;\; i = 1, 2, 3, \tag{2.13}$$

where b_i^R are the trihedral coordinates of Rv with respect to $\{R\mathbf{v_1}, R\mathbf{v_2}, R\mathbf{v_3}\}$.

PROOF. Multiplying (2.9) by R, we have

$$R\mathbf{v} = b_1 R\mathbf{v_1} + b_2 R\mathbf{v_2} + b_3 R\mathbf{v_3}.$$

THEOREM 2.7. The three planes spanned by pairs of the $\mathbf{v_i}$'s divide $\mathbb{R}^3$ into eight trihedra. The functions b_1, b_2, b_3 have constant signs on each of the eight trihedra. In particular, $v \in \mathcal{T}$ if and only if $b_i \geq 0, \;\; i = 1, 2, 3$.

PROOF. Let $\mathcal{T}^{ijk}$ denote a trihedron generated by $\{(-1)^i\mathbf{v_1}, (-1)^j\mathbf{v_2}, (-1)^k\mathbf{v_3}\}$, $i, j, k \in \{0, 1\}$. Note that $\mathcal{T}^{000} = \mathcal{T}$ and each of the eight trihedra can be described this way. Fix i, j, k. We show that for all v in the interior of $\mathcal{T}^{ijk}$ b_1^{000} ($b_1(v)$ with respect to $\mathcal{T}$) has a constant sign.

Let b_1^{ijk} be the first trihedral coordinate of v in the interior of $\mathcal{T}^{ijk}$ with respect to $\mathcal{T}^{ijk}$. Note that by Lemma 2.5, $b_1^{ijk}(v) > 0$ for any such v. Then

$$b_1^{ijk}(v) = \frac{\det(v, (-1)^j v_2, (-1)^k v_3)}{\det((-1)^i v_1, (-1)^j v_2, (-1)^k v_3)}$$

$$= (-1)^i \frac{\det(v, v_2, v_3)}{\det(v_1, v_2, v_3)} = (-1)^i b_1^{000}(v).$$

Since $b_1^{ijk}(v) > 0$ by above b_1^{000} has a constant sign in the interior of $\mathcal{T}^{ijk}$.

Next we define spherical barycentric coordinates and relate their properties to the set of trihedral coordinates.

The intersection of $\mathbb{S}^2$ with the trihedron $\mathcal{T}$ generated by V is a spherical triangle τ.

DEFINITION 2.4. The spherical barycentric coordinates of a point v on $\mathbb{S}^2$ relative to τ are the unique real numbers b_1, b_2, b_3 such that

$$\mathbf{v} = b_1\mathbf{v_1} + b_2\mathbf{v_2} + b_3\mathbf{v_3}. \tag{2.14}$$

It is clear that the spherical barycentric coordinates of a point v with respect to τ are exactly the same as the trihedral coordinates of v with respect to $\mathcal{T}$. This implies they have the following properties:

LEMMA 2.8.

1) $b_i(v_j) = \delta_{ij}, i, j = 1, 2, 3,$
2) For all v in the interior of τ, $b_i(v) > 0$,
3) In contrast to the usual barycentric coordinates on the planar triangles which always sum to 1, $b_1(v) + b_2(v) + b_3(v) > 1$, if $v \in \tau$ and $v \neq v_1, v_2, v_3$,
4) If the edges of a spherical triangle τ are extended to great circles, the sphere is divided into eight regions. The spherical barycentric coordinates b_1, b_2, b_3 have constant signs on each of these eight regions,
5) If a point v lies on an edge of τ, then one of its spherical barycentric coordinates vanishes. The remaining two spherical barycentric coordinates are ratios of sines of geodesic distances, rather then ratios of geodesic distances,
6) Spherical barycentric coordinates are infinitely differentiable functions of v,

7) The spherical barycentric coordinates of a point v on the sphere relative to one spherical triangle τ can be computed from those relative to another spherical triangle by matrix multiplication,

8) The b_i are ratios of volumes of tetrahedra,

9) The spherical barycentric coordinates of a point v are invariant under rotation, i.e., they depend only on the relative positions of v and v_1, v_2, v_3 to each other,

10) The span of the spherical barycentric coordinates $b_1(v), b_2(v), b_3(v)$ relative to any triangle is always the three-dimensional linear space obtained by restricting the space $\mathcal{L}$ of linear homogeneous polynomials on $\mathbb{R}^3$ to the sphere $\mathbb{S}^2$, and is thus independent of the triangle.

Proof. Apply Lemma 2.5, Theorem 2.6 and Theorem 2.7.

We now show that spherical barycentric coordinates can also be expressed in terms of certain natural angles associated with the geometry. Let $\mathbf{n_i}$ denote the unit normal vectors to the planes $P_i := \text{span}(V \backslash \mathbf{v_i}), i = 1, 2, 3$. The orientation of these vectors is chosen to be consistent with the orientation of the vectors $\mathbf{v_i}$ relative to P_i, i.e.,

$$\text{sgn}\ \det(v_1, v_2, v_3) = \text{sgn}\ \det(n_1, v_2, v_3) =$$

$$\text{sgn}\ \det(v_1, n_2, v_3) = \text{sgn}\ \det(v_1, v_2, n_3).$$

For a point $v \in \mathbb{S}^2$, let the angles $\alpha_i,\ \beta_i$, be defined by the dot products

$$\sin \alpha_i := \mathbf{v} \cdot \mathbf{n_i}, \qquad \sin \beta_i := \mathbf{v_i} \cdot \mathbf{n_i}, \quad i = 1, 2, 3.$$

The α_i represent oriented angles between the vector $\mathbf{v}$ and the planes P_i, while the β_i are the analogous angles between $\mathbf{v_i}$ and P_i. For nontrivial spherical triangles, $\det(v_1, v_2, v_3) \neq 0$, and therefore $\sin \beta_i \neq 0, i = 1, 2, 3$.

THEOREM 2.9. The spherical barycentric coordinates of a point $v \in \mathbb{S}^2$ with respect to a triangle τ are given by

$$b_i(v) = \frac{\sin \alpha_i}{\sin \beta_i}, \;\; i = 1, 2, 3. \tag{2.15}$$

PROOF. Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote the unit coordinate vectors and $\|\cdot\|$ the usual Euclidean norm. Define

$$\mathbf{d_1} := \det \begin{bmatrix} \mathbf{i} & v_2^x & v_3^x \\ \mathbf{j} & v_2^y & v_3^y \\ \mathbf{k} & v_2^z & v_3^z \end{bmatrix},$$

$$\mathbf{d_2} := \det \begin{bmatrix} v_1^x & \mathbf{i} & v_3^x \\ v_1^y & \mathbf{j} & v_3^y \\ v_1^z & \mathbf{k} & v_3^z \end{bmatrix},$$

$$\mathbf{d_3} := \det \begin{bmatrix} v_1^x & v_2^x & \mathbf{i} \\ v_1^y & v_2^y & \mathbf{j} \\ v_1^z & v_2^z & \mathbf{k} \end{bmatrix}.$$

Then $\mathbf{n_1} = \mathbf{d_i}/\|\mathbf{d_i}\|$, and thus

$$\frac{\sin \alpha_i}{\sin \beta_i} = \frac{\mathbf{v} \cdot \mathbf{n_i}}{\mathbf{v_i} \cdot \mathbf{n_i}} = \frac{\mathbf{v} \cdot \mathbf{d_i}/\|\mathbf{d_i}\|}{\mathbf{v_i} \cdot \mathbf{d_i}/\|\mathbf{d_i}\|} = \frac{\mathbf{v} \cdot \mathbf{d_i}}{\mathbf{v_i} \cdot \mathbf{d_i}}. \tag{2.16}$$

It is easy to check that

$$\mathbf{v_i} \cdot \mathbf{d_i} = \det(v_1, v_2, v_3), \;\; i = 1, 2, 3,$$

and that

$$\mathbf{v} \cdot \mathbf{d_1} = \det(v, v_2, v_3),$$

$$\mathbf{v} \cdot \mathbf{d_2} = \det(v_1, v, v_3),$$

$$\mathbf{v} \cdot \mathbf{d_3} = \det(v_1, v_2, v).$$

Then by (2.16) and the property (2.10) of trihedral coordinates we get (2.15).

LEMMA 2.10. Let C be the unit circle in $\mathbb{R}^2$ centered at the origin, and let A be a circular arc with vertices $v_1 \neq v_2$ which are not antipodal. Let b_1, b_2 denote the circular barycentric coordinates of $v \in C$ relative to A. Then

$$\begin{aligned} b_1(v) &= \frac{\sin(\theta_2 - \theta)}{\sin(\theta_2 - \theta_1)}, \\ b_2(v) &= \frac{\sin(\theta - \theta_1)}{\sin(\theta_2 - \theta_1)}, \end{aligned} \tag{2.17}$$

where $\theta,\ \theta_1,\ \theta_2$ are the polar coordinates of $\mathbf{v}, \mathbf{v_1}, \mathbf{v_2}$ respectively.

Proof. Since

$$\begin{aligned} \mathbf{v_1} &= (\cos\theta_1, \sin\theta_1)^T \\ \mathbf{v_2} &= (\cos\theta_2, \sin\theta_2)^T \\ \mathbf{v}\ &= (\cos\theta\ , \sin\theta\ )^T \end{aligned}$$

and

$$\mathbf{v} = b_1\mathbf{v_1} + b_2\mathbf{v_2},$$

the circular barycentric coordinates of v are solving the system:

$$\begin{bmatrix} \cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

We immediately get the result.

Theorem 2.11. For each $i = 1, 2, 3$, let C_i be the great circle passing through the points $v \in \mathbb{S}^2$ and $v_i \in V$, and let y_i denote the intersection of C_i with the edge of τ opposite to v_i. Then the spherical barycentric coordinates of v can be computed as

$$b_i = \frac{\sin\delta_i}{\sin(\delta_i + \gamma_i)}, i = 1, 2, 3, \tag{2.18}$$

where δ_i is the signed geodesic distance (measured along C_i) from y_i to v, and γ_i is the signed geodesic distance from v to v_i.

Proof. It suffices to prove (2.18) for $i = 1$. By Lemma 2.10 if $v \in C_1$ it can be expressed relatively to y_1 and v_1 as:

$$\mathbf{v} = \frac{\sin\delta_1}{\sin(\delta_1 + \gamma_1)}\mathbf{v_1} + \frac{\sin\gamma_1}{\sin(\delta_1 + \gamma_1)}\mathbf{y_1}.$$

By the same lemma we can write $\mathbf{y_1}$ as a linear combination of $\mathbf{v_2}$ and $\mathbf{v_3}$ only. Then by the uniqueness of barycentric coordinates

$$b_1 = \frac{\sin \delta_1}{\sin(\delta_1 + \gamma_1)}.$$

Similarly, we can show the result for $i = 2, 3$.

2.3 Homogeneous Bernstein-Bézier polynomials

Let $\mathcal{P}_d$ denote the space of polynomials of total degree d on $\mathbb{R}^3$. Recall that the dimension of $\mathcal{P}_d$ is $\binom{d+3}{3}$ and that the set of classical Bernstein polynomials

$$B^d_{ijkl}(v) := \frac{d!}{i!j!k!\ell!} b_1^i b_2^j b_3^k b_4^\ell, \qquad i + j + k + \ell = d \tag{2.19}$$

forms a basis for $\mathcal{P}_d$ (cf. [2]).

Let $\mathcal{H}_d$ denote the space of polynomials of degree d which are homogeneous of degree d.

Lemma 2.12. The space $\mathcal{H}_d$ is an $\binom{d+2}{2}$ dimensional subspace of $\mathcal{P}_d$. Moreover, if we choose v_4 to be the origin in the above construction of the Bernstein polynomials, then the set $\{B^d_{ijk0} : i + j + k = d\}$ forms a basis for $\mathcal{H}_d$.

Proof. Let $f, g \in \mathcal{H}_d$, and $\alpha \in \mathbb{R}$. Then

(i) $(f + g)(\alpha v) = f(\alpha v) + g(\alpha v) = \alpha f(v) + \alpha g(v) = \alpha(f + g)(v)$

(ii) $\forall \beta \in \mathbb{R},\ \beta f(\alpha v) = \beta\alpha f(v) = \alpha(\beta f)(v)$.

Thus $\mathcal{H}_d$ is a subspace of $\mathcal{P}_d$.

Let $f = \sum_{0 \le i+j+k \le d} c_{ijk} x^i y^i z^k$ be in $\mathcal{H}_d$. Since f is homogeneous of degree d we must have for all $\alpha \in \mathbb{R}$

$$\alpha^d \sum_{0 \le i+j+k \le d} c_{ijk} x^i y^i z^k = \sum_{0 \le i+j+k \le d} \alpha^{i+j+k} c_{ijk} x^i y^i z^k$$

and thus

$$\sum_{0 \le i+j+k \le d} (\alpha^d - \alpha^{i+j+k}) c_{ijk} x^i y^i z^k = 0.$$

Since $\{x^i, y^i, z^k, 0 \le i+j+k \le d\}$ is a linearly independent set

$$(\alpha^d - \alpha^{i+j+k})c_{ijk} = 0 \tag{2.20}$$

Choose $\alpha \neq 1$. Then (2.20) implies

$$c_{ijk} = 0, \ \forall i+j+k \neq d,$$

and

$$f = \sum_{i+j+k=d} c_{ijk} x^i y^i z^k.$$

It follows that $\{x^i, y^i, z^k, \ i+j+k=d\}$ spans $\mathcal{H}_d$ and thus $\dim(\mathcal{H}_d) = \binom{d+2}{2}$.

Next, we show that the set $\{B^n_{ijk0} : i+j+k=d\}$ forms a basis for $\mathcal{H}_d$. Since $\{B^d_{ijkl} : i+j+k+\ell = d\}$ is a linearly independent set, so is $\{B^d_{ijk0} : i+j+k=d\}$. Each B^d_{ijk0} is a homogeneous polynomial of degree d, thus

$$\text{span}\{B^d_{ijk0} : i+j+k=d\} \subset \mathcal{H}_d.$$

Since

$$\dim\{\text{span}\{B^d_{ijk0} : i+j+k=d\}\} = \binom{d+2}{2} = \dim(\mathcal{H}_d)$$

the proof is complete.

For ease of notation, it is convenient to drop the last subscript and introduce the following definition.

Definition 2.5. Let $\mathcal{T}$ be a trihedron generated by $\{v_1, v_2, v_3\}$, and let $b_1(v)$, $b_2(v)$, $b_3(v)$ denote the trihedral coordinates as functions of $v \in \mathbb{R}^3$. Given an integer $d \ge 0$, we define the homogeneous Bernstein-Bézier basis polynomials of degree d on $\mathcal{T}$ to be the set of polynomials

$$B^d_{ijk}(v) := \frac{d!}{i!j!k!} b_1^i(v) b_2^j(v) b_3^k(v), \qquad i+j+k=d. \tag{2.21}$$

We call

$$P(v) := \sum_{i+j+k=d} c_{ijk} B^d_{ijk}(v) \tag{2.22}$$

a homogeneous Bernstein-Bézier (HBB-) polynomial of degree d.

Many properties of classical, planar, Bernstein-Bézier polynomials hold for HBB-polynomials. We present several of major importance.

For example, to evaluate P at points in $\mathbb{R}^3$ we have classical de Casteljau algorithm:

THEOREM 2.13. Suppose we want to evaluate the HBB-polynomial at a point w with trihedral coordinates b_1, b_2, b_3.

Set $c^0_{ijk} := c_{ijk},\ i+j+k=d$.

For $\ell = 1$ to d

For $i+j+k = d-\ell$

$c^{\ell}_{ijk} := b_1 c^{\ell-1}_{i+1,j,k} + b_2 c^{\ell-1}_{i,j+1,k} + b_3 c^{\ell-1}_{i,j,k+1}$.

Then $P(w) = c^d_{000}$.

PROOF. Let $B^0_{000}(w) = 1$. Suppose

$$c^{\ell-1}_{ijk} = \sum_{r+s+t=\ell-1} c_{i+r,j+s,k+t} B^{\ell-1}_{rst}(w)$$

for some ℓ and all i, j, k such that $i+j+k = d-\ell+1$. By the definition

$$c^{\ell}_{ijk} = b_1 c^{\ell-1}_{i+1,j,k} + b_2 c^{\ell-1}_{i,j+1,k} + b_3 c^{\ell-1}_{i,j,k+1} = b_1 \sum_{r+s+t=\ell-1} c_{i+1+r,j+s,k+t} B^{\ell-1}_{rst} +$$

$$b_2 \sum_{r+s+t=\ell-1} c_{i+r,j+1+s,k+t} B^{\ell-1}_{rst} + b_3 \sum_{r+s+t=\ell-1} c_{i+r,j+s,k+1+t} B^{\ell-1}_{rst} =$$

$$\sum_{r+s+t=\ell-1} (b_1 c_{i+1+r,j+s,k+t} + b_2 c_{i+r,j+1+s,k+t} + b_3 c_{i+r,j+s,k+1+t}) \frac{(\ell-1)!}{r!s!t!} b_1^r b_2^s b_3^t =$$

$$\sum_{r+s+t=\ell-1} c_{i+1+r,j+s,k+t} b_1^{r+1} b_2^s b_3^t \frac{(\ell-1)!}{r!s!t!} + \sum_{r+s+t=\ell-1} c_{i+r,j+1+s,k+t} b_1^r b_2^{s+1} b_3^t \frac{(\ell-1)!}{r!s!t!} +$$

$$\sum_{r+s+t=\ell-1} c_{i+r,j+s,k+1+t} b_1^r b_2^s b_3^{t+1} \frac{(\ell-1)!}{r!s!t!} =$$

$$\sum_{r+1+s+t=\ell} \frac{r+1}{\ell} c_{i+1+r,j+s,k+t} b_1^{r+1} b_2^s b_3^t \frac{\ell!}{(r+1)!s!t!} +$$

$$\sum_{r+1+s+t=\ell} \frac{s+1}{\ell} c_{i+r,j+1+s,k+t} b_1^r b_2^{s+1} b_3^t \frac{\ell!}{r!(s+1)!t!} +$$

$$\sum_{r+1+s+t=\ell} \frac{t+1}{\ell} c_{i+r,j+s,k+1+t} b_1^r b_2^s b_3^{t+1} \frac{\ell!}{r!s!(t+1)!} =$$

$$\sum_{r'+s+t=\ell} \frac{r'}{\ell} c_{i+r',j+s,k+t} b_1^{r'} b_2^s b_3^t \frac{\ell!}{r'!s!t!} + \sum_{r+s'+t=\ell} \frac{s'}{\ell} c_{i+r,j+s',k+t} b_1^r b_2^{s'} b_3^t \frac{\ell!}{r!s'!t!} +$$

$$\sum_{r+s+t'=\ell} \frac{t'}{\ell} c_{i+r,j+s,k+t'} b_1^r b_2^s b_3^{t'} \frac{\ell!}{r!s!t'!} = \sum_{r+s+t=\ell} \frac{r+s+t}{\ell} c_{i+r,j+s,k+t} B_{rst}^{\ell} =$$

$$\sum_{r+s+t=\ell} c_{i+r,j+s,k+t} B_{rst}^{\ell} \quad .$$

Then

$$c_{000}^d = \sum_{r+s+t=d} c_{r,s,t} B_{rst}^d(w) = P(w).$$

The following is the analog of the classical subdivision algorithm for bivariate BB-polynomials.

THEOREM 2.14. Let $\{c_{ijk}^{\ell}\}$ be the coefficients produced by de Casteljau algorithm using trihedral coordinates b_1, b_2, b_3 of a point $w \in \mathcal{T}$ with vertices $\{v_1, v_2, v_3\}$. Then

$$P(v) = \begin{cases} \sum_{i+j+k=d} c_{0,j,k}^i B_{ijk;1}^d(v), & v \in \mathcal{T}_1 = \{w, v_2, v_3\} \\ \sum_{i+j+k=d} c_{i,0,k}^j B_{ijk;2}^d(v), & v \in \mathcal{T}_2 = \{v_1, w, v_3\} \\ \sum_{i+j+k=d} c_{i,j,0}^k B_{ijk;3}^d(v), & v \in \mathcal{T}_3 = \{v_1, v_2, w\}, \end{cases} \tag{2.23}$$

where $B_{ijk;\nu}^d$ are Bernstein-Bézier polynomials associated with the trihedron $\mathcal{T}_\nu,\ \nu = 1, 2, 3$.

PROOF. Suppose $v \in \mathcal{T}_1$, and

$$P(v) = \sum_{i+j+k=d} c_{i,j,k} B_{ijk}^d(v) \tag{2.24}$$

with respect to $\mathcal{T}$, and

$$P(v) = \sum_{i+j+k=d} c_{i,j,k;1} B_{ijk;1}^d(v)$$

with respect to $\mathcal{T}_1$. We claim that $c_{ijk;1} = c^i_{0,j,k}$. The trihedral coordinates of w with respect to $\mathcal{T}$ are determined by

$$w = a_1 v_1 + a_2 v_2 + a_3 v_3.$$

The trihedral coordinates of v with respect to $\mathcal{T}$ are determined by

$$v = b_1 v_1 + b_2 v_2 + b_3 v_3$$

and with respect to $\mathcal{T}_1$ are determined by

$$v = c_1 w + c_2 v_2 + c_3 v_3.$$

Then

$$v = c_1(a_1 v_1 + a_2 v_2 + a_3 v_3) + c_2 v_2 + c_3 v_3 =$$

$$c_1 a_1 v_1 + (c_1 a_2 + c_2) v_2 + (c_1 a_3 + c_3) v_3.$$

The uniqueness of barycentric coordinates implies that

$$b_1 = c_1 a_1,$$

$$b_2 = c_1 a_2 + c_2,$$

$$b_3 = c_1 a_3 + c_3.$$

By (2.24)

$$P(v) = \sum_{i+j+k=d} c_{ijk} \frac{d!}{i!j!k!} b_1^i b_2^j b_3^k =$$

$$\sum_{i+j+k=d} c_{ijk} \frac{d!}{i!j!k!} c_1^i a_1^i (c_1 a_2 + c_2)^j (c_1 a_3 + c_3)^k.$$

Using binomial expansion and rearranging the terms we get

$$P(v) = \sum_{i+j+k=d} c_{ijk} \frac{d!}{i!j!k!} c_1^i a_1^i \Big(\sum_{r+s=j} \frac{j!}{r!s!} c_1^r a_2^r c_2^s \Big) \Big(\sum_{\ell+m=k} \frac{k!}{\ell!m!} c_1^l a_3^l c_3^m \Big) =$$

$$\sum_{i+j+k=d} \sum_{r+s=j} \sum_{\ell+m=k} c_{ijk} \frac{d!}{i!r!s!\ell!m!} c_1^{i+r+\ell} c_2^s c_3^m a_1^i a_2^r a_3^\ell =$$

$$\sum_{i+j+k=d}\sum_{r+s=j}\sum_{\ell+m=k} c_{ijk}\frac{(i+r+\ell)!}{i!r!\ell!}B^d_{i+r+\ell,s,m;1}a_1^i a_2^r a_3^\ell =$$
$$\sum_{i+j+k=d}\sum_{r+s=j}\sum_{\ell+m=k} c_{i,r+s,\ell+m}\frac{(i+r+\ell)!}{i!r!\ell!}a_1^i a_2^r a_3^l B^d_{i+r+\ell,s,m;1} =$$
$$\sum_{i+j+k=d}\sum_{r+s=j}\sum_{\ell+m=k} c_{i,r+s,\ell+m}B^{i+r+\ell}_{i,r,\ell}B^d_{i+r+\ell,s,m;1}.$$

Introducing a new index of summation $p = i + r + \ell$, and since

$$\sum_{i+r+\ell=p} c_{i,r+s,\ell+m}B^{i+r+\ell}_{i,r,\ell} = c^{i+r+\ell}_{0,s,m}$$

we have

$$P(v) = \sum_{p+s+m=d}(\sum_{i+r+\ell=p} c_{i,r+s,\ell+m}B^p_{i,r,m})B^d_{p,s,m;1} =$$
$$\sum_{p+s+m=d} C^p_{0,s,m}B^d_{p,s,m;1}.$$

Similar proof works for $v \in \mathcal{T}_2$ and for $v \in \mathcal{T}_3$.

We now establish necessary and sufficient conditions for two HBB-polynomials to join together smoothly across a plane trough the origin in the sense that the polynomials and their usual directional derivatives as trivariate functions are continuous as we cross the plane.

Theorem 2.15. Let $\mathcal{T}$ and $\hat{\mathcal{T}}$ be trihedra generated by vertices $V = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ and $\hat{V} = \{\mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$. Let

$$P(v) = \sum_{i+j+k=d} c_{ijk}B^d_{ijk}(v)$$

and

$$\hat{P}(v) = \sum_{i+j+k=d} \hat{c}_{ijk}\hat{B}^d_{ijk}(v),$$

where $\{B^d_{ijkl}\}$ and $\{\hat{B}^d_{ijkl}\}$ are the Bernstein-Bézier basis functions associated with $\mathcal{T}$ and $\hat{\mathcal{T}}$. Then P and $\hat{P}$ and all of their derivatives up to order m agree on the face shared by $\mathcal{T}$ and $\hat{\mathcal{T}}$ if and only if

$$\hat{c}_{ijk} = \sum_{r+s+t=i} c_{r,j+s,k+t}B^i_{rst}(v_4) \tag{2.25}$$

for all $i = 0, ..., m$ and all j, k such that $i + j + k = d$.

PROOF. Suppose

$$Q(v) = \sum_{i+j+k+\ell=d} C_{ijkl} B^d_{ijkl}(v) \tag{2.26}$$

and

$$\hat{Q}(v) = \sum_{i+j+k+\ell=d} \hat{C}_{ijkl} \hat{B}^d_{ijkl}(v), \tag{2.27}$$

where

$$C_{ijkl} := \begin{cases} c_{ijk}, \ if \ \ell = 0 \\ 0, \ \text{otherwise} \end{cases} \tag{2.28}$$

and

$$\hat{C}_{ijkl} := \begin{cases} \hat{c}_{ijk}, \ if \ \ell = 0 \\ 0, \ \text{otherwise} \end{cases} \tag{2.29}$$

and $B^d_{ijkl}(v)$ are the usual BB-polynomials of degree d associated with the trihedron with vertices $\{v_1, v_2, v_3, 0\}$ and $\hat{B}^d_{ijkl}(v)$ are those associated with the trihedron with vertices $\{v_4, v_2, v_3, 0\}$. It is well-known that these polynomials join with C^m continuity if and only if

$$\hat{C}_{ijkl} = \sum_{r+s+t+u=i} C_{r,j+s,k+t,\ell+u} B^i_{rstu}(v_4), \ i = 0, ..., m. \tag{2.30}$$

In view of (2.28, 2.29) we can choose $\ell = u = 0$. In this case, (2.30) holds if and only if (2.25) holds. But $P = Q$ and $\hat{P} = \hat{Q}$, proof is complete.

2.4 Spherical Bernstein-Bézier polynomials

In this section we discuss properties of BB-polynomials restricted to the sphere $\mathbb{S}^2$. We start by stating the existence of homogeneous extensions.

LEMMA 2.16. Suppose f is a function defined on $\mathbb{S}^2$ and let $t \in \mathbb{R}$. Then

$$F_t(v) := \|v\|^t f(v/\|v\|) \tag{2.31}$$

is the unique homogeneous extension of f of degree t to all of $\mathbb{R}^3\backslash\{0\}$, i.e., $F_t|_{\mathbb{S}^2} = f$, and F_t is homogeneous of degree t.

PROOF. The assertion is an immediate consequence of the definition.

DEFINITION 2.6. The restriction of an HBB-polynomial of degree d to the points on the unit sphere is called a spherical Bernstein-Bézier (SBB-) polynomial of degree d.

Many properties of SBB-polynomials follow naturally from the properties of HBB-polynomials.

THEOREM 2.17. The polynomials $\{B^d_{ijk}, i+j+k=d\}$ restricted to $\mathbb{S}^2$ are linearly independent.

PROOF. Suppose

$$P(v) = \sum_{i+j+k=n} c_{ijk} B^d_{ijk}(v) = 0$$

for all $v \in \mathbb{S}^2$. By Lemma 2.16 there exists the unique homogeneous extension of $P(v)$ to all of $\mathbb{R}^3$ of degree d. Then $P(v) = 0$ for all $v \in \mathbb{R}^3$. The linear independence of the B^d_{ijk}'s implies that $c_{ijk} = 0, i+j+k=d$ and thus the B^d_{ijk}'s restricted to $\mathbb{S}^2$ are linearly independent.

De Casteljau and subdivision algorithms can also be applied to the restricted polynomials.

We now consider the question when two polynomials on adjoining surface triangles join smoothly across a common edge e.

THEOREM 2.18. Suppose Q and $\hat{Q}$ are polynomials as in (2.26) and (2.27) and let τ and $\hat{\tau}$ be the surface triangles with a common edge e. Then the restrictions of Q and $\hat{Q}$ to $\mathbb{S}^2$, P and $\hat{P}$, along with their derivatives up to order m join continuously along e, i.e., for every point $v \in e$ and every curve $c \in \hat{S}$ crossing e at v,

$$D^j_c P(v) = D^j_c \hat{P}(v), \qquad j = 0, ..., m, \tag{2.32}$$

if and only if

$$\hat{c}_{ijk} = \sum_{r+s+t=i} c_{r,j+s,k+t} B^i_{rst}(v_4), \tag{2.33}$$

for all $i = 0, ..., m$ and all j, k such that $i + j + k = d$.

PROOF. Suppose (2.32) holds for all $v \in e$ and for all $c \in \mathbb{S}^2$ crossing e at v. Since P and $\hat{P}$ are polynomials of degree d, by Lemma 2.16 there exist unique homogeneous extensions of degree d which thus must be our Q and $\hat{Q}$. Since $Q|_{\mathbb{S}^2} = P$ and $\hat{Q}|_{\mathbb{S}^2} = \hat{P}$

$$D^j_c Q(v) = D^j_c \hat{Q}(v), \qquad j = 0, ..., m, \tag{2.34}$$

for every point $v \in e$ and every curve $c \in \mathbb{S}^2$ crossing e at v. Now we claim that (2.34) holds for any v on the common face of tetrahedras corresponding to τ and $\hat{\tau}$. Let v belong to the common face of $\mathcal{T}$ and $\hat{\mathcal{T}}$. Clearly, if $v \neq 0$, there exist $v' \in e$ and $\lambda \in R$, such that $v = \lambda v'$. Since Q and $\hat{Q}$ are homogeneous of degree d

$$Q(v) = Q(\lambda v') = \lambda^d Q(v'),$$

and similarly for $\hat{Q}$. Then we have

$$D^j_c Q(v) = \lambda^d D^j_c Q(v') = \lambda^d D^j_c \hat{Q}(v') = D^j_c \hat{Q}(v), \qquad j = 0, ..., m.$$

By the Theorem 2.15

$$\hat{c}_{ijk} = \sum_{r+s+t=i} c_{r,j+s,k+t} B^i_{rst}(v_4).$$

For the other direction, suppose (2.33) holds. Then by Theorem 2.15 $Q(v)$ and $\hat{Q}(v)$ join smoothly across the common face, i.e.,

$$D^j Q(v) = D^j \hat{Q}(v), \qquad j = 0, ..., m, \tag{2.35}$$

for any v on the face. This condition holds for any curve on the common face and thus for the edge e as well. Since $Q(v)|e = P(v)$ and $\hat{Q}(v)|e = \hat{P}(v)$ (2.35) holds for the restrictions. In particular,

$$\nabla P(v) = \nabla \hat{P}(v), \qquad v \in e.$$

Now let c be a curve on the sphere-like surface $\mathbb{S}^2$, then by the chain rule

$$\nabla P(v) = \nabla c D_c P(v) = \nabla c D_c \hat{P}(v) = \nabla \hat{P}(v),$$

and so on. Thus we have the result for any $v \in e$ and any curve c crossing e at v.

Now we turn to a question how to compute derivatives of spherical functions and in particular SBB-polynomials. Let us define what we mean by the derivatives of a spherical function.

Definition 2.7. We define the directional derivative $D_g f$ of f at a point $v \in \mathbb{S}^2$ by

$$D_g f(v) := D_g F(v) = g^T \nabla F(v), \tag{2.36}$$

where F is some homogeneous extension of f, and ∇F is the gradient of the trivariate function F.

While a polynomial of degree d has a natural homogeneous extension to $\mathbb{R}^3$, a general function f on $\mathbb{S}^2$ has infinitely many different extensions. The value of its derivative may depend on which extension we take. The following result identifies an important case where it does not matter which extension we take.

Lemma 2.19. Suppose f is a function on $\mathbb{S}^2$ and g is a tangent vector to $\mathbb{S}^2$ at a point v. Then the value of $D_g f(v)$ can be computed from (2.36) using any homogeneous extension of f.

Proof. Let F be a homogeneous extension of f, and let C be a C^1 smooth curve on $\mathbb{S}^2$ passing through the point v, parameterized by a parameter θ such that $C(\theta) = v$ and $C'(\theta) = g$ for $\theta = 0$. By the chain rule we obtain

$$\frac{df(C(\theta))}{d\theta}|_{\theta=0} = \frac{dF(C(\theta))}{d\theta}|_{\theta=0} = g^T \nabla F(v) = D_g F(v).$$

This shows that $D_g F(v)$ does not depend on the degree of homogeneity of F since the left-hand side clearly depends only on $f = F|_{\mathbb{S}^2}$.

Let us continue with directional derivatives of barycentric coordinates.

LEMMA 2.20. Let g be a given unit vector in $\mathbb{R}^3$. Then

$$D_g b_i = b_i(g). \tag{2.37}$$

PROOF. Say $i = 1$. Let $\tau = \{v_1, v_2, v_3\}$ and $v \in \mathbb{S}^2$. By (2.36) and (5.3)

$$D_g b_1 = g^T \nabla b_1 = \frac{\det(g, v_2, v_3)}{\det(v_1, v_2, v_3)} = b_1(g).$$

PROPOSITION 2.21. Suppose P is an SBB-polynomial. Then

$$D_g P(v) = b^T(g) \nabla_b P, \tag{2.38}$$

where

$$\nabla_b := (\frac{\partial}{\partial b_1}, \frac{\partial}{\partial b_2}, \frac{\partial}{\partial b_3})^T. \tag{2.39}$$

PROOF. By the definition

$$D_g P(v) = g^T \nabla P(v) = g^T \begin{bmatrix} \frac{\partial b_1}{\partial x} & \frac{\partial b_2}{\partial x} & \frac{\partial b_3}{\partial x} \\ \frac{\partial b_1}{\partial y} & \frac{\partial b_2}{\partial y} & \frac{\partial b_3}{\partial y} \\ \frac{\partial b_1}{\partial z} & \frac{\partial b_2}{\partial z} & \frac{\partial b_3}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial P}{\partial b_1} \\ \frac{\partial P}{\partial b_2} \\ \frac{\partial P}{\partial b_3} \end{bmatrix} =$$

$$\begin{bmatrix} g^T \nabla b_1 \\ g^T \nabla b_2 \\ g^T \nabla b_3 \end{bmatrix}^T \nabla_b P = \begin{bmatrix} b_1(g) \\ b_2(g) \\ b_3(g) \end{bmatrix}^T \nabla_b P.$$

We now turn to the problem of computing higher derivatives of SBB-polynomials. Let $c^0_{ijk} := c_{ijk}$ be the Bézier coefficients of P of degree d, and let $g_1, ..., g_m$, $1 \le m \le d$, be a set of direction vectors. For each $1 \le \ell \le m$, let $c^\ell_{ijk}, i + j + k = d - \ell$, be the intermediate values obtained in carrying out de Casteljau algorithm using $b(g_\ell)$. That is, c^ℓ_{ijk} is obtained from the recursion

$$c^\ell_{ijk} = b_1(g_\ell) c^{\ell-1}_{i+1,j,k} + b_2(g_\ell) c^{\ell-1}_{i,j+1,k} + b_1(g_\ell) c^{\ell-1}_{i,j,k+1}, \ell = 1, ..., m.$$

It follows that c^{ℓ}_{ijk} depend on the vectors $g_1, ..., g_\ell$, but not on their ordering.

THEOREM 2.22. For any $0 \le m \le d$,

$$D_{g_1,\ldots,g_m} P(v) := D_{g_1} \cdots D_{g_m} P(v) = \frac{d!}{(d-m)!} \sum_{i+j+k=d-m} c^m_{ijk} B^{d-m}_{ijk}(v). \tag{2.40}$$

PROOF. By Lemma 2.19, for $i + j + k = d$,

$$D_{g_1} B^d_{ijk}(v) = \frac{d!}{i!j!k!}[ib_1^{i-1} b_2^j b_3^k D_{g_1} b_1 + jb_1^i b_2^{j-1} b_3^k D_{g_1} b_2 + kb_1^i b_2^j b_3^{k-1} D_{g_1} b_3] =$$

$$d[B^{d-1}_{i-1,j,k}(v) b_1(g_1) + B^{d-1}_{i,j-1,k}(v) b_2(g_1) + B^{d-1}_{i,j,k-1}(v) b_3(g_1)].$$

Substituting this in

$$D_{g_1} P(v) = \sum_{i+j+k=d} c_{ijk} D_{g_1} B^d_{ijk}(v)$$

and rearranging terms we get (2.40) for $m = 1$. The general result follows by induction.

It is clear from the properties of trihedral coordinates that the values of an SBB-polynomial P at the vertices of its domain triangle are given by $P(v_1) = c_{d00}$, $P(v_2) = c_{0d0}$, $P(v_3) = c_{00d}$. The derivatives of P at the vertices of τ also have a simple form.

PROPOSITION 2.23. For all $0 \le m \le d$,

$$\begin{aligned} D_{g_1,\ldots,g_m} P(v_1) &= \frac{d!}{(d-m)!} c^m_{d-m,0,0}, \\ D_{g_1,\ldots,g_m} P(v_2) &= \frac{d!}{(d-m)!} c^m_{0,d-m,0}, \\ D_{g_1,\ldots,g_m} P(v_3) &= \frac{d!}{(d-m)!} c^m_{0,0,d-m}. \end{aligned} \tag{2.41}$$

PROOF. Consider $P(v_1)$. By Theorem 2.22

$$D_{g_1,\ldots,g_m} P(v_1) = \frac{d!}{(d-m)!} \sum_{i+j+k=d-m} c^m_{ijk} B^{d-m}_{ijk}(v_1),$$

where

$$B^{d-m}_{ijk}(v_1) = \frac{(d-m)!}{i!j!k!} b_1(v_1)^i b_2(v_1)^j b_3(v_1)^k = \frac{(d-m)!}{i!j!k!} 1^i 0^j 0^k = 1,$$

if $i = d - m, j = 0, k = 0$ and is 0 otherwise. Thus

$$D_{g_1,\ldots,g_m} P(v_1) = \frac{d!}{(d-m)!} c^m_{d-m,0,0}.$$

In many applications it is necessary to compute integrals of piecewise polynomial functions. Evaluating integrals of spherical polynomials is considerably more difficult than in the planar case. Recall that for planar triangles, the integral of a Bernstein basis polynomial of degree d is equal to the area of the corresponding triangle divided by $d + 1$. Thus, the value of the integral does not depend on the particular basis polynomial or on the precise shape of the triangle. Unfortunately, this attractive property does not carry over to spherical polynomials. In general, for two different triangles, the values of the integrals are different unless the two triangles are similar. Moreover, the integrals of the Bernstein basis polynomials of degree d associated with a single triangle are also different in general.

To compute integrals in this case we propose a mapping of a surface triangle τ to a planar triangle $\bar{\tau}$ by means of radial projection defined in Section 2.1. This will enable us to use a standard integration technique for planar triangles.

Lemma 2.24. Let τ be a spherical triangle and $\bar{\tau}$ its radial projection as in Section 2.1. Suppose $|\tau| \leq 1$ and R_τ denotes the radial projection defined by $R_\tau \bar{\omega} := \frac{\bar{\omega}}{|\bar{\omega}|}$ for $\bar{\omega} \in \bar{\tau}$. If σ and $\bar{\sigma}$ denote the Lebesgue measures on τ and $\bar{\tau}$ correspondingly then

$$\int_\tau f(\omega) d\sigma(\omega) = \int_{\bar{\tau}} f(R_\tau \bar{\omega}) |\bar{\omega}|^{-3} d\bar{\sigma}(\bar{\omega}). \tag{2.42}$$

Proof. Without loss of generality assume that the tangent plane $\mathbf{T}_\tau$ is $z = 1$. Recall that $\frac{\bar{\omega}}{|\bar{\omega}|} = \omega$, and for $\omega = (x, y, z)$ we can write $\bar{\omega} = (x', y', 1)$ with $x' = x/z$ and $y' = y/z$. Then $d\bar{\sigma} = dx'dy'$. For the spherical measure recall that $d\sigma = \sin\phi d\phi d\theta$, where ϕ and θ are spherical coordinates of ω defined by

$$x = \cos\theta \sin\phi$$

$$y = \sin\theta \sin\phi$$

$$z = \cos\phi.$$

Therefore

$$x' = \cos\theta\tan\phi$$

$$y' = \sin\theta\tan\phi.$$

We can compute the partial derivatives

$$\frac{\partial x'}{\partial\theta} = -\sin\theta\tan\phi,$$

$$\frac{\partial x'}{\partial\phi} = \cos\theta\sec^2\phi,$$

$$\frac{\partial y'}{\partial\theta} = \cos\theta\tan\phi,$$

$$\frac{\partial y'}{\partial\phi} = \sin\theta\sec^2\phi.$$

Then

$$|\frac{\partial(x',y')}{\partial(\theta,\phi)}| = \frac{\sin\phi}{\cos^3\phi}$$

and hence using $\cos\phi = z = |\bar{\omega}|^{-1}$ we get (2.42).

2.5 Non-homogeneous spherical polynomials

Let us now define non-homogeneous spherical polynomials and trace their properties to the properties outlined above for homogeneous polynomials.

It was shown in [19] that $\mathcal{H}_d \oplus \mathcal{H}_{d-1}$ restricted to the unit sphere is identical to the space $\mathcal{P}_d$ of trivariate non-homogeneous polynomials of degree d restricted to the unit sphere. Therefore the set $\{B^d_{ijk},\ i+j+k=d\} \cup \{B^{d-1}_{ijk},\ i+j+k=d-1\}$ forms a basis for $\mathcal{P}_d$. We can express a non-homogeneous spherical polynomial P in terms of BB-basis functions as

$$P(v) = \sum_{i+j+k=d} a_{ijk}B^d_{ijk}(v) + \sum_{i+j+k=d-1} c_{ijk}B^{d-1}_{ijk}(v).$$

With this definition it is easy to see that evaluating (de Casteljau's algorithm), taking derivatives and computing integrals with homogeneous polynomials can be easily adapted for non-homogeneous polynomials.

2.6 Spherical Sobolev Space Semi-norms

In this section we start by following the construction in [23] to define Sobolev-type norms and semi-norms for functions on the unit sphere. This construction uses a concept of a homogeneous extension. Recall that a trivariate function $f(v)$ is homogeneous of degree n if

$$f(\alpha v) = \alpha^n f(v), \forall v \in \mathbb{R}^3 \backslash \{0\}, \alpha \neq 0. \tag{2.43}$$

Recall next that by Lemma 2.16, every spherical function f has a unique homogeneous extension of degree n to $\mathbb{R}^3\backslash\{0\}$ defined by

$$f_n(u) = |u|^n f\left(\frac{u}{|u|}\right). \tag{2.44}$$

Let Ω be a domain on $\mathbb{S}^2$ such that $|\Omega| \leq 1$, and let $\bar{\Omega}$ denote the image of Ω under the inverse radial projection as defined in Section 2.1. We will be relating properties of a spherical function f defined on Ω to the properties of its homogeneous extension f_n restricted to $\bar{\Omega}$. Such a restriction is denoted by $\bar{f}_n$.

Fix $1 \leq p \leq \infty$, k nonnegative integer and let B denote an open set in $\mathbb{R}^2$. Recall that the corresponding classical Sobolev space $W^{k,p}(B)$ is the space of functions on B whose derivatives up to order k belong to $L_p(B)$ [1]. A norm on $W^{k,p}(B)$ can be defined as

$$\|g\|_{k,p,B} := \sum_{\gamma_1+\gamma_2 \leq k} ||D_\xi^{\gamma_1} D_\eta^{\gamma_2} g||_{p,B}, \tag{2.45}$$

where $D_\xi^{\gamma_1} D_\eta^{\gamma_2} = \frac{\partial^{\gamma_1+\gamma_2}}{\partial\xi^{\gamma_1}\partial\eta^{\gamma_2}}$.

Suppose that $\{(\Gamma_j, \phi_j)\}$ is an atlas for Ω. Let $\{\alpha_j\}$ be a partition of unity subordinate to the atlas. We define spherical Sobolev spaces $W^{k,p}(\Omega)$ as follows:

$$W^{k,p}(\Omega) := \{f : (\alpha_j f) \circ \phi_j^{-1} \in W^{k,p}(\phi_j(\Gamma_j)), \text{for all } j\}. \tag{2.46}$$

Let $f \in W^{k,p}(\Omega)$. Then

$$|f|_{k,p,\Omega} := \sum_{|\alpha|=k} \|D^\alpha f_{k-1}\|_{p,\Omega} \tag{2.47}$$

is a Sobolev-type semi-norm of f on $W^{k,p}(\Omega)$. Here $\|D^\alpha f_{k-1}\|_{p,\Omega}$ is understood as the L_p-norm of the restriction of the trivariate function $D^\alpha f_{k-1}$ to Ω.

In addition to the semi-norm defined above we will use another semi-norm defined analogously to it:

$$|f|'_{k,p,\Omega} := \sum_{|\alpha|=k} \|D^\alpha f_{k-2}\|_{p,\Omega} \tag{2.48}$$

Lemma 2.25. Let $f \in W^{k,p}(\Omega)$ for some $k \geq 1$ with $|\Omega| \leq 1$. Then $(D^\alpha f_{k-2})|_\Omega \in L_p(\Omega)$ for all multi-indices α such that $|\alpha| = k$.

Proof. By Lemma 3.3 [23] $(D^\beta f_{k-2})|_\Omega \in L_p(\Omega)$ for all $|\beta| = k-1$ and

$$D_x^{\beta_1} D_y^{\beta_2} D_z^{\beta_3} f_{k-2} = (-z)^{-\beta_3} \sum_{\ell=0}^{\beta_3} \binom{\beta_3}{\ell} x^\ell y^{\beta_3-\ell} D_x^{\beta_1+\ell} D_y^{\beta_2+\beta_3-\ell} f_{k-2}$$

for $|\beta| = \beta_1 + \beta_2 + \beta_3 = k-1$. Then

$$D_x D^\beta f_{k-2} = (-z)^{-\beta_3} (\beta_3 \sum_{\ell=0}^{\beta_3-1} \binom{\beta_3-1}{\ell} x^\ell y^{\beta_3-\ell-1} D_x^{\beta_1+\ell+1} D_y^{\beta_2+\beta_3-\ell-1} +$$

$$\sum_{\ell=0}^{\beta_3} \binom{\beta_3}{\ell} x^\ell y^{\beta_3-\ell} D_x^{\beta_1+\ell+1} D_y^{\beta_2+\beta_3-\ell}) f_{k-2}.$$

Recall from Lemma 3.3 [23] that $|\binom{\beta_3}{\ell} x^\ell y^{\beta_3-\ell}| \leq (|x|+|y|)^{\beta_3} \leq (2M_\Omega)^{\beta_3}$. Since $z=1$

$$\|D_x^{\beta_1+1} D_y^{\beta_2} D_z^{\beta_3} f_{k-2}\|_{p,\bar{\Omega}} \leq \beta_3 (2M_\Omega)^{\beta_3-1} \sum_{\ell=0}^{\beta_3-1} \|D_x^{\beta_1+\ell+1} D_y^{\beta_2+\beta_3-\ell-1} f_{k-2}\|_{p,\bar{\Omega}} +$$

$$(2M_\Omega)^{\beta_3} \sum_{\ell=0}^{\beta_3} \|D_x^{\beta_1+\ell+1} D_y^{\beta_2+\beta_3-\ell} f_{k-2}\|_{p,\bar{\Omega}} = \beta_3 (2M_\Omega)^{\beta_3-1} \sum_{\gamma_1+\gamma_2=k-1} \|D_x^{\gamma_1} D_y^{\gamma_2} \bar{f}_{k-2}\|_{p,\bar{\Omega}} +$$

$$(2M_\Omega)^{\beta_3} \sum_{\gamma_1+\gamma_2=k} \|D_x^{\gamma_1} D_y^{\gamma_2} \bar{f}_{k-2}\|_{p,\bar{\Omega}} \leq \beta_3 (2M_\Omega)^{\beta_3 - 1} \|\bar{f}_{k-2}\|_{k-1,p,\bar{\Omega}} + (2M_\Omega)^{\beta_3} \|\bar{f}_{k-2}\|_{k,p,\bar{\Omega}}$$

for every $f \in W^{k,p}(\Omega)$.

Similarly,

$$D_y D^\beta f_{k-2} = (-z)^{-\beta_3} (\sum_{\ell=0}^{\beta_3-1} \beta_3 \binom{\beta_3 - 1}{\ell} x^\ell y^{\beta_3-\ell-1} D_x^{\beta_1+\ell} D_y^{\beta_2+\beta_3-\ell} +$$

$$\sum_{\ell=0}^{\beta_3} \binom{\beta_3}{\ell} x^\ell y^{\beta_3-\ell} D_x^{\beta_1+\ell} D_y^{\beta_2+\beta_3-\ell+1}) f_{k-2}$$

and

$$\|D_x^{\beta_1} D_y^{\beta_2+1} D_z^{\beta_3} f_{k-2}\|_{p,\bar{\Omega}} \leq \beta_3 (2M_\Omega)^{\beta_3-1} \|\bar{f}_{k-2}\|_{k-1,p,\bar{\Omega}} + (2M_\Omega)^{\beta_3} \|\bar{f}_{k-2}\|_{k,p,\bar{\Omega}}.$$

Finally,

$$D_z D^\beta f_{k-2} = (-z)^{-\beta_3-1} (\sum_{\ell=0}^{\beta_3} \beta_3 \binom{\beta_3}{\ell} x^\ell y^{\beta_3-\ell} D_x^{\beta_1+\ell} D_y^{\beta_2+\beta_3-\ell} +$$

$$\sum_{\ell=0}^{\beta_3} \binom{\beta_3}{\ell} x^{\ell+1} y^{\beta_3-\ell} D_x^{\beta_1+\ell+1} D_y^{\beta_2+\beta_3-\ell} +$$

$$\sum_{\ell=0}^{\beta_3} \binom{\beta_3}{\ell} x^\ell y^{\beta_3-\ell+1} D_x^{\beta_1+\ell} D_y^{\beta_2+\beta_3-\ell+1}) f_{k-2}$$

and therefore

$$\|D_x^{\beta_1} D_y^{\beta_2} D_z^{\beta_3+1} f_{k-2}\|_{p,\bar{\Omega}} \leq \beta_3 (2M_\Omega)^{\beta_3} \|\bar{f}_{k-2}\|_{k-1,p,\bar{\Omega}} + (2M_\Omega)^{\beta_3+1} \|\bar{f}_{k-2}\|_{k,p,\bar{\Omega}},$$

since $|x| \leq M_\Omega$ and $|y| \leq M_\Omega$. By Lemma 3.2 [23] $\bar{f}_{k-1} \in W_{k,p}(\bar{\Omega})$. Hence $\|D^\alpha f_{k-2}\|_{p,\bar{\Omega}} < \infty$ for every $|\alpha| = k$. By Lemma 3.1 [23] $\|D^\alpha f_{k-2}\|_{p,\Omega} < \infty$ as well.

Our next result relates Sobolev-type semi-norms for a spherical function f defined on Ω with Sobolev type semi-norms of corresponding planar functions $\bar{f}_{k-2}$, $\bar{f}_{k-1}$ defined on $\bar{\Omega}$.

PROPOSITION 2.26. There exist positive constants A_2, A_3, A_4, A_5 and A_6 depending only on k and p such that for every $f \in W_{k,p}(\Omega)$

$$A_2|f|_{k,p,\Omega} \leq |\bar{f}_{k-1}|_{k,p,\bar{\Omega}} \leq A_3|f|_{k,p,\Omega} \tag{2.49}$$

and

$$A_4|\bar{f}_{k-2}|_{k,p,\bar{\Omega}} \leq |f|'_{k,p,\Omega} \leq A_5|\bar{f}_{k-2}|_{k-1,p,\bar{\Omega}} + A_6|\bar{f}_{k-2}|_{k,p,\bar{\Omega}}. \tag{2.50}$$

PROOF. The first assertion is Proposition 3.4 in [23]. The proof of (2.50) is similar and follows from Lemma 2.25.

Our next proposition shows that the semi-norms defined by (2.47) and (2.48) annihilate certain homogeneous polynomials.

PROPOSITION 2.27. Suppose Ω is an open connected subset of $\mathbb{S}^2$. Let $f \in W^{k,p}(\Omega)$ and $k \geq 2$. $|f|_{k,p,\Omega} = 0$ if and only if f is a homogeneous spherical polynomial of degree $k-1$. $|f|'_{k,p,\Omega} = 0$ if and only if f is a homogeneous spherical polynomial of degree $k-2$.

PROOF. The first part of this proposition is Proposition 3.5 in [23]. For the second assertion note first that if f is a homogeneous spherical polynomial of degree $k-2$, then so is its $(k-2)$nd extension. Then all of the partial derivatives of f_{k-2} of order k are zero. Suppose now $|f|'_{k,p,\Omega} = 0$. Then for every multi-index α with $|\alpha| = k$ $D^\alpha f_{k-2}|_\Omega = 0$. Denote $g = D^\alpha f_{k-2}|_\Omega$ and consider the homogeneous extension of g of degree -2. Since f_{k-2} is homogeneous of degree $k-2$ and $|\alpha| = k$, $D^\alpha f_{k-2}$ is homogeneous of degree -2. By the uniqueness of homogeneous extension $g_{-2} = D^\alpha f_{k-2}$. On the other hand by the definition $g_{-2}(v) = |v|^{-2} g(v/|v|)$ and thus g_{-2} is zero on $\mathbb{R}^3 \backslash \{0\}$. Therefore $D^\alpha f_{k-2} = 0$ on $\mathbb{R}^3 \backslash \{0\}$. Therefore f_{k-2} is a polynomial of degree at most $k-1$. The homogeneity of f_{k-2} implies that f_{k-2} is in fact a homogeneous polynomial of degree exactly $k-2$. Therefore so is f.

2.7 Basic inequalities

Given a homogeneous trivariate polynomial P in BB form (2.22), let c be a vector of its coefficients. Let $\|c\|_{\infty,\tau}$ and $\|c\|_{p,\tau}$ denote its ℓ_∞ and ℓ_p norms on a spherical triangle τ respectively.

Lemma 2.28. Any homogeneous polynomial P of degree d in Bernstein-Bézier form (2.22) with respect to a spherical triangle τ with $|\tau| \le 1$ satisfies the property

$$A_7 \; \|c\|_{\infty,\tau} \le \|P\|_{\infty,\tau} \le A_8 \|c\|_{\infty,\tau} \tag{2.51}$$

and

$$A_9 A_\tau^{1/p} \; \|c\|_{p,\tau} \le \|P\|_{p,\tau} \le A_8 A_\tau^{1/p} \|c\|_{p,\tau} \tag{2.52}$$

for any $1 \le p < \infty$. Here A_7, A_8 are positive constants independent of τ, P and p. A_9 depends d, p and the smallest angle of τ.

Proof. Proof of (2.51) can be found in [23]. For (2.52) fix $1 \le p < \infty$. By Lemma 4.4 in [23] there exists a positive constant K_3 depending on d, p and the smallest angle Θ_τ of τ such that

$$A_\tau^{-1/p} \|P\|_{p,\tau} \le \|P\|_{\infty,\tau} \le K_3 A_\tau^{-1/p} \|P\|_{p,\tau}. \tag{2.53}$$

Then using (2.51) we get

$$\frac{A_\tau^{1/p}}{K_3} A_7 \binom{d+2}{2}^{-1/p} \|c\|_{p,\tau} \le \frac{A_\tau^{1/p}}{K_3} A_7 \|c\|_{\infty,\tau} \le \frac{A_\tau^{1/p}}{K_3} \|P\|_{\infty,\tau} \le \|P\|_{p,\tau}.$$

Similarly, by (2.53)

$$\|P\|_{p,\tau} \le A_\tau^{1/p} \|P\|_{\infty,\tau} \le A_8 A_\tau^{1/p} \|c\|_{\infty,\tau} \le A_8 A_\tau^{1/p} \|c\|_{p,\tau}.$$

Therefore we obtain (2.52) with $A_9 := \frac{A_7}{K_3} \binom{d+2}{2}^{-1/p}$.

Next we need Markov-type inequalities.

Lemma 2.29. Let P be a trivariate homogeneous polynomial of degree d defined on a spherical triangle τ with $|\tau| \leq 1$. There exist constants A_{10}, A'_{10} depending on d and Θ_τ only, and A_{11}, A'_{11} depending on d, such that

$$\begin{aligned} |P|_{k,\infty,\tau} &\leq \frac{A_{10}}{(\tan\frac{\rho_\tau}{2})^k}\|P\|_{\infty,\tau}, \\ |P|'_{k,\infty,\tau} &\leq \frac{A'_{10}}{(\tan\frac{\rho_\tau}{2})^k}\|P\|_{\infty,\tau}, \end{aligned} \tag{2.54}$$

and

$$\begin{aligned} |P|_{k,p,\tau} &\leq \frac{A_{11}}{(\tan\frac{\rho_\tau}{2})^k}\|P\|_{p,\tau}, \\ |P|'_{k,p,\tau} &\leq \frac{A'_{11}}{(\tan\frac{\rho_\tau}{2})^k}\|P\|_{p,\tau} \end{aligned} \tag{2.55}$$

for $1 \leq p < \infty$. Here ρ_τ is a the diameter of the largest spherical cap contained in τ.

Proof. For the first equation in (2.55) we modify the proof of Proposition 4.3 in [23] by replacing (2.1) with (2.3). Similar proof works for the second equation with the key inequality using Lemma 3.6 [23]

$$|P|'_{k,p,\Omega} \leq A_5|\bar{P}_{k-2}|_{k-1,p,\bar{\Omega}} + A_6|\bar{P}_{k-2}|_{k,p,\bar{\Omega}} \leq$$

$$\max\{A_5, A_6\}\|\bar{P}_{k-2}\|_{k,p,\bar{\Omega}} \leq$$

$$\max\{A_5, A_6\}K_4\|\bar{P}_d\|_{k,p,\bar{\Omega}}.$$

To prove (2.54) we apply (2.53) to both sides of (2.55) to get

$$|P|_{k,\infty,\tau} \leq \frac{A_{11}K_5}{(\tan\frac{\rho_\tau}{2})^k}\|P\|_{\infty,\tau}$$

for some K_5 depending on $d-k$ and Θ_Δ.

Finally we express a bound on the values of certain spherical functions in terms of its 2nd Sobolev semi-norm over a spherical triangle.

Lemma 2.30. Let τ be a spherical triangle such that $|\tau| \leq 1$ and suppose $f \in W^{2,p}(\tau)$ vanishes at the vertices of τ, that is $f(v_i) = 0, i = 1, 2, 3$. Then for all $v \in \tau$,

$$|f(v)| \leq A_{12}\left(\tan\frac{|\tau|}{2}\right)^2 |f|_{2,\infty,\tau},$$

$$|f(v)| \le A'_{12} \left(\tan \frac{|\tau|}{2}\right)^2 |f|'_{2,\infty,\tau} \tag{2.56}$$

for some positive constants $A_{12},\ A'_{12}$ independent of f and τ. Moreover, if f is a homogeneous polynomial of degree d, then

$$|f(v)| \le A_{13} A_\tau^{-1/p} \left(\tan \frac{|\tau|}{2}\right)^2 |f|_{2,p,\tau}$$
$$|f(v)| \le A'_{13} A_\tau^{-1/p} \left(\tan \frac{|\tau|}{2}\right)^2 |f|'_{2,p,\tau} \tag{2.57}$$

for some positive constants $A_{13},\ A'_{13}$ dependent only on d, p and the smallest angle in τ.

Proof. Let R_τ be the radial projection defined before. Let $\bar{v}_i, i = 1, 2, 3$ denote the vertices of a planar triangle $\bar{\tau}$, which is the image of τ under the inverse of R_τ and $\bar{v} = R_\tau^{-1} v$ for $v \in \tau$. Recall that $|\bar{\tau}| = 2 \tan \frac{|\tau|}{2}$ by Lemma 2.1.

Let $f_\delta(v) = |v|^\delta f\left(\frac{v}{|v|}\right)$ be the homogeneous extension of f to $\mathbb{R}^3 \backslash \{0\}$ of degree $\delta = 0$ or 1, and let $\bar{f}_\delta$ denote its restriction to the planar triangle $\bar{\tau}$. By Lemma 3.2 in [23], $\bar{f}_\delta$ belongs to $W^{2,p}(\bar{\tau})$. Note also that $\bar{f}_\delta(\bar{v}_i) = |\bar{v}_i|^\delta f(v_i) = 0, i = 1, 2, 3$. Therefore by Lemma 6.1 in [15], we have for every $\bar{v} \in \bar{\tau}$

$$|\bar{f}_\delta(\bar{v})| \le 12 |\bar{\tau}|^2 |\bar{f}_\delta|_{2,\infty,\bar{\tau}}. \tag{2.58}$$

Since $f(v) = \frac{\bar{f}_\delta(\bar{v})}{|\bar{v}|^\delta}$ and $|\bar{v}|^\delta \ge 1$ for all $\bar{v} \in \bar{\tau}$,

$$|f(v)| \le |\bar{f}_\delta(\bar{v})| \le 48 \left(\tan \frac{|\tau|}{2}\right)^2 |\bar{f}_\delta|_{2,\infty,\bar{\tau}},$$

by (2.58). By Proposition 2.26 we get (2.57) with $A_{12} = 48K_6$ and $A'_{12} = 48K_7$.

If f is a homogeneous polynomial, then its second derivatives are homogeneous polynomials and by (2.53) we have

$$|f|_{2,\infty,\tau} \le K_8 A_\tau^{-1/p} |f|_{2,p,\tau}$$

and

$$|f|'_{2,\infty,\tau} \le K_8 A_\tau^{-1/p} |f|'_{2,p,\tau}$$

for some K_8 depending on d, p and the smallest angle in τ. Hence

$$|f(v)| \leq 48K_6 \left(\tan\frac{|\tau|}{2}\right)^2 |f|_{2,\infty,\tau} \leq A_{13}A_\tau^{-1/p}\left(\tan\frac{|\tau|}{2}\right)^2 |f|_{2,p,\tau}$$

and

$$|f(v)| \leq A'_{13}A_\tau^{-1/p}\left(\tan\frac{|\tau|}{2}\right)^2 |f|'_{2,p,\tau}.$$

This completes the proof with $A_{13} = 48K_6K_8$ and $A'_{13} = 48K_7K_8$.

2.8 Stable local basis

We now describe the stable local bases that the spline spaces poses. We shall use the spline spaces that have a local basis to solve the interpolation problem on the sphere.

Let

$$\mathcal{D} := \cup_{\tau\in\Delta}\{\xi^\tau_{ijk}, i+j+k=d\}, \tag{2.59}$$

with $\xi^\tau_{ijk} := \frac{iu+jv+kw}{d}$ for $\tau =< u, v, w >$ be the set of domain points associated with Δ and d. It is well known that each spline in $S^0_d(\Delta)$ is uniquely determined by associating one Bézier coefficient with each domain point. A subset $\mathcal{M} \subset \mathcal{D}$ is called a minimal determining set for $S^r_d(\Delta)$ if the values of the coefficients of $s \in S^r_d(\Delta)$ associated with domain points in $\mathcal{M}$ uniquely determine all of the coefficients of s.

Definition 2.8. A basis $\{B_\xi\}_{\xi\in\mathcal{M}}$ for a space $\mathcal{S}$ of splines on a triangulation Δ is a stable local basis, if there exists an integer ℓ and constants $0 < C_1 < C_2 < \infty$ depending only on d and the smallest angle θ_Δ in the triangulation Δ such that

1) for each $\xi \in \mathcal{M}$, $\mathrm{supp}(B_\xi) \subseteq \mathrm{star}^\ell(v_\xi)$ for some v_ξ of Δ,

2) for all $\{c_\xi\}_{\xi\in\mathcal{M}}$,

$$C_1 \max_{\xi\in\mathcal{M}}|c_\xi| \leq \|\sum_{\xi\in\mathcal{M}} c_\xi B_\xi\|_{\infty,\mathbb{S}^2} \leq C_2 \max_{\xi\in\mathcal{M}}|c_\xi|. \tag{2.60}$$

A construction of a stable local basis using the Bernstein-Bézier representation of splines in $S_d^r(\Delta)$ when $d \geq 3r+2$ is outlined in [23] with a reference to [10]. Given a minimal determining set, we can construct a basis $\{B_\xi\}_{\xi\in\mathcal{M}}$ for $S_d^r(\Delta)$ by requiring

$$\mu_\eta B_\xi = \delta_{\xi,\eta}, \quad \eta \in \mathcal{M}, \tag{2.61}$$

where μ_η is the linear functional which picks the coefficient associated with the domain point η. In particular, B_ξ has the property that the coefficient associated with ξ is 1 while the coefficients associated with all other points in $\mathcal{M}$ are zero. The remaining coefficients of B_ξ are computed using smoothness conditions.

For any given spline space $S_d^r(\Delta)$, there are many possible choices for a minimal determining set $\mathcal{M}$. A choice of $\mathcal{M}$ presented in [10] leads to a basis with the following properties, where for each ξ, $\Omega_\xi := \mathrm{supp}(B_\xi)$ and τ_ξ is the triangle in which ξ lies.

Proposition 2.31. Let $\{B_\xi\}_{\xi\in\mathcal{M}}$ be the basis for $S_d^r(\Delta)$ corresponding to the minimal determining set $\mathcal{M}$ described in [10]. Then there exist constants $C_3, ..., C_9$ depending only on d, p and the minimal angle in Δ such that for each $\xi \in \mathcal{M}$,

1) there exists a vertex $v_\xi \in \Delta$ such that $\Omega_\xi \subseteq \mathrm{star}^3(v_\xi)$,

2) $\|B_\xi\|_{\infty,\mathbb{S}^2} \leq C_3$,

3) $|\mu_\xi s| \leq C_4 \|s\|_{\infty,\tau_\xi}$, for all $s \in S_d^r(\Delta)$,

4) $|\mu_\xi s| \leq C_5 A_{\tau_\xi}^{-1/p} \|s\|_{p,\tau_\xi}$, for all $s \in S_d^r(\Delta)$, and for every $\tau \in \Delta$,

5) $\|B_\xi\|_{p,\tau} \leq C_6 A_\tau^{1/p}$,

6) $\#I_\tau \leq C_7$, where $I_\tau := \{\xi : \tau \subset \Omega_\xi\}$,

7) $|B_\xi|_{k,\infty,\tau} \leq C_8 \rho_\tau^{-k}$, for all $0 \leq k \leq d$

8) $|B_\xi|_{k,p,\tau} \leq C_9 \rho_\tau^{-k} A_\tau^{1/p}$, for all $0 \leq k \leq d$.

The proof of the above lemma can be found in [23] with $\delta = k-1$ used for 7) and 8). It works the same way with $\delta = k-2$ by means of the results presented in Sections 2.6 and 2.7. Further analysis of the proof of 8) of the above lemma leads to a refinement of 8) as follows. Using (2.1) instead of (2.3) in [23] one gets

$$|B_\xi|_{k,p,\tau} \leq C_9 \left(\tan \frac{\rho_\tau}{2}\right)^{-k} A_\tau^{1/p} \tag{2.62}$$

with $C_9 = A_7 C_6$.

It was shown in [23] that with the basis defined above one can construct a quasi-interpolation operator $Q : L_p(\mathbb{S}^2) \to S_d^r(\Delta)$ which achieves the optimal approximation property. Indeed, extend the linear functionals μ_ξ to all of $L_p(\mathbb{S}^2)$ using Hahn-Banach theorem. Then for every $f \in L_p(\tau_\xi)$,

$$|\mu_\xi f| \leq C_5 A_{\tau_\xi}^{-1/p} \|f\|_{p,\tau_\xi}, \xi \in \mathcal{M}. \tag{2.63}$$

This inequality implies that for each ξ, the carrier of the extended functional μ_ξ is contained in τ_ξ, i.e., if $f \equiv 0$ on τ_ξ, then $\mu_\xi f = 0$. With (2.62) in mind we modify the proof of Proposition 5.2 in [23] accordingly to get the following

Proposition 2.32. For each $f \in L_p(\mathbb{S}^2)$, let

$$Qf := \sum_{\xi \in \mathcal{M}} (\mu_\xi f) B_\xi. \tag{2.64}$$

Then $Qg = g$ for all $g \in \mathcal{H}_d(\mathbb{S}^2)$. Moreover, there exists a constant C_{10} depending only on d, p and the smallest angle in Δ such that for each triangle $\tau \in \Delta$,

$$|Qf|_{k,p,\tau} \leq C_{10} \left(\tan \frac{\rho_\tau}{2}\right)^{-k} \|f\|_{p,\Omega_\tau}, \tag{2.65}$$

where $\Omega_\tau := \cup_{\xi \in I_\tau} \Omega_\xi$ and $I_\tau := \{\xi : \tau \subset \Omega_\xi\}$.

Proof. The proof can be found in [23].

Theorem 4.2 in [23] states the existence of a spherical polynomial of degree d approximating $f \in W^{d+1,p}(\tau)$ for $|\tau| \leq 1$ satisfying

$$|f - s|_{k,p,\tau} \leq K_9' |\tau|^{d+1-k} |f|_{d+1,p,\tau}.$$

for some positive constant K_9 depending on d, p and the smallest angle of τ. With a little modification in the proof we can see that in fact

$$|f-s|_{k,p,\tau} \leq K_9(\tan\frac{|\tau|}{2})^{d+1-k}|f|_{d+1,p,\tau} \tag{2.66}$$

for a positive constant K_9 depending on d, p and the smallest angle of τ. Using this inequality we can prove the following result.

THEOREM 2.33. Suppose $\tau \in \Delta$ is a spherical triangle with $|\tau| \leq 1$. Let $f \in W^{m+1,p}(\tau)$ for $0 \leq m \leq d$ such that $(d-m) \bmod 2 = 0$. There exists a spherical homogeneous polynomial s of degree d such that for every $0 \leq k \leq m$

$$|f-s|_{k,p,\tau} \leq C_{11}\left(\tan\frac{|\tau|}{2}\right)^{m+1-k}|f|_{m+1,p,\tau}. \tag{2.67}$$

Here C_{11} is a constant that depends on p, m and θ_Δ. Moreover

$$|f-s|_{k,p,\Omega_\tau} \leq C_{11}\left(\tan\frac{|T'|}{2}\right)^{m+1-k}|f|_{m+1,p,\Omega_\tau}. \tag{2.68}$$

Here T' is the largest triangle in Ω_τ, i.e. $|T| = \max\{|T| : T \in \Omega_\tau\}$.

PROOF. Fix m. By Theorem 4.2 in [23], there exists a spherical homogeneous polynomial s' of degree m such that for every $0 \leq k \leq m$

$$|f-s'|_{k,p,\tau} \leq C_{11}|\tau|^{m+1-k}|f|_{m+1,p,\tau}. \tag{2.69}$$

If we slightly modify the proof of Theorem 4.2 [23], i.e. replace (2.1) by (2.2), we can get

$$|f-s'|_{k,p,\tau} \leq C_{11}\left(\tan\frac{|\tau|}{2}\right)^{m+1-k}|f|_{m+1,p,\tau}. \tag{2.70}$$

Since $(d-m) \bmod 2 = 0$, $s = |v|^{d-m}s'$ is a homogeneous spherical polynomial of degree d. Since on the unit sphere $s' \equiv s$, their $k-1$-st extensions are the same, and we have (2.67). To get (2.68), sum (2.67) over triangles in Ω_τ. This completes the proof.

THEOREM 2.34. Let Δ be a β-quasi-uniform spherical triangulation with $|\Delta| \leq 1$.

Let $1 \leq p \leq \infty$, $d \geq 3r+2$, and $0 \leq k \leq d$. Then there exists a constant C_{12} depending only on d, p and the smallest angle in Δ, such that

$$|f - Qf|_{k,p,\tau} \leq C_{12} \left(\tan \frac{|T'|}{2} \right)^{m+1-k} |f|_{m+1,p,\Omega_\tau}, \tag{2.71}$$

for all $f \in W^{m+1,p}(\mathbb{S}^2)$ and all $\tau \in \Delta$. Moreover, there exists a constant C_{13} such that

$$|f - Qf|_{k,p,\mathbb{S}^2} \leq C_{13} \left(\tan \frac{|\Delta|}{2} \right)^{m+1-k} |f|_{m+1,p,\mathbb{S}^2}, \tag{2.72}$$

for all $f \in W^{m+1,p}(\mathbb{S}^2)$ and all $0 \leq k \leq d$ such that $Qf \in W^{k,p}(\mathbb{S}^2)$. Here m is taken between 0 and d with $(d-m) \bmod 2 = 0$.

Proof. Let $\tau \in \Delta$ with $|\tau| \leq 1$. By Theorem 2.33 there exists a spherical homogeneous polynomial s of degree d such that (2.67) holds. By the linearity of Q and the fact that Q reproduces polynomials of degree d we can write

$$|f - Qf|_{k,p,\tau} \leq |f - s|_{k,p,\tau} + |Q(f-s)|_{k,p,\tau}.$$

We now consider the last term in the above inequality. By (2.65)

$$|Q(f-s)|_{k,p,\tau} \leq C_{10} \left(\tan \frac{\rho_\tau}{2} \right)^{-k} \|f - s\|_{p,\Omega_\tau}.$$

Since Δ is assumed to be β-quasi-uniform $|\rho_\tau| \geq \frac{|T'|}{\beta}$ and therefore

$$\tan \frac{\rho_\tau}{2} \geq \tan \frac{|T'|}{2\beta} \geq \frac{1}{\beta^2} \tan \frac{|T'|}{2}.$$

By Theorem 2.33

$$\begin{aligned} |Q(f-s)|_{k,p,\tau} &\leq C_{10} C_{11} (\beta)^{2k} \left(\tan \frac{|T'|}{2} \right)^{-k} \left(\tan \frac{|T'|}{2} \right)^{m+1} |f|_{m+1,p,\Omega_\tau} \\ &\leq C_{10} C_{11} (\beta)^{2k} \left(\tan \frac{|T'|}{2} \right)^{m+1-k} |f|_{m+1,p,\Omega_\tau}. \end{aligned}$$

Therefore we get (2.71) with $C_{12} = C_{11}(1 + C_{10}\beta^{2k})$.

To prove (2.72), we sum (2.71) over all triangles in Δ.

$$|f - Qf|_{k,p,\mathbb{S}^2} = \sum_{\tau \in \Delta} |f - Qf|_{k,p,\tau} \leq C_{12} \left(\tan \frac{|\Delta|}{2} \right)^{m+1-k} \sum_{\tau \in \Delta} |f|_{m+1,p,\Omega_\tau}$$

$$
\begin{aligned}
&\leq C_{12}\left(\tan\frac{|\Delta|}{2}\right)^{m+1-k}\sum_{\tau\in\Delta}\sum_{\tau'\subset\Omega_\tau}|f|_{k,p,\tau'}\\
&= C_{12}\left(\tan\frac{|\Delta|}{2}\right)^{m+1-k}\sum_{\tau'\in\Delta}\#\{\tau:\tau'\subset\Omega_\tau\}|f|_{m+1,p,\tau'}\\
&\leq C_{12}K_{10}\left(\tan\frac{|\Delta|}{2}\right)^{m+1-k}\sum_{\tau'\in\Delta}|f|_{m+1,p,\tau'}.
\end{aligned}
$$

Here $K_{10} := \max\{\#\{\tau : \tau' \subset \Omega_\tau\}, \tau' \in \Delta\}$ which is bounded by Lemma 2.4. Therefore (2.72) holds with $C_{13} = C_{12}K_{10}$. This completes the proof.

Corollary 2.35. Let Δ be a β-quasi-uniform spherical triangulation with $|\Delta| \leq 1$. Let $1 \leq p \leq \infty$, $d \geq 3r+2$, and $0 \leq k \leq d$. Then there exists a constant C_{14} depending only on d, p and the smallest angle in Δ, such that

$$
|f - Qf|'_{k,p,\tau} \leq C_{14}\sum_{\ell=0}^{k}\left(\tan\frac{|T'|}{2}\right)^{m+1-\ell}|f|_{m+1,p,\Omega_\tau}, \tag{2.73}
$$

for all $f \in W^{m+1,p}(\mathbb{S}^2)$ and all $\tau \in \Delta$. Moreover, there exists a constant C_{15} such that

$$
|f - Qf|'_{k,p,\mathbb{S}^2} \leq C_{15}\sum_{\ell=0}^{k}\left(\tan\frac{|\Delta|}{2}\right)^{m+1-\ell}|f|_{m+1,p,\mathbb{S}^2}, \tag{2.74}
$$

for all $f \in W^{m+1,p}(\mathbb{S}^2)$ and all $0 \leq k \leq d$ such that $Qf \in W^{k,p}(\mathbb{S}^2)$. Here m is taken between 0 and d with $(d-m) \bmod 2 = 0$.

Proof. The key step in the proof is to note that by Proposition 2.26 for any function h

$$
|h|'_{k,p,\tau} \leq A_5|\bar{h}_{k-2}|_{k-1,p,\bar{\tau}} + A_6|\bar{h}_{k-2}|_{k,p,\bar{\tau}} \leq \max\{A_5, A_6\}\|\bar{h}_{k-2}\|_{k,p,\bar{\tau}}.
$$

By Lemma 3.6 [23] $\|\bar{h}_{k-2}\|_{k,p,\bar{\tau}} \leq K_9\|\bar{h}_{k-1}\|_{k,p,\bar{\tau}}$. Then

$$
|h|'_{k,p,\tau} \leq \max\{A_5, A_6\}K_9(|\bar{h}_{k-1}|_{k,p,\bar{\tau}} + \|\bar{h}_{k-1}\|_{k-1,p,\bar{\tau}}).
$$

By Proposition 2.26 and Lemma 3.6 [23]

$$
|h|'_{k,p,\tau} \leq \max\{A_5, A_6\}K_9(A_2|h|_{k,p,\tau} + K_{11}\|\bar{h}_{k-2}\|_{k-1,p,\bar{\tau}}).
$$

Repeating the process and choosing a suitable constant we obtain for $h = f - Qf$:

$$|f - Qf|'_{k,p,\tau} \leq K_{12} \sum_{\ell=0}^{k} |f - Qf|_{\ell,p,\tau}.$$

By Theorem 2.34 for each $0 \leq \ell \leq k$

$$|f - Qf|_{\ell,p,\tau} \leq C_{12} \left(\tan \frac{|T'|}{2} \right)^{m+1-\ell} |f|_{m+1,p,\Omega_\tau}$$

and we have (2.73). To obtain (2.74) we sum over triangles in Δ.

Chapter 3

Global Spline Approximation on the Sphere

3.1 Minimal energy interpolating spline. Linear extension.

Suppose we are given values $\{f(v), v \in \mathcal{V}\}$ of an unknown function f at a set $\mathcal{V}$ of scattered points on the unit sphere. To find a homogeneous spline approximation of f, we choose a linear space $\mathcal{S} \subseteq S_d^r(\Delta)$ of polynomial splines of degree d and smoothness r defined on a triangulation Δ with vertices at the points of $\mathcal{V}$. Define

$$\Gamma(f) := \{s \in \mathcal{S} : s(v) = f(v), v \in \mathcal{V}\}$$

to be the set of all splines in $\mathcal{S}$ that interpolate f at the points in $\mathcal{V}$. Assume that $\mathcal{S}$ is big enough, so that $\Gamma(f)$ is not empty. We choose a spline S_f such that

$$\mathcal{E}(S_f) = \min_{s \in \Gamma(f)} \mathcal{E}(s), \tag{3.1}$$

where for a spherical triangle $\tau \in \Delta$

$$\mathcal{E}_\tau(s) := \sum_{|\alpha|=2} \|D^\alpha s_1\|_{2,\tau}^2 \tag{3.2}$$

and

$$\mathcal{E}(s) := \sum_{\tau \in \Delta} \mathcal{E}_\tau(s). \tag{3.3}$$

Here s_1 is the linear homogeneous extension of s to $\mathbb{R}^3 \backslash \{0\}$, α is a triple index with entries running through x, y, z, e.g., $D^{(1,1,1)} = D_x D_y D_z$, and $\| \cdot \|_{2,\tau}$ is the usual L_2 norm on τ. We call S_f the minimal energy interpolation spline. Let $B(\mathbb{S}^2)$ be the set of all bounded real-valued functions on the sphere. Define

$$\mathcal{X} := \{f \in B(\mathbb{S}^2) : f|_\tau \in C^3(\tau), \forall \tau \in \Delta\}.$$

For each triangle $\tau \in \Delta$, let

$$\langle f, g \rangle_\tau := \int_\tau \sum_{|\alpha|=2} D^\alpha f_1 \; D^\alpha g_1.$$

Then

$$\langle f, g \rangle := \langle f, g \rangle_{\mathbb{S}^2} = \sum_{\tau \in \Delta} \langle f, g \rangle_\tau$$

is a semi-definite inner product on $\mathcal{X}$. Let $\|f\|_\tau$ and $\|f\|$ be the associated semi-norms. We refer to them as energy or $\mathcal{X}$-norms.

It is easy to see that $\langle \cdot, \cdot \rangle$ is an inner product on the linear space

$$\mathcal{W} := \{s \in \mathcal{S} : s(v) = 0, v \in \mathcal{V}\}. \tag{3.4}$$

Indeed, if $\langle w, w \rangle = 0$ for some $w \in \mathcal{W}$, then w is a linear homogeneous polynomial on Δ and since w vanishes at all vertices, $w \equiv 0$. Since $\mathcal{W}$ is finite-dimensional, it follows that $\mathcal{W}$ equipped with the inner product $\langle \cdot, \cdot \rangle$ is a Hilbert space.

Given f, suppose s_f is any spline in the set $\Gamma(f)$ defined above. Then it is easy to see that the solution S_f to the minimal energy problem is equal to $s_f - \mathcal{P}s_f$, where $\mathcal{P}$ is the linear projector $\mathcal{P} : \mathcal{X} \to \mathcal{W}$ defined by

$$\mathcal{E}(f - \mathcal{P}f) = \min_{w \in \mathcal{W}} \mathcal{E}(f - w), \tag{3.5}$$

for all $f \in \mathcal{X}$. Since $\mathcal{W}$ is a Hilbert space with respect to $\langle \cdot, \cdot \rangle$, $\mathcal{P}f$ is uniquely defined and characterized by

$$\langle f - \mathcal{P}f, w \rangle = 0, \forall w \in \mathcal{W}. \tag{3.6}$$

Moreover

$$\|\mathcal{P}f\| \leq \|f\| \tag{3.7}$$

for all $f \in \mathcal{X}$.

We now establish a lemma showing the equivalence of certain semi-norms on the space $\mathcal{X}$ defined above.

LEMMA 3.1. Let τ be a spherical triangle with $|\tau| \leq 1$ and let $f \in \mathcal{X}$. Let $\mathcal{E}_\tau$ be the functional defined in (3.2). There exists a positive constant D_1 such that

$$D_1 |f|^2_{2,2,\tau} \leq \mathcal{E}_\tau(f) \leq |f|^2_{2,2,\tau}, \forall f \in \mathcal{X}. \tag{3.8}$$

PROOF. By the definition

$$|f|^2_{2,2,\tau} = \left(\sum_{|\alpha|=2} \|D^\alpha f_1\|_{2,\tau} \right)^2 \geq \sum_{|\alpha|=2} \|D^\alpha f_1\|^2_{2,\tau} = \mathcal{E}_\tau(f).$$

Since the number of elements in the sum (3.2) is 9,

$$|f|^2_{2,2,\tau} = \left(\sum_{|\alpha|=2} \|D^\alpha f_1\|_{2,\tau} \right)^2 \leq 9 \sum_{|\alpha|=2} \|D^\alpha f_1\|^2_{2,\tau} = 9\ \mathcal{E}_\tau(f).$$

Next we establish a reproductive property of the energy functional $\mathcal{E}_\tau$.

LEMMA 3.2. Let τ be a spherical triangle with $|\tau| \leq 1$. Suppose $f \in \mathcal{X}$. Then $\mathcal{E}_\tau(f) = 0$ if and only if f is a trivariate homogeneous linear polynomial on τ.

PROOF. If f is a trivariate linear homogeneous polynomial, so is its linear extension. Naturally its second order derivatives vanish on $\mathbb{R}^3$. If $\mathcal{E}_\tau(f) = 0$, then for every α with $|\alpha| = 2$ we have $D^\alpha f_1|_\tau = 0$. Since f_1 is linear homogeneous, $D^\alpha f_1$ is homogeneous of degree -1, therefore, by the uniqueness of homogeneous extensions $(D^\alpha f_1|_\tau)_{-1} = D^\alpha f_1$. On the other hand, by the definition $(D^\alpha f_1|_\tau)_{-1}(v) = |v|^{-1}(D^\alpha f_1|_\tau)\left(\frac{v}{|v|}\right)$. As we noted above $D^\alpha f_1|_\tau = 0$, and therefore $D^\alpha f_1 = 0$ on $\mathbb{R}^3$. Hence f_1 is a polynomial of degree at most 1. Since it is a homogeneous linear function f_1 must be a homogeneous linear polynomial on $\mathbb{R}^3$. Therefore by the uniqueness of a homogeneous extension f is a linear homogeneous polynomial on τ.

In addition to Lemma 3.2 we need to establish the equivalence of energy and L_2 norms on the Hilbert space $\mathcal{W}$.

THEOREM 3.3. Suppose $\mathcal{S} \subseteq S^0_d(\Delta)$ is a spline space defined on a β-quasi-uniform triangulation Δ with $|\Delta| \leq 1$. Let $\mathcal{W}$ be the associated Hilbert space (3.4). There exist constants $0 < D_2 \le D_3 < \infty$ depending only d and β such that

$$D_2 \|s\|_{2,\mathbb{S}^2}^2 \le \left(\tan \frac{|\Delta|}{2}\right)^4 \|s\|^2 \le D_3 \|s\|_{2,\mathbb{S}^2}^2, \tag{3.9}$$

for all $s \in \mathcal{W}$.

Proof. By Lemmas 2.30 and 3.1 for every $s \in \mathcal{W}$

$$\int_\tau |s|^2 \le A_{13}^2 \left(\tan \frac{|\tau|}{2}\right)^4 |s|_{2,2,\tau}^2 \le D_1{}^{-1} A_{13}^2 \left(\tan \frac{|\tau|}{2}\right)^4 \mathcal{E}_\tau(s).$$

Summing over all $\tau \in \Delta$ we get

$$\int_{\mathbb{S}^2} |s|^2 \le D_1{}^{-1} A_{13}^2 \left(\tan \frac{|\Delta|}{2}\right)^4 \mathcal{E}(s).$$

By Lemma 3.1 and Lemma 2.29

$$\mathcal{E}_\tau(s) \le |s|_{2,2,\tau}^2 \le \frac{A_{11}^2}{(\tan \frac{\rho_\tau}{2})^4} \|s\|_{2,\tau}^2.$$

Sum over $\tau \in \Delta$ to get

$$\mathcal{E}(s) \le \frac{A_{11}^2}{(\tan \frac{\rho_\Delta}{2})^4} \|s\|_{2,\mathbb{S}^2}^2.$$

Since Δ is β-quasi-uniform $|\rho_\Delta| \ge \frac{|\Delta|}{\beta}$ and therefore

$$\tan \frac{\rho_\Delta}{2} \ge \tan \frac{|\Delta|}{2\beta} \ge \frac{1}{\beta^2} \left(\tan \frac{|\Delta|}{2}\right).$$

Then

$$\mathcal{E}(s) \le \frac{A_{11}^2 \beta^8}{(\tan \frac{\Delta}{2})^4} \|s\|_{2,\mathbb{S}^2}^2.$$

Let $D_2 := D_1 A_{13}^{-2}$ and $D_3 := A_{11}^2 \beta^8$ to get the result.

Next we want to show that under certain conditions on $\mathcal{S}$, the $\mathcal{X}$-norm on the Hilbert space $\mathcal{W}$ is also equivalent to a certain coefficient norm.

Corollary 3.4. Suppose $\mathcal{S} \subseteq S_d^r(\Delta)$ is a spline space defined on a β-quasi-uniform triangulation Δ, and that $\{B_\xi\}_{\xi \in \mathcal{M}}$ is a stable local basis for $\mathcal{S}$ defined in Proposition 2.31. Then $\{B_\xi\}_{\xi \in \mathcal{N}}$ is a Riesz basis (with respect to the $\mathcal{X}$-norm) for the linear space $\mathcal{W}$ defined in (3.4). Here $\mathcal{N}$ is the subset of the minimal determining set $\mathcal{M}$

excluding the set of vertices $\mathcal{V}$ of Δ. In particular, there exist positive constants D_4, D_5 depending on d, β and ℓ such that

$$D_4 \min_{\tau\in\Delta} A_\tau \sum_{\xi\in\mathcal{N}} |c_\xi|^2 \leq \left(\tan\frac{|\Delta|}{2}\right)^4 \|\sum_{\xi\in\mathcal{N}} c_\xi B_\xi\|^2 \leq D_5 \max_{\tau\in\Delta} A_\tau \sum_{\xi\in\mathcal{N}} |c_\xi|^2, \tag{3.10}$$

for all $\{c_\xi\}_{\xi\in\mathcal{N}}$.

Proof. Let us note first that for any spline $s \in \mathcal{W}$, $s = \sum_{\xi\in\mathcal{M}} c_\xi B_\xi = \sum_{\xi\in\mathcal{N}} c_\xi B_\xi$ due to the zero interpolating conditions and (2.64). Denote $\mathcal{N}_\tau := \{\xi : \tau \subseteq \mathrm{supp}(B_\xi)\}$, the set of domain points ξ with the support of corresponding basis functions B_ξ containing τ. By Proposition 2.31, 4) there exists a positive constant C_5 depending only on d and θ_Δ, such that for each coefficient

$$|c_{\xi'}|^2 \leq C_5^2 A_\tau^{-1} \int_\tau |s|^2,$$

where τ contains ξ'. Note that since basis functions have local support, $s|_\tau = \sum_{\xi\in\mathcal{N}_\tau} c_\xi B_\xi$. Therefore

$$\sum_{\xi\in\mathcal{N}\cap\tau} |c_\xi|^2 \leq C_5^2 \binom{d+2}{2} A_\tau^{-1} \int_\tau |\sum_{\xi\in\mathcal{N}_\tau} c_\xi B_\xi|^2.$$

Then summing over $\tau \in \Delta$

$$\frac{1}{\binom{d+2}{2} C_5^2} \min_{\tau\in\Delta} A_\tau \sum_{\xi\in\mathcal{N}} |c_\xi|^2 \leq \int_{\mathbb{S}^2} |\sum_{\xi\in\mathcal{N}} c_\xi B_\xi|^2. \tag{3.11}$$

Similarly, by Proposition 2.31, 5) there exists a positive constant C_6, depending only on d and θ_Δ, such that

$$\int_\tau |B_\xi|^2 \leq C_6^2 A_\tau$$

for any ξ and τ. Then

$$\int_\tau |\sum_{\xi\in\mathcal{N}_\tau} c_\xi B_\xi|^2 \leq \int_\tau \sum_{\xi\in\mathcal{N}_\tau} |c_\xi|^2 \sum_{\xi\in\mathcal{N}_\tau} |B_\xi|^2 \leq n_\ell(\tau) \binom{d+2}{2} C_6^2 A_\tau \sum_{\xi\in\mathcal{N}_\tau} |c_\xi|^2,$$

where $n_\ell(\tau)$ is the number of triangles in $\mathrm{star}^\ell(\tau)$. Summing over triangles in Δ

$$\int_{\mathbb{S}^2} |\sum_{\xi\in\mathcal{N}} c_\xi B_\xi|^2 \leq \max_{\tau\in\Delta} A_\tau \binom{d+2}{2} C_6^2 \sum_{\tau\in\Delta} n_\ell(\tau) \sum_{\xi\in\mathcal{N}_\tau} |c_\xi|^2.$$

Since $\mathcal{N}_\tau \subseteq \{\xi \in \mathrm{star}^\ell(\tau)\}$

$$\int_{\mathbb{S}^2} |\sum_{\xi\in\mathcal{N}} c_\xi B_\xi|^2 \le \binom{d+2}{2} C_6^2 \max_{\tau\in\Delta} A_\tau n_\ell^2 \sum_{\xi\in\mathcal{N}} |c_\xi|^2,$$

where $n_\ell := \max_{\tau\in\Delta}\{n_\ell(\tau)\}$. By Lemma 2.4

$$n_\ell \le \frac{5\beta^2(2\ell+1)^2}{4}.$$

Hence

$$\int_{\mathbb{S}^2} |\sum_{\xi\in\mathcal{N}} c_\xi B_\xi|^2 \le K_{12} \max_{\tau\in\Delta} A_\tau \sum_{\xi\in\mathcal{N}} |c_\xi|^2, \tag{3.12}$$

with K_{12} depending on d, β and ℓ. By Theorem 3.3, and (3.11), (3.12) above, we get

$$\frac{D_2}{\binom{d+2}{2}C_5^2} \min_{\tau\in\Delta} A_\tau \sum_{\xi\in\mathcal{N}} |c_\xi|^2 \le \left(\tan\frac{|\Delta|}{2}\right)^4 \|s\|^2 \le D_3 K_{12} \max_{\tau\in\Delta} A_\tau \sum_{\xi\in\mathcal{N}} |c_\xi|^2.$$

Therefore, we obtain (3.10) with $D_4 = \frac{D_2}{\binom{d+2}{2}C_5^2}$ and $D_5 = D_3 K_{12}$.

Next, we estimate $\mathcal{X}$-norm of the projection operator $\mathcal{P}$ in (3.6) outside of support of $f \in \mathcal{X}$. Here we follow a similar result for bivariate splines that can be found in [16], making several adjustments for the spherical splines. Before we proceed with the result we need the following lemma, which can be found in [9].

Lemma 3.5. If the sequence $\{a_i\}_{i=1}^\infty$ satisfies

$$|a_m| \ge \gamma \sum_{j\ge m+1} |a_j|$$

for all $m \ge 0$ and some $\gamma \in (0,1)$, then

$$|a_m| \le a_0 \frac{(1-\gamma)^m}{\gamma}.$$

Proof. See [9].

It is established in Section 5 of [23] that $\{B_\xi\}_{\xi\in\mathcal{M}}$ is a local basis with a local support size ℓ equal to 3. The following theorem, however, holds in general for any fixed ℓ.

THEOREM 3.6. There exist constants $0 < \sigma < 1$ and D_6, depending only on ℓ, d, β, such that for any triangle $T \in \Delta$ and any function $f \in \mathcal{X}$ with $\mathrm{supp}(f) \subseteq T$

$$\|\mathcal{P}f\|_\tau \leq D_6 \sigma^k \|f\|, \tag{3.13}$$

whenever $\tau \in \mathrm{star}^{2(k+2)\ell+1}(T) \backslash \mathrm{star}^{2(k+1)\ell+1}(T)$ and $k \geq 1$.

PROOF. Let

$$\begin{aligned} \mathcal{M}_0^T : &= \{\xi \in \mathcal{M} : \mathrm{supp}(B_\xi) \cap T \neq \emptyset\}, \\ \mathcal{M}_k^T : &= \{\xi \in \mathcal{M} : \mathrm{supp}(B_\xi) \cap \mathrm{star}^{2k\ell}(T) \neq \emptyset\}, \\ \mathcal{N}_0^T : &= \mathcal{M}_0^T, \\ \mathcal{N}_k^T : &= \mathcal{M}_k^T \backslash \mathcal{M}_{k-1}^T. \end{aligned}$$

Suppose $\mathcal{P}f = \sum_{\xi \in \mathcal{M}} c_\xi B_\xi$, and let

$$u_k := \sum_{\xi \in \mathcal{M}_k^T} c_\xi B_\xi, \quad w_k := \mathcal{P}f - u_k, \quad a_k := \sum_{\xi \in \mathcal{N}_k^T} c_\xi^2,$$

for $k \geq 0$. Since $\mathcal{P}f \in \mathcal{W}$, by Corollary 3.4

$$\sum_{j \geq k+1} a_j = \sum_{\xi \notin \mathcal{M}_k^T} c_\xi^2 \leq \left(\tan \frac{|\Delta|}{2}\right)^4 (D_4 \min_{\tau \in \Delta} A_\tau)^{-1} \|w_k\|^2.$$

Note that $w_k \in \mathcal{W}$ as well, then using (3.6) we have $\langle f - \mathcal{P}f, w_k \rangle = 0$. Moreover, $\langle f, w_k \rangle = 0$, since $\mathrm{supp}(f) \subseteq T$ and $\mathrm{supp}(w_k)$ lies outside T. In fact, $\mathrm{supp}(w_k) \cap \cup_{\xi \in \mathcal{M}_{k-1}^T} \mathrm{supp}(B_\xi) = \emptyset$ for $k \geq 1$, it follows that

$$\|w_k\|^2 = \langle \mathcal{P}f - u_k, w_k \rangle = \langle f - u_k, w_k \rangle = -\langle u_k, w_k \rangle =$$

$$-\langle \sum_{\xi \in \mathcal{N}_k^T} c_\xi B_\xi, w_k \rangle \leq \| \sum_{\xi \in \mathcal{N}_k^T} c_\xi B_\xi \| \, \|w_k\|,$$

and therefore by (3.10)

$$\|w_k\|^2 \leq \| \sum_{\xi \in \mathcal{N}_k^T} c_\xi B_\xi \|^2 \leq \frac{D_5 \max_{\tau \in \Delta} A_\tau}{(\tan \frac{|\Delta|}{2})^4} a_k.$$

Hence

$$\sum_{j\ge k+1} a_j \le \frac{D_5}{D_4}\frac{\max_{\tau\in\Delta} A_\tau}{\min_{\tau\in\Delta} A_\tau} a_k.$$

By Lemma 2.3

$$\frac{\max_{\tau\in\Delta} A_\tau}{\min_{\tau\in\Delta} A_\tau} \le \frac{5}{4}\frac{|\Delta|^2}{\rho_\Delta^2} \le \frac{5}{4}\beta^2,$$

and thus

$$\sum_{j\ge k+1} a_j \le \frac{5D_5\beta^2}{4D_4} a_k.$$

Let $\gamma := \frac{4D_4}{4D_4+5D_5\beta^2}$. Then by Lemma 3.5

$$a_k \le a_0\frac{(1-\gamma)^k}{\gamma} = \frac{a_0}{\gamma}\sigma^{2k},$$

with $\sigma := \sqrt{1-\gamma}$. It is easy to see that both γ and σ are positive and bounded above by 1. Since (3.7) holds for f, by Corollary 3.4 we have

$$a_0 \le \sum_{j\ge 0} a_j = \sum_{\xi\in\mathcal{M}} c_\xi^2 \le \frac{(\tan\frac{|\Delta|}{2})^4}{D_4 \min_{\tau\in\Delta} A_\tau}\|\mathcal{P}f\|^2 \le \frac{(\tan\frac{|\Delta|}{2})^4}{D_4 \min_{\tau\in\Delta} A_\tau}\|f\|^2.$$

Let $\tau \in \mathrm{star}^{2(k+2)\ell+1}(T)\backslash\mathrm{star}^{2(k+1)\ell+1}(T)$ for some $k \ge 1$. If $\xi \in \mathcal{M}_k^T$, then $\mathrm{supp}(B_\xi) \subseteq \mathrm{star}^{2(k+1)\ell}(T)$, and therefore $\tau\cap\mathrm{supp}(B_\xi) = \emptyset$. Using (3.10) again,

$$\|\mathcal{P}f\cdot\chi_\tau\|^2 \le \|\sum_{\xi\notin\mathcal{M}_k^T} c_\xi B_\xi\|^2 \le$$

$$\frac{D_5\max_{\tau\in\Delta} A_\tau}{(\tan\frac{|\Delta|}{2})^4}\sum_{\xi\notin\mathcal{M}_k^T} c_\xi^2 = \frac{D_5\max_{\tau\in\Delta} A_\tau}{(\tan\frac{|\Delta|}{2})^4}\sum_{j\ge k+1} a_j \le \frac{5D_5^2\beta^2}{4\gamma D_4^2}\sigma^{2k}\|f\|^2.$$

We obtained (3.13) with $D_6 = \sqrt{\frac{5}{4\gamma}}\frac{D_5\beta}{D_4}$.

As a consequence of the last result, we can now compare Sobolev semi-norms of $\mathcal{P}f$ and f. Analogous result for bivariate polynomials can be found in [15], and similar proof holds.

THEOREM 3.7. There exists a constant D_7 depending only on d, ℓ and β, such that for every $f \in \mathcal{X}$

$$|\mathcal{P}f|_{2,\infty,\mathbb{S}^2} \le D_7|f|_{2,\infty,\mathbb{S}^2}. \tag{3.14}$$

PROOF. Let τ be a fixed triangle in Δ, and let

$$\Omega_0^\tau := \text{star}^{4\ell+1}(\tau), \quad \Omega_k^\tau := \text{star}^{2(k+2)\ell+1}(\tau)\backslash\text{star}^{2(k+1)\ell+1}(\tau).$$

Let n_k denote the number of triangles in $\Omega_k^\tau,\ k \geq 0$. For a homogeneous polynomial of degree with d we have by Lemma 3.1 and (2.53)

$$\|P\|_\tau^2 \geq D_1|P|_{2,2,\tau}^2 \geq \frac{D_1 A_\tau}{K_{13}^2}|P|_{2,\infty,\tau}^2,$$

for K_{13} depending on d and Θ_τ. Similarly, for any function $f \in \mathcal{X}$ and any triangle $\tau \in \Delta$ by Lemma 3.1 and (2.56) we have

$$\|f\|_\tau^2 \leq A_\tau|f|_{2,\infty,\tau}^2. \tag{3.15}$$

Write $f = \sum_{\tau\in\Delta} f_\tau$ with $\text{supp}(f_\tau) \subseteq \tau$. Since $\mathcal{P}$ is a linear operator,

$$|\mathcal{P}f|_{2,\infty,\tau} \leq \sum_{\tau\in\Delta}|\mathcal{P}f_\tau|_{2,\infty,\tau} \leq \frac{K_{13}}{(D_1A_\tau)^{1/2}}\sum_{\tau\in\Delta}\|\mathcal{P}f_\tau\|_\tau.$$

Then by (3.13), (3.7) and (3.15)

$$\begin{aligned}
|\mathcal{P}f|_{2,\infty,\tau} &\leq \frac{K_{13}}{(D_1A_\tau)^{1/2}}\sum_{k\geq 0}\sum_{\tau\in\Omega_k^\tau}\|\mathcal{P}f_\tau\|_\tau \\
&\leq \frac{K_{13}}{(D_1A_\tau)^{1/2}}\Big(\sum_{\tau\in\Omega_0^\tau}\|f_\tau\| + \sum_{k\geq 1}\sum_{T\in\Omega_k^\tau} D_6\sigma^k\|f_\tau\|\Big) \\
&\leq \frac{K_{13}}{(D_1A_\tau)^{1/2}}(\max_{\tau\in\Delta} A_T^{1/2})(n_0 + D_6\sum_{k\geq 1}\sigma^k n_k)|f|_{2,\infty,\tau}.
\end{aligned}$$

By Lemma 2.4 each n_k is bounded by a constant depending on β and k. Since $\sigma < 1$, and number of rings around τ is bounded $\sum_{k\geq 1}\sigma^k < \infty$. Also, as above

$$\frac{\max_{\tau\in\Delta} A_\tau^{1/2}}{\min_{\tau\in\Delta} A_\tau^{1/2}} \leq \sqrt{\frac{5}{4}\frac{|\Delta|}{\rho_\Delta}} \leq \sqrt{\frac{5}{4}}\beta$$

by Lemma 2.3. Then (3.14) follows by taking the supremum over all $\tau \in \Delta$.

We are finally in a position to prove the main result of this section.

THEOREM 3.8. Suppose $\mathcal{S} \subseteq S_d^r(\Delta)$ is a spline space defined on a β-quasi-uniform triangulation Δ with $|\Delta| \leq 1$, $d \geq 3r+2$. For d odd there exists a constant D_8 depending only on d and β, such that the minimal energy interpolant S_f, defined in (2.73), satisfies

$$\|f - S_f\|_{\infty,\mathbb{S}^2} \leq D_8 \left(\tan\frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2}, \tag{3.16}$$

for all $f \in C^2(\mathbb{S}^2)$. For d even there exist constants D_9 and D_{10} depending only on d and β, such that the minimal energy interpolant S_f satisfies

$$\|f - S_f\|_{\infty,\mathbb{S}^2} \leq D_9 \left(\tan\frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2} + D_{10} \left(\tan\frac{|\Delta|}{2}\right)^3 |f|_{3,\infty,\mathbb{S}^2}, \tag{3.17}$$

for all $f \in C^3(\mathbb{S}^2)$.

Proof. Given a function $f \in \mathcal{X}$, let $s_f \in \Gamma(f)$ be the quasi-interpolant defined in Section 2.8. If d is odd by Theorem 2.34 there exists a constant C_{13} depending on d and the smallest angle of Δ such that

$$\|f - s_f\|_{\infty,\mathbb{S}^2} \leq C_{13} \left(\tan\frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2}, \tag{3.18}$$

and

$$|f - s_f|_{2,\infty,\mathbb{S}^2} \leq C_{13} |f|_{2,\infty,\mathbb{S}^2}.$$

Then

$$\begin{aligned} |s_f|_{2,\infty,\mathbb{S}^2} &\leq |f - s_f|_{2,\infty,\mathbb{S}^2} + |f|_{2,\infty,\mathbb{S}^2} \\ &\leq (C_{13}+1)|f|_{2,\infty,\mathbb{S}^2}. \end{aligned} \tag{3.19}$$

Since $\mathcal{P}s_f = s_f - S_f$, by Theorem 3.7

$$|s_f - S_f|_{2,\infty,\mathbb{S}^2} = |\mathcal{P}s_f|_{2,\infty,\mathbb{S}^2} \leq D_7 |s_f|_{2,\infty,\mathbb{S}^2},$$

and by (3.19)

$$|s_f - S_f|_{2,\infty,\mathbb{S}^2} \leq D_7(C_{13}+1)|f|_{2,\infty,\mathbb{S}^2}.$$

Since both S_f and s_f interpolate f, their difference satisfies the hypothesis of Lemma 2.30 and thus

$$\|s_f - S_f\|_{\infty,\mathbb{S}^2} \leq A_8 \left(\tan\frac{|\Delta|}{2}\right)^2 |s_f - S_f|_{2,\infty,\mathbb{S}^2}$$

$$\leq \quad A_8 D_7 (C_{13}+1) \left(\tan \frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2}.$$

Then by (3.18)

$$\begin{aligned}
\|f - S_f\|_{\infty,\mathbb{S}^2} &\leq \|f - s_f\|_{\infty,\mathbb{S}^2} + \|s_f - S_f\|_{\infty,\mathbb{S}^2} \\
&\leq C_{13} \left(\tan \frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2} + A_8 D_7 (C_{13}+1) \left(\tan \frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2} \\
&= D_8 \left(\tan \frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2}
\end{aligned}$$

With $D_8 = C_{13} + A_8 D_7 (C_{13}+1)$ we get the desired result. Similarly, if d is even by Theorem 2.34 there exists a constant C_{13} depending on d and the smallest angle of Δ such that

$$\|f - s_f\|_{\infty,\mathbb{S}^2} \leq C_{13} \left(\tan \frac{|\Delta|}{2}\right)^3 |f|_{3,\infty,\mathbb{S}^2}, \tag{3.20}$$

and

$$|f - s_f|_{2,\infty,\mathbb{S}^2} \leq C_{13} \left(\tan \frac{|\Delta|}{2}\right) |f|_{3,\infty,\mathbb{S}^2}.$$

Then

$$\begin{aligned}
|s_f|_{2,\infty,\mathbb{S}^2} &\leq |f - s_f|_{2,\infty,\mathbb{S}^2} + |f|_{2,\infty,\mathbb{S}^2} \\
&\leq C_{13} \left(\tan \frac{|\Delta|}{2}\right) |f|_{3,\infty,\mathbb{S}^2} + |f|_{2,\infty,\mathbb{S}^2}.
\end{aligned} \tag{3.21}$$

Since $\mathcal{P} s_f = s_f - S_f$, by Theorem 3.7

$$|s_f - S_f|_{2,\infty,\mathbb{S}^2} = |\mathcal{P} s_f|_{2,\infty,\mathbb{S}^2} \leq D_7 |s_f|_{2,\infty,\mathbb{S}^2},$$

and by (3.19)

$$|s_f - S_f|_{2,\infty,\mathbb{S}^2} \leq D_7 \left(C_{13} \left(\tan \frac{|\Delta|}{2}\right) |f|_{3,\infty,\mathbb{S}^2} + |f|_{2,\infty,\mathbb{S}^2} \right).$$

Since both S_f and s_f interpolate f, their difference satisfies the hypothesis of Lemma 2.30 and thus

$$\|s_f - S_f\|_{\infty,\mathbb{S}^2} \quad \leq \quad A_8 \left(\tan \frac{|\Delta|}{2}\right)^2 |s_f - S_f|_{2,\infty,\mathbb{S}^2}$$

$$\leq \quad A_8 D_7 \left(\tan\frac{|\Delta|}{2}\right)^2 \left(C_{13}\left(\tan\frac{|\Delta|}{2}\right)|f|_{3,\infty,\mathbb{S}^2} + |f|_{2,\infty,\mathbb{S}^2}\right).$$

Then by (3.20)

$$\begin{aligned}
\|f - S_f\|_{\infty,\mathbb{S}^2} &\leq \|f - s_f\|_{\infty,\mathbb{S}^2} + \|s_f - S_f\|_{\infty,\mathbb{S}^2} \\
&\leq D_9\left(\tan\frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2} + D_{10}\left(\tan\frac{|\Delta|}{2}\right)^3 |f|_{3,\infty,\mathbb{S}^2}
\end{aligned}$$

and we get the desired result.

3.2 Constant extensions

In this section we derive error bounds for interpolating splines minimizing the energy functional defined in terms of constant extensions. Such a functional allows reproduction of homogeneous polynomials of even degree by even degree splines. Let us begin by reintroducing our notation and deriving results similar to those in the previous section.

Suppose we are given values $\{f(v), v \in \mathcal{V}\}$ of an unknown function f at a set $\mathcal{V}$ of scattered points on the unit sphere. To find a homogeneous spline approximation of f, we choose a linear space $\mathcal{S} \subseteq S_d^r(\Delta)$ of polynomial splines of degree d and smoothness r defined on a triangulation Δ with vertices at the points of $\mathcal{V}$. Let

$$\Gamma(f) := \{s \in \mathcal{S} : s(v) = f(v), v \in \mathcal{V}\}$$

be the set of all splines in $\mathcal{S}$ that interpolate f at the points of $\mathcal{V}$. Assume that $\mathcal{S}$ is big enough so that $\Gamma(f)$ is not empty. We choose a spline S_f such that

$$\mathcal{E}'(S_f) = \min_{s\in\Gamma(f)} \mathcal{E}'(s), \tag{3.22}$$

where for a spherical triangle $\tau \in \Delta$

$$\mathcal{E}'_\tau(s) := \sum_{|\alpha|=2} \|D^\alpha s_0\|_{2,\tau}^2 \tag{3.23}$$

and

$$\mathcal{E}'(s) := \sum_{\tau\in\Delta} \mathcal{E}'_\tau(s). \tag{3.24}$$

Here s_0 is the constant homogeneous extension of s to $\mathbb{R}^3\backslash\{0\}$. We call S_f the minimal energy interpolating spline. Again, let

$$\mathcal{X} := \{f \in B(\mathbb{S}^2) : f|_\tau \in C^3(\tau), \forall\tau \in \Delta\},$$

where $B(\mathbb{S}^2)$ is the set of all bounded real-valued functions on the sphere. For each triangle $\tau \in \Delta$, define

$$\langle f, g\rangle'_\tau := \int_\tau \sum_{|\alpha|=2} D^\alpha f_0 \; D^\alpha g_0.$$

Then

$$\langle f, g\rangle' := \langle f, g\rangle'_{\mathbb{S}^2} = \sum_{\tau\in\Delta} \langle f, g\rangle'_\tau$$

is a semi-definite inner product on $\mathcal{X}$. Let $\|\cdot\|'_\tau$ and $\|\cdot\|'$ be the associated semi-norms.

Again, $\langle\cdot,\cdot\rangle'$ is an inner product on the linear space

$$\mathcal{W} := \{s \in \mathcal{S} : s(v) = 0, v \in \mathcal{V}\}. \tag{3.25}$$

Since $\mathcal{W}$ is finite-dimensional, it follows that $\mathcal{W}$ equipped with the inner product $\langle\cdot,\cdot\rangle'$ is a Hilbert space.

For given f let s_f be any spline in the set $\Gamma(f)$. The solution S_f to the minimal energy problem is equal to $s_f - \mathcal{P}s_f$, where $\mathcal{P}$ is the linear projector $\mathcal{P} : \mathcal{X} \to \mathcal{W}$ defined for $f \in \mathcal{X}$ by

$$\mathcal{E}'(f - \mathcal{P}f) = \min_{w\in\mathcal{W}} \mathcal{E}'(f - w). \tag{3.26}$$

Since $\mathcal{W}$ is a Hilbert space with respect to $\langle\cdot,\cdot\rangle'$, $\mathcal{P}f$ is uniquely defined and characterized by

$$\langle f - \mathcal{P}f, w\rangle' = 0, \forall w \in \mathcal{W}. \tag{3.27}$$

Moreover

$$\|\mathcal{P}f\|' \leq \|f\|' \tag{3.28}$$

for all $f \in \mathcal{X}$.

We now establish a lemma showing the equivalence of certain semi-norms on the space $\mathcal{X}$ defined above.

LEMMA 3.9. Let τ be a spherical triangle with $|\tau| \leq 1$ and $f \in \mathcal{X}$. Let $\mathcal{E}'_\tau$ be the functional defined in (3.23). There exists a positive constant D'_1 such that

$$D'_1(|f|'_{2,2,\tau})^2 \leq \mathcal{E}'_\tau(f) \leq (|f|'_{2,2,\tau})^2. \tag{3.29}$$

PROOF. Similar to the proof of Lemma 3.1.

Next we establish reproductive property of the energy functional $\mathcal{E}'_\tau$.

LEMMA 3.10. Let τ be a spherical triangle with $|\tau| \leq 1$. Suppose $f \in \mathcal{X}$. Then $\mathcal{E}'_\tau(f) = 0$ if and only if f is a constant on τ.

PROOF. Similar to the proof of Lemma 3.2.

Next we establish the equivalence of energy and L_2 norms on the Hilbert space $\mathcal{W}$.

THEOREM 3.11. Suppose $\mathcal{S} \subseteq S^0_d(\Delta)$ is a spline space defined on a β-quasi-uniform triangulation Δ with $|\Delta| \leq 1$, and let $\mathcal{W}$ be the associated Hilbert space (3.25). Then there exist constants $0 < D'_2 < D'_3 < \infty$ depending only d and β such that

$$D'_2\|s\|^2_{2,\mathbb{S}^2} \leq \left(\tan\frac{|\Delta|}{2}\right)^4 (\|s\|')^2 \leq D'_3\|s\|^2_{2,\mathbb{S}^2}, \tag{3.30}$$

for all $s \in \mathcal{W}$.

PROOF. Similar to the proof of Theorem 3.3.

COROLLARY 3.12. Suppose $\mathcal{S} \subseteq S^r_d(\Delta)$ is a spline space defined on a β-quasi-uniform triangulation Δ, and that $\{B_\xi\}_{\xi\in\mathcal{M}}$ is a stable local basis for $\mathcal{S}$ defined in Proposition 2.31. Then $\{B_\xi\}_{\xi\in\mathcal{N}}$ is a Riesz basis (with respect to the $\mathcal{X}$-norm) for the linear space $\mathcal{W}$ defined in (3.25). Here $\mathcal{N}$ is the subset of the minimal determining set $\mathcal{M}$ excluding the set of vertices $\mathcal{V}$ of Δ. In particular, there exist positive constants

D_4', D_5' depending on d, β, and ℓ such that

$$D_4' \min_{\tau\in\Delta} A_\tau \sum_{\xi\in\mathcal{N}} |c_\xi|^2 \le \left(\tan\frac{|\Delta|}{2}\right)^4 \left(\|\sum_{\xi\in\mathcal{N}} c_\xi B_\xi\|'\right)^2 \le D_5' \max_{\tau\in\Delta} A_\tau \sum_{\xi\in\mathcal{N}} |c_\xi|^2, \quad (3.31)$$

for all $\{c_\xi\}_{\xi\in\mathcal{N}}$.

Proof. Similar to Corollary 3.4.

Next, we estimate $\mathcal{X}$-norm of the projection operator $\mathcal{P}$ in (3.27) outside of support of $f \in \mathcal{X}$.

Theorem 3.13. There exist constants $0 < \sigma' < 1$ and D_6', depending only on ℓ, d, β, such that for any triangle $T \in \Delta$ and any function $f \in \mathcal{X}$ with $\text{supp}(f) \subseteq T$

$$\|\mathcal{P}f\|_\tau' \le D_6'(\sigma')^k \|f\|', \quad (3.32)$$

whenever $\tau \in \text{star}^{2(k+2)\ell+1}(T)\backslash\text{star}^{2(k+1)\ell+1}(T)$ and $k \ge 1$.

Proof. Similar to Theorem 3.6.

As a consequence of the last result, we can now compare Sobolev semi-norms of $\mathcal{P}f$ and f.

Theorem 3.14. There exists a constant D_7' depending only on d, ℓ and β, such that for every $f \in \mathcal{X}$

$$|\mathcal{P}f|_{2,\infty,\mathbb{S}^2}' \le D_7' |f|_{2,\infty,\mathbb{S}^2}'. \quad (3.33)$$

Proof. Similar to Theorem 3.7.

We are finally in a position to prove the main result of this section.

Theorem 3.15. Suppose $\mathcal{S} \subseteq S_d^r(\Delta)$ is a spline space defined on a β-quasi-uniform triangulation Δ with $|\Delta| \le 1$, $d \ge 3r+2$. If d is even there exist constants D_8' and D_9' depending only on d and β, such that the minimal energy interpolant S_f, defined in (3.23), satisfies

$$\|f - S_f\|_{\infty,\mathbb{S}^2} \le D_8' \left(\tan\frac{|\Delta|}{2}\right)^3 |f|_{3,\infty,\mathbb{S}^2} + D_9' \left(\tan\frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2}', \quad (3.34)$$

for all $f \in C^3(\mathbb{S}^2)$. If d is odd then S_f minimizing (3.23) satisfies

$$\|f - S_f\|_{\infty,\mathbb{S}^2} \leq D'_{10}\left(\tan\frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2} + D'_9\left(\tan\frac{|\Delta|}{2}\right)^2 |f|'_{2,\infty,\mathbb{S}^2}. \tag{3.35}$$

for all $f \in C^2(\mathbb{S}^2)$.

Proof. Given a function $f \in \mathcal{X}$, let $s_f \in \Gamma(f)$ be the quasi-interpolant defined in Section 2.8. If d is even by Theorem 2.34 there exists a constant C_{13} depending on d and the smallest angle of Δ such that

$$\|f - s_f\|_{\infty,\mathbb{S}^2} \leq C_{13}\left(\tan\frac{|\Delta|}{2}\right)^3 |f|_{3,\infty,\mathbb{S}^2}, \tag{3.36}$$

and by Corollary 2.35

$$|f - s_f|'_{2,\infty,\mathbb{S}^2} \leq C_{15}\sum_{\ell=0}^{2}\left(\tan\frac{|\Delta|}{2}\right)^{3-\ell} |f|_{3,\infty,\mathbb{S}^2}.$$

Then

$$\begin{aligned} |s_f|'_{2,\infty,\mathbb{S}^2} &\leq |f - s_f|'_{2,\infty,\mathbb{S}^2} + |f|'_{2,\infty,\mathbb{S}^2} \\ &\leq C_{15}\sum_{\ell=0}^{2}\left(\tan\frac{|\Delta|}{2}\right)^{3-\ell} |f|_{3,\infty,\mathbb{S}^2} + |f|'_{2,\infty,\mathbb{S}^2}. \end{aligned} \tag{3.37}$$

Since $\mathcal{P}s_f = s_f - S_f$, by Theorem 3.14

$$|s_f - S_f|'_{2,\infty,\mathbb{S}^2} = |\mathcal{P}s_f|'_{2,\infty,\mathbb{S}^2} \leq D'_7 |s_f|'_{2,\infty,\mathbb{S}^2},$$

and by (3.37)

$$|s_f - S_f|'_{2,\infty,\mathbb{S}^2} \leq D'_7\left(C_{15}\sum_{\ell=0}^{2}\left(\tan\frac{|\Delta|}{2}\right)^{3-\ell} |f|_{3,\infty,\mathbb{S}^2} + |f|'_{2,\infty,\mathbb{S}^2}\right).$$

Since both S_f and s_f interpolate f, their difference satisfies the hypothesis of Lemma 2.30 and thus

$$\begin{aligned} \|s_f - S_f\|_{\infty,\mathbb{S}^2} &\leq A'_8\left(\tan\frac{|\Delta|}{2}\right)^2 |s_f - S_f|'_{2,\infty,\mathbb{S}^2} \\ &\leq A'_8 D_7\left(C_{15}\sum_{\ell=0}^{2}\left(\tan\frac{|\Delta|}{2}\right)^{5-\ell} |f|_{3,\infty,\mathbb{S}^2} + \left(\tan\frac{|\Delta|}{2}\right)^2 |f|'_{2,\infty,\mathbb{S}^2}\right). \end{aligned}$$

Then by (3.36)

$$\begin{aligned}
\|f - S_f\|_{\infty,\mathbb{S}^2} &\leq \|f - s_f\|_{\infty,\mathbb{S}^2} + \|s_f - S_f\|_{\infty,\mathbb{S}^2} \\
&\leq C_{13}\left(\tan\frac{|\Delta|}{2}\right)^3 |f|_{3,\infty,\mathbb{S}^2} \\
&+ A_8' D_7 \left(C_{15}\sum_{\ell=0}^{2}\left(\tan\frac{|\Delta|}{2}\right)^{5-\ell} |f|_{3,\infty,\mathbb{S}^2} + \left(\tan\frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2}' \right) \\
&\leq D_8'\left(\tan\frac{|\Delta|}{2}\right)^3 |f|_{3,\infty,\mathbb{S}^2} + D_9'\left(\tan\frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2}'.
\end{aligned}$$

If d is odd by Theorem 2.34 there exists a constant C_{13} depending on d and the smallest angle of Δ such that

$$\|f - s_f\|_{\infty,\mathbb{S}^2} \leq C_{13}\left(\tan\frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2}, \tag{3.38}$$

and by Corollary 2.35

$$|f - s_f|_{2,\infty,\mathbb{S}^2}' \leq C_{15}\sum_{\ell=0}^{2}\left(\tan\frac{|\Delta|}{2}\right)^{2-\ell} |f|_{2,\infty,\mathbb{S}^2}.$$

Then

$$\begin{aligned}
|s_f|_{2,\infty,\mathbb{S}^2}' &\leq |f - s_f|_{2,\infty,\mathbb{S}^2}' + |f|_{2,\infty,\mathbb{S}^2}' \\
&\leq C_{15}\sum_{\ell=0}^{2}\left(\tan\frac{|\Delta|}{2}\right)^{2-\ell} |f|_{2,\infty,\mathbb{S}^2} + |f|_{2,\infty,\mathbb{S}^2}'.
\end{aligned} \tag{3.39}$$

Since $\mathcal{P}s_f = s_f - S_f$, by Theorem 3.14

$$|s_f - S_f|_{2,\infty,\mathbb{S}^2}' = |\mathcal{P}s_f|_{2,\infty,\mathbb{S}^2}' \leq D_7'|s_f|_{2,\infty,\mathbb{S}^2}',$$

and by (3.39)

$$|s_f - S_f|_{2,\infty,\mathbb{S}^2}' \leq D_7'\left(C_{15}\sum_{\ell=0}^{2}\left(\tan\frac{|\Delta|}{2}\right)^{2-\ell} |f|_{2,\infty,\mathbb{S}^2} + |f|_{2,\infty,\mathbb{S}^2}' \right).$$

Since both S_f and s_f interpolate f, their difference satisfies the hypothesis of Lemma 2.30 and thus

$$\|s_f - S_f\|_{\infty,\mathbb{S}^2} \leq A_8'\left(\tan\frac{|\Delta|}{2}\right)^2 |s_f - S_f|_{2,\infty,\mathbb{S}^2}'$$

$$\leq \quad A_8' D_7 \left(C_{15} \sum_{\ell=0}^{2} \left(\tan \frac{|\Delta|}{2} \right)^{4-\ell} |f|_{2,\infty,\mathbb{S}^2} + \left(\tan \frac{|\Delta|}{2} \right)^{2} |f|'_{2,\infty,\mathbb{S}^2} \right).$$

Then by (3.38)

$$\begin{aligned}
\|f - S_f\|_{\infty,\mathbb{S}^2} &\leq \|f - s_f\|_{\infty,\mathbb{S}^2} + \|s_f - S_f\|_{\infty,\mathbb{S}^2} \\
&\leq C_{13} \left(\tan \frac{|\Delta|}{2} \right)^{2} |f|_{2,\infty,\mathbb{S}^2} \\
&+ A_8' D_7 \left(C_{15} \sum_{\ell=0}^{2} \left(\tan \frac{|\Delta|}{2} \right)^{4-\ell} |f|_{2,\infty,\mathbb{S}^2} + \left(\tan \frac{|\Delta|}{2} \right)^{2} |f|'_{2,\infty,\mathbb{S}^2} \right) \\
&\leq D_8' \left(\tan \frac{|\Delta|}{2} \right)^{2} |f|_{2,\infty,\mathbb{S}^2} + D_9' \left(\tan \frac{|\Delta|}{2} \right)^{2} |f|'_{2,\infty,\mathbb{S}^2}.
\end{aligned}$$

The constants are taken $D_8' = C_{13} + 3A_8' D_7 C_{15}$ and $D_9' = A_8' D_7$.

3.3 Discrete Least Squares Splines

In this section we derive error bounds for the discrete least squares spline approximation on the sphere. Suppose $\mathcal{V} = \{v_i, i = 1, \cdots, n\}$ are the given data sites over the unit sphere $\mathbb{S}^2$ and Δ is a triangulation of $\mathbb{S}^2$ whose vertices may not relate to the data sites. Fix a spline space $\mathcal{S} \subseteq S_d^r(\Delta)$. Let

$$\mathcal{X} := \{f \in B(\mathbb{S}^2) : f|_\tau \in C^m(\tau), \forall \tau \in \Delta\},$$

for some $m \leq d$. Given a function f in $\mathcal{X}$ we are interested in error bounds for $f - S_f$, where S_f is defined by

$$\|f - S_f\| = \min_{s \in \mathcal{S}} \|f - s\|. \tag{3.40}$$

We refer to S_f as the discrete least squares spline approximating f. Here $\|\cdot\|$ is the ℓ_2 norm corresponding to

$$\langle f, g \rangle = \sum_{i=1}^{n} f(v_i) g(v_i), \tag{3.41}$$

which is a semi-definite inner product on $\mathcal{X}$. Note that $\langle \cdot, \cdot \rangle$ has the following properties.

Lemma 3.16. Let $f, g \in \mathcal{X}$. For $\langle \cdot, \cdot \rangle$ defined in (3.41) we have

1) $\langle f,g\rangle = 0$, whenever $fg = 0$ on $\mathbb{S}^2$;

2) $\|f\| \le \|g\|$, whenever $|f(v)| \le |g(v)|, \forall v \in \mathbb{S}^2$;

3) $f \cdot \chi_\tau \in \mathcal{X}$, for every $f \in \mathcal{X}$ and $\tau \in \Delta$;

4) $f = \sum_{\tau\in\Delta} f_\tau$ for some $f_\tau \in \mathcal{X}$ with $\mathrm{supp}(f_\tau) \subseteq \tau$;

5) $\|f \cdot \chi_\tau\| \le G_1\|f\|_{\infty,\tau}$ and $\|f_\tau\| \le G_1\|f\|_{\infty,\tau}$ for every $f \in \mathcal{X}$ and $\tau \in \Delta$.

Here G_1 is a positive constant independent of f and τ.

PROOF. 1)-3) follow directly from the definitions of $\mathcal{X}$ and $\langle\cdot,\cdot\rangle$. Next note that $f_\tau := f \cdot \chi_\tau$ satisfies 4). For 5) let $G_1 := \sqrt{\max\#\{\mathcal{V}\cap\tau\}}$ and consider

$$\|f \cdot \chi_\tau\|^2 = \sum_{\mathcal{V}\cap\tau} |f(v)|^2 \le \sum_{\mathcal{V}\cap\tau} \|f\|^2_{\infty,\tau} \le G_1^2\|f\|^2_{\infty,\tau}.$$

Now we can establish the following result similar to Corollaries 3.4 and 3.12.

LEMMA 3.17. Let $\{B_\xi\}_{\xi\in\mathcal{M}}$ be the basis for $\mathcal{S}$ corresponding to the minimal determining set $\mathcal{M}$ introduced in Section 2.8. Suppose that the data set $\mathcal{V}$ has a property that for every $s \in \mathcal{S}$ and every $\tau \in \Delta$

$$G_2\|s\|_{\infty,\tau} \le (\sum_{\mathcal{V}\cap\tau} s(v)^2)^{1/2} = \|s \cdot \chi_\tau\| \tag{3.42}$$

for some positive constant G_2. Then there exist positive constants G_3, G_4 depending on d, ℓ and the smallest angle of Δ such that

$$G_3 \sum_{\xi\in\mathcal{M}} |c_\xi|^2 \le \|\sum_{\xi\in\mathcal{M}} c_\xi B_\xi\|^2 \le G_4 \sum_{\xi\in\mathcal{M}} |c_\xi|^2, \tag{3.43}$$

for all $\{c_\xi\}_{\xi\in\mathcal{M}}$.

PROOF. Let $s = \sum_{\xi\in\mathcal{M}} c_\xi B_\xi$. By Proposition 2.31 3) there exists a constant C_4 depending on d and the smallest angle of Δ such that

$$\sum_{\xi\in\mathcal{M}} c_\xi^2 \le C_4^2 \sum_{\xi\in\mathcal{M}} \|s\|^2_{\infty,\tau_\xi}.$$

Recall that τ_ξ is a triangle in Δ containing ξ. By property (3.42) above we have

$$G_2^2 \|s\|_{\infty,\tau_\xi}^2 \leq \|s \cdot \chi_{\tau_\xi}\|^2.$$

Note that there may be more than one domain point $\eta \in \mathcal{M}$ in τ_ξ. Let $K_{14} = \max_{\tau\in\Delta} \#\{\xi \in \mathcal{M} : \tau \cap \operatorname{supp}(B_\xi) \neq \emptyset\}$. As we see in the proof of Corollary 3.4 K_{14} depends on d, ℓ and β. Then

$$\sum_{\xi\in\mathcal{M}} c_\xi^2 \leq C_4^2 G_2^{-2} \sum_{\xi\in\mathcal{M}} \|s \cdot \chi_{\tau_\xi}\|^2 \leq C_4^2 G_2^{-2} K_{14} \|s\|^2.$$

Therefore we have the left hand side of (3.43) with $G_3 = C_4^{-2} G_2^2 K_{14}^{-1}$. Now let $\mathcal{M}_\tau := \{\xi \in \mathcal{M} : \|B_\xi \cdot \chi_\tau\| > 0\}$. Then $\#\mathcal{M}_\tau \leq K_{14}$, and using Lemma 3.16 5) and Proposition 2.31 2) we get

$$\|s\|^2 \leq \sum_{\tau\in\Delta} \|s \cdot \chi_\tau\|^2 = \sum_{\tau\in\Delta} \left\| \left(\sum_{\xi\in\mathcal{M}_\tau} c_\xi B_\xi \right) \cdot \chi_\tau \right\|^2$$

$$\leq G_1^2 \sum_{\tau\in\Delta} \left\| \left(\sum_{\xi\in\mathcal{M}_\tau} c_\xi B_\xi \right) \right\|_{\infty,\tau}^2 \leq G_1^2 C_3 \sum_{\tau\in\Delta} \left(\sum_{\xi\in\mathcal{M}_\tau} |c_\xi| \right)^2$$

$$\leq G_1^2 C_3 \sum_{\tau\in\Delta} \#\mathcal{M}_\tau \sum_{\xi\in\mathcal{M}_\tau} |c_\xi|^2 \leq G_1^2 C_3 K_{14} \sum_{\tau\in\Delta} \sum_{\xi\in\mathcal{M}_\tau} |c_\xi|^2.$$

Since in the last sum ξ may be repeated more than once let $K_{15} := \max_{\xi\in\mathcal{M}} \#\{\tau \in \Delta : \|B_\xi \cdot \chi_\tau\| > 0\}$. Then

$$\|s\|^2 \leq G_1^2 C_3 K_{14} K_{15} \sum_{\xi\in\mathcal{M}} |c_\xi|^2$$

and the proof is complete.

Define a projection operator $\mathcal{P} : \mathcal{X} -> \mathcal{S}$ by $\mathcal{P}f = S_f$. Note that

$$\langle S_f - f, s \rangle = 0, \forall s \in \mathcal{S}. \tag{3.44}$$

The next result is similar to Theorems 3.6 and 3.13.

THEOREM 3.18. Suppose that the data set $\mathcal{V}$ satisfies the property (3.42). There exist constants $0 < \sigma < 1$ and G_5, depending only on d, β and ℓ such that for any triangle $T \in \Delta$ and any function $f \in \mathcal{X}$ with $\text{supp}(f) \subseteq T$

$$\|\mathcal{P}f \cdot \chi_\tau\| \leq G_5 \sigma^k \|f\|, \tag{3.45}$$

whenever $\tau \in \text{star}^{2(k+2)\ell+1}(T) \backslash \text{star}^{2(k+1)\ell+1}(T)$ with $k \geq 1$.

PROOF. Proof is similar to the one of Theorem 3.6. Let

$$\begin{aligned}
\mathcal{M}_0^T : &= \{\xi \in \mathcal{M} : \text{supp}(B_\xi) \cap T \neq \emptyset\}, \\
\mathcal{M}_k^T : &= \{\xi \in \mathcal{M} : \text{supp}(B_\xi) \cap \text{star}^{2k\ell}(T) \neq \emptyset\}, \\
\mathcal{N}_0^T : &= \mathcal{M}_0^T, \\
\mathcal{N}_k^T : &= \mathcal{M}_k^T \backslash \mathcal{M}_{k-1}^T.
\end{aligned}$$

Suppose $\mathcal{P}f = \sum_{\xi \in \mathcal{M}} c_\xi B_\xi$, and let

$$u_k := \sum_{\xi \in \mathcal{M}_k^T} c_\xi B_\xi, \quad w_k := \mathcal{P}f - u_k, \quad a_k := \sum_{\xi \in \mathcal{N}_k^T} c_\xi^2,$$

for $k \geq 0$. Since $\mathcal{P}f \in \mathcal{S}$, by Lemma 3.17

$$\sum_{j \geq k+1} a_j = \sum_{\xi \notin \mathcal{M}_k^T} c_\xi^2 \leq G_3^{-1} \|w_k\|^2.$$

Note that since $w_k \in \mathcal{S}$, using (3.44) we have $\langle f - \mathcal{P}f, w_k \rangle = 0$. Moreover, $\langle f, w_k \rangle = 0$, since $\text{supp}(f) \subseteq T$ and $\text{supp}(w_k)$ lies outside T. In fact, $\text{supp}(w_k) \cap \cup_{\xi \in \mathcal{M}_{k-1}^T} \text{supp}(B_\xi) = \emptyset$ for $k \geq 1$, it follows that

$$\|w_k\|^2 = \langle \mathcal{P}f - u_k, w_k \rangle = \langle f - u_k, w_k \rangle = -\langle u_k, w_k \rangle =$$

$$-\langle \sum_{\xi \in \mathcal{N}_k^T} c_\xi B_\xi, w_k \rangle \leq \| \sum_{\xi \in \mathcal{N}_k^T} c_\xi B_\xi \| \, \|w_k\|,$$

and therefore by (3.43)

$$\|w_k\|^2 \leq \| \sum_{\xi \in \mathcal{N}_k^T} c_\xi B_\xi \|^2 \leq G_4 \sum_{\xi \in \mathcal{N}_k^T} |c_\xi|^2 = G_4 \, a_k.$$

Hence

$$\sum_{j\geq k+1} a_j \leq G_3^{-1} G_4 \; a_k.$$

Let $\gamma := \frac{G_3}{G_4}$. Then by Lemma 3.5

$$a_k \leq a_0 \frac{(1-\gamma)^k}{\gamma} = \frac{a_0}{\gamma}\sigma^{2k},$$

with $\sigma := \sqrt{1-\gamma}$. It is easy to see that both γ and σ are positive and bounded above by 1. Note that (3.44) implies that $\|\mathcal{P}f\| \leq \|f\|$ for every $f \in \mathcal{X}$. By Lemma 3.17

$$a_0 \leq \sum_{j\geq 0} a_j = \sum_{\xi\in\mathcal{M}} c_\xi^2 \leq G_3^{-1}\|\mathcal{P}f\|^2 \leq G_3^{-1}\|f\|^2.$$

Let $\tau \in \text{star}^{2(k+2)\ell+1}(T)\backslash\text{star}^{2(k+1)\ell+1}(T)$ for some $k \geq 1$. If $\xi \in \mathcal{M}_k^T$, then $\text{supp}(B_\xi) \subseteq \text{star}^{2(k+1)\ell}(T)$, and therefore $\tau \cap \text{supp}(B_\xi) = \emptyset$. Using Lemma 3.17 again we obtain

$$\|\mathcal{P}f\cdot\chi_\tau\|^2 \leq \|\sum_{\xi\notin\mathcal{M}_k^T} c_\xi B_\xi\|^2 \leq$$

$$G_4 \sum_{\xi\notin\mathcal{M}_k^T} |c_\xi|^2 = G_4 \sum_{j\geq k+1} a_j \leq \frac{G_4^2}{G_3^2\gamma}\sigma^{2k}\|f\|^2.$$

We are now ready to compare the supremum norms of f and $\mathcal{P}f$.

THEOREM 3.19. Suppose that the data set $\mathcal{V}$ satisfies the property (3.42). Suppose Δ is a β-quasi-uniform triangulation with $|\Delta| \leq 1$. The projection $\mathcal{P}$ defined by (3.40) satisfies

$$\|\mathcal{P}f\|_{\infty,\mathbb{S}^2} \leq G_6\|f\|_{\infty,\mathbb{S}^2}$$

and therefore

$$\|\mathcal{P}\|_{\infty,\mathbb{S}^2} \leq G_6. \tag{3.46}$$

G_6 depends on d, ℓ and β.

PROOF. Let τ be a fixed triangle in Δ, and let

$$\Omega_0^\tau := \text{star}^{4\ell+1}(\tau), \quad \Omega_k^\tau := \text{star}^{2(k+2)\ell+1}(\tau)\backslash\text{star}^{2(k+1)\ell+1}(\tau)$$

(where $\ell = 3$). Let m_k denote the maximal number of triangles in $\Omega_k^\tau,\ k \geq 0$. By Lemma 2.4

$$m_k \leq \frac{5\beta^2}{4}(4(k+2)\ell + 2)^2 - \frac{2}{\pi\beta^2}(4(k+1)\ell+2)^2.$$

Write $f = \sum_{T\in\Delta} f_T$ with $\operatorname{supp}(f_T) \in T$ and consider

$$\sum_{T\in\Delta} \|\mathcal{P} f_T \cdot \chi_\tau\| = \sum_{k\geq 0}\sum_{T\in\Omega_k^\tau} \|\mathcal{P} f_T \cdot \chi_\tau\| \leq \sum_{T\in\Omega_0^\tau} \|f_T\| + \sum_{k\geq 1}\sum_{T\in\Omega_k^\tau} G_5\sigma^k \|f_T\|.$$

By Lemma 3.16 5)

$$\sum_{T\in\Delta} \|\mathcal{P} f_T \cdot \chi_\tau\| \leq G_1(m_0 + G_5 \sum_{k\geq 1} \sigma^k m_k)\|f\|_{\infty,\mathbb{S}^2}. \tag{3.47}$$

Since $\mathcal{P}$ is a linear operator using (3.42) we get

$$\|\mathcal{P} f\|_{\infty,\tau} \leq \sum_{T\in\Delta} \|\mathcal{P} f_T\|_{\infty,\tau} \leq G_2^{-1} \sum_{T\in\Delta} \|\mathcal{P} f_T \cdot \chi_\tau\|.$$

Then (3.47) implies

$$\|\mathcal{P} f\|_{\infty,\tau} \leq \frac{G_1}{G_2}(m_0 + G_5 \sum_{k\geq 1} \sigma^k m_k)\|f\|_{\infty,\mathbb{S}^2}.$$

Taking the supremum over all $\tau \in \Delta$ and all $f \in \mathcal{X}$ we get (3.46) with $G_6 = \frac{G_1}{G_2}(m_0 + G_5 \sum_{k\geq 1} \sigma^k m_k)$ depending on d and β.

We are finally in the position to prove the main result of this section.

THEOREM 3.20. Let Δ be a β-quasi-uniform spherical triangulation with $|\Delta| \leq 1$. Suppose that the data set $\mathcal{V}$ satisfies the property (3.42). Let $d \geq 3r+2$, and $0 \leq m \leq d$. Then there exists a constant G_7 depending only on d and the smallest angle in Δ, such that for every function f in $W^{m+1,\infty}(\mathbb{S}^2)$

$$\|f - \mathcal{P} f\|_{\infty,\mathbb{S}^2} \leq G_7 \left(\tan\frac{|\Delta|}{2}\right)^{m+1} |f|_{m+1,\infty,\mathbb{S}^2}. \tag{3.48}$$

Here m is taken to satisfy $(d-m) \bmod 2 = 0$.

PROOF. Let s_f be a quasi-interpolant defined in Section 2.8. Since s_f is a homogeneous polynomial of degree d, $\mathcal{P} s_f = s_f$ and therefore

$$\|f - \mathcal{P} f\|_{\infty,\mathbb{S}^2} \leq \|f - s_f\|_{\infty,\mathbb{S}^2} + \|s_f - \mathcal{P} f\|_{\infty,\mathbb{S}^2} \leq \|f - s_f\|_{\infty,\mathbb{S}^2} + \|\mathcal{P} s_f - \mathcal{P} f\|_{\infty,\mathbb{S}^2}.$$

Since $\mathcal{P}$ is linear

$$\|f-\mathcal{P}f\|_{\infty,\mathbb{S}^2} \leq \|f-s_f\|_{\infty,\mathbb{S}^2} + \|\mathcal{P}(s_f-f)\|_{\infty,\mathbb{S}^2} \leq (1+\|\mathcal{P}\|_{\infty,\mathbb{S}^2})\|f-s_f\|_{\infty,\mathbb{S}^2}.$$

By Theorem 2.34

$$\|f-s_f\|_{\infty,\mathbb{S}^2} \leq C_{13}\left(\tan\frac{|\Delta|}{2}\right)^{m+1}|f|_{m+1,\infty,\mathbb{S}^2}.$$

By Theorem 3.18 $\|\mathcal{P}\|_{\infty,\mathbb{S}^2} \leq G_6$. Therefore

$$\|f-\mathcal{P}f\|_{\infty,\mathbb{S}^2} \leq G_7\left(\tan\frac{|\Delta|}{2}\right)^{m+1}|f|_{m+1,\infty,\mathbb{S}^2}$$

with $G_7 = (1+G_6)C_{13}$ depending only on d, p and the smallest angle in Δ.

Note that if $\mathcal{P}f$ is the discrete least squares solution approximating f in a spline space of odd degree d then the convergence rate for a function $f \in W^{2,\infty}(\mathbb{S}^2)$ is quadratic. If we are working in a space of even degree then for $f \in W^{2,\infty}(\mathbb{S}^2)$, m must be even, $m+1$ odd, and therefore we can at most get linear convergence. Higher convergence rate will be expected for functions of higher smoothness.

3.4 Penalized least squares splines

In this section we discuss the last of the three global data fitting methods: penalized least squares approximation. A general penalized least squares problem was treated in [18] and can be stated as follows below.

Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{S}$ be linear spaces of functions on $\mathbb{R}^n$ where $\mathcal{S} \subseteq \mathcal{Y} \subseteq \mathcal{X}$. Suppose $\|\cdot\|_{\mathcal{X}} : \mathcal{X} \to \mathbb{R}$ and $\|\cdot\|_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{R}$ are semi-norms induced by semi-definite inner products $\langle\cdot,\cdot\rangle$ on $\mathcal{X}$ and $[\cdot,\cdot]$ on $\mathcal{Y}$ respectively. Given $f \in \mathcal{X}$ and $\lambda > 0$ we seek $S_{\lambda,f} \in \mathcal{S}$ such that

$$\Phi(S_{\lambda,f}) = \min_{s\in\mathcal{S}}\Phi(s),$$

where

$$\Phi(s) := \|f-s\|^2_{\mathcal{X}} + \lambda\|s\|^2_{\mathcal{Y}}.$$

Then $S_{\lambda,f}$ is called a penalized least squares fit of f corresponding to λ.

Let us introduce a non-penalized least squares fit as well. $S_f \in \mathcal{S}$ is called a non-penalized least squares fit of f if

$$\|f - S_f\|_{\mathcal{X}}^2 = \min_{s\in\mathcal{S}} \|f - s\|_{\mathcal{X}}^2.$$

Let us present here a version of the result in [18], which we will be needing in our work.

THEOREM 3.21. Suppose $\mathcal{X}, \mathcal{Y}, \mathcal{S}$ are function spaces on some set $\Omega \in \mathbb{R}^n$. Suppose $\mathcal{X} \subseteq L_\infty(\Omega)$ and let

$$K_{\mathcal{S}} := \sup\{\frac{\|s\|_{\mathcal{Y}}}{\|s\|_{\mathcal{X}}} : s \in \mathcal{S}, s \neq 0\} < \infty,$$

$$k_{\mathcal{S}} := \sup\{\frac{\|s\|_{\infty,\Omega}}{\|s\|_{\mathcal{X}}} : s \in \mathcal{S}, s \neq 0\} < \infty.$$

Then

$$\|S_f - S_{\lambda,f}\|_{\infty,\Omega} \le \lambda \; k_{\mathcal{S}} K_{\mathcal{S}} \|S_f\|_{\mathcal{Y}}.$$

PROOF. Recall that

$$\langle f - S_f, s\rangle = 0, \forall s \in \mathcal{S},$$

and note that $S_{\lambda,f}$ is characterized by

$$\langle f - S_{\lambda,f}, s\rangle = \lambda[S_{\lambda,f}, s], \forall s \in \mathcal{S}.$$

Subtracting the two equations we obtain

$$\langle S_f - S_{\lambda,f}, s\rangle = \lambda[S_{\lambda,f}, s], \forall s \in \mathcal{S}.$$

In particular, let $s = S_f - S_{\lambda,f}$, then

$$0 \le \|S_f - S_{\lambda,f}\|_{\mathcal{X}}^2 = \lambda[S_{\lambda,f}, S_f - S_{\lambda,f}] = \lambda[S_{\lambda,f}, S_f] - \lambda[S_{\lambda,f}, S_{\lambda,f}], \tag{3.49}$$

from what follows

$$[S_{\lambda,f}, S_{\lambda,f}] \le [S_{\lambda,f}, S_f].$$

By Cauchy-Schwarz inequality

$$[S_{\lambda,f}, S_f] \le \|S_{\lambda,f}\|_{\mathcal{Y}} \|S_f\|_{\mathcal{Y}},$$

therefore

$$\|S_{\lambda,f}\|_{\mathcal{Y}}^2 = [S_{\lambda,f}, S_{\lambda,f}] \le \|S_{\lambda,f}\|_{\mathcal{Y}} \|S_f\|_{\mathcal{Y}}.$$

Dividing both sides by the $\mathcal{Y}$-norm of $S_{\lambda,f}$ we get

$$\|S_{\lambda,f}\|_{\mathcal{Y}} \le \|S_f\|_{\mathcal{Y}}. \tag{3.50}$$

In addition, (3.49) implies

$$\|S_f - S_{\lambda,f}\|_{\mathcal{X}}^2 \le \lambda[S_{\lambda,f}, S_f] \le \lambda \|S_{\lambda,f}\|_{\mathcal{Y}} \|S_f\|_{\mathcal{Y}} \le \lambda \|S_f\|_{\mathcal{Y}}$$

by (3.50). On the other hand (3.49) and Cauchy-Schwarz inequality imply

$$\|S_f - S_{\lambda,f}\|_{\mathcal{X}}^2 = \lambda[S_{\lambda,f}, S_f - S_{\lambda,f}] \le \lambda \|S_{\lambda,f}\|_{\mathcal{Y}} \|S_f - S_{\lambda,f}\|_{\mathcal{Y}}.$$

By the definition of $K_{\mathcal{S}}$ and (3.50) in the last inequality we get

$$\|S_f - S_{\lambda,f}\|_{\mathcal{X}}^2 \le \lambda \|S_f\|_{\mathcal{Y}} K_{\mathcal{S}} \|S_f - S_{\lambda,f}\|_{\mathcal{X}}.$$

Dividing both sides by the $\mathcal{X}$-norm of $S_f - S_{\lambda,f}$ we get

$$\|S_f - S_{\lambda,f}\|_{\mathcal{X}} \le \lambda K_{\mathcal{S}} \|S_f\|_{\mathcal{Y}}.$$

By the definition of $k_{\mathcal{S}}$ we finally obtain

$$\|S_f - S_{\lambda,f}\|_{\infty,\Omega} \le \lambda\ k_{\mathcal{S}} K_{\mathcal{S}} \|S_f\|_{\mathcal{Y}}.$$

Let us now describe the framework for penalized least squares splines on spherical triangulations. Suppose we are given a set $\mathcal{V}$ of locations on the unit sphere along with corresponding values $\{f(v), v \in \mathcal{V}\}$ for some function f. Let Δ be a regular triangulation of the sphere $\mathbb{S}^2$ whose vertices form a subset of the data sites $\mathcal{V}$.

Consider the spline space $S_d^r(\Delta)$ of degree d and smoothness r with $3r+2 \le d$. We seek a spline function $S_{\lambda,f} \in S_d^r(\Delta)$ satisfying

$$\mathcal{P}_\lambda(S_{\lambda,f}) = \min\{\mathcal{P}_\lambda(s) : s \in S_d^r(\Delta)\}, \tag{3.51}$$

where λ is a positive weight and

$$\mathcal{P}_\lambda(s) := \mathcal{L}(s) + \lambda\mathcal{E}(s) \tag{3.52}$$

or

$$\mathcal{P}'_\lambda(s) := \mathcal{L}(s) + \lambda\mathcal{E}'(s). \tag{3.53}$$

Here the least squares functional $\mathcal{L}$ and energy functionals $\mathcal{E}$, $\mathcal{E}'$ are defined in previous sections. Let

$$\mathcal{X} := \{f \in B(\mathbb{S}^2) : f|_\tau \in C^m(\tau), \forall \tau \in \Delta\},$$

for some $m \le d$, $\mathcal{Y} := \mathcal{X}$ and $\mathcal{S} \subseteq S_d^r(\Delta)$. Then the semi-definite inner products on $\mathcal{X}$ and $\mathcal{Y}$ are defined as

$$\langle f, g\rangle := \sum_{v\in\mathcal{V}} f(v)g(v),$$

$$[f,g] := \int_{\mathbb{S}^2} \sum_{|\alpha|=2} D^\alpha f_1 D^\alpha g_1$$

and

$$[f,g]' := \int_{\mathbb{S}^2} \sum_{|\alpha|=2} D^\alpha f_0 D^\alpha g_0.$$

Define a linear operator $Q_\lambda : \mathcal{X} \to \mathcal{S}$ by $Q_\lambda f := S_{\lambda,f}$. We need to investigate the behavior of $\|f - Q_\lambda f\|_{\infty,\mathbb{S}^2}$ as a function of λ and the approximation properties of $\mathcal{S}$.

Note now, that the non-penalized least squares spline S_f is in fact the discrete least squares spline minimizing

$$\mathcal{L}(s) = \sum_{v\in\mathcal{V}} (s(v) - f(v))^2$$

over the splines $s \in \mathcal{S}$.

THEOREM 3.22. Let Δ be a β-quasi-uniform triangulation of the sphere $\mathbb{S}^2$ whose vertices form a subset of the data sites $\mathcal{V}$ and $|\Delta| \leq 1$. Suppose that the data set $\mathcal{V}$ has a property that for every $s \in S_d^r(\Delta)$ and every $\tau \in \Delta$

$$F_1 \|s\|_{\infty,\tau} \leq \left(\sum_{\mathcal{V}\cap\tau} s(v)^2 \right)^{1/2} = \|s \cdot \chi_\tau\|_{\mathcal{X}} \tag{3.54}$$

for some positive constant F_1. Let $S_{\lambda,f}$ be the spline minimizing $\mathcal{P}_\lambda$. Then

$$\|f - S_{\lambda,f}\|_{\infty,\mathbb{S}^2} \leq F_2 \left(\tan \frac{|\Delta|}{2} \right)^{m+1} |f|_{m+1,\infty,\mathbb{S}^2} + F_3 \lambda \|f\|_{\infty,\mathbb{S}^2}. \tag{3.55}$$

for every function f in $W^{m+1,\infty}(\mathbb{S}^2)$. Here m is taken between 0 and d with $(d - m) \bmod 2 = 0$, the constants F_2, F_3 depend on d, λ, F_1, ρ_Δ, β and cardinality of $\mathcal{V}$.

PROOF. For any $s \in \mathcal{S}$

$$\|s\|_{\mathcal{Y}}^2 = \mathcal{E}(s)$$

and then by Lemma 3.1

$$\mathcal{E}_\tau(s) \leq |s|_{2,2,\tau}^2.$$

Since s is a homogeneous polynomial of degree d By Lemma 2.29

$$|s|_{2,2,\tau}^2 \leq \frac{K_{16}^2}{(\tan \frac{\rho_\tau}{2})^4} \|s\|_{2,\tau}^2$$

for some constant K_{16} depending on d. By Lemma 4.4 [23]

$$\|s\|_{2,\tau}^2 \leq A_\tau \|s\|_{\infty,\tau}^2.$$

Then

$$\|s\|_{\mathcal{Y}}^2 = \sum_{\tau\in\Delta} \|s|_\tau\|_{\mathcal{Y}}^2 \leq \sum_{\tau\in\Delta} \frac{A_\tau K_{16}^2}{(\tan \frac{\rho_\tau}{2})^4} \|s\|_{\infty,\tau}^2.$$

Since $\rho_\Delta \leq \rho_\tau$

$$\|s\|_{\mathcal{Y}}^2 \leq \frac{K_{16}^2}{(\tan \frac{\rho_\Delta}{2})^4} \left(\sum_{\tau\in\Delta} A_\tau \right) \|s\|_{\infty,\mathbb{S}^2}^2 \leq \frac{K_{16}^2 4\pi}{(\tan \frac{\rho_\Delta}{2})^4} \|s\|_{\infty,\mathbb{S}^2}^2,$$

and therefore

$$\|s\|_{\mathcal{Y}} \leq \frac{K_{16}\sqrt{4\pi}}{(\tan\frac{\rho_\Delta}{2})^2}\|s\|_{\infty,\mathbb{S}^2}.$$

By (3.54)

$$\|s\|_{\infty,\tau} \leq F_1^{-1}\|s\cdot\chi_\tau\|_{\mathcal{X}} \leq F_1^{-1}\|s\|_{\mathcal{X}}$$

and therefore

$$\|s\|_{\infty,\mathbb{S}^2} \leq F_1^{-1}\|s\|_{\mathcal{X}}. \tag{3.56}$$

Then

$$\|s\|_{\mathcal{Y}} \leq \frac{K_{16}\sqrt{4\pi}}{F_1(\tan\frac{\rho_\Delta}{2})^2}\|s\|_{\mathcal{X}},$$

and hence

$$K_{\mathcal{S}} := \sup\{\frac{\|s\|_{\mathcal{Y}}}{\|s\|_{\mathcal{X}}} : s \in \mathcal{S}, s \neq 0\} \leq \frac{K_{16}\sqrt{4\pi}}{F_1(\tan\frac{\rho_\Delta}{2})^2}.$$

Also, equation (3.56) implies that

$$k_{\mathcal{S}} := \sup\{\frac{\|s\|_{\infty,\mathbb{S}^2}}{\|s\|_{\mathcal{X}}} : s \in \mathcal{S}, s \neq 0\} \leq F_1^{-1}.$$

In addition, since

$$\|s\|_{\mathcal{X}}^2 = \sum_{\tau\in\Delta}\|s\cdot\chi_\tau\|_{\mathcal{X}}^2 = \sum_{\tau\in\Delta}\sum_{v\in\mathcal{V}\cap\tau} s(v)^2 \leq \sum_{\tau\in\Delta} n_\tau\|s\|_{\infty,\tau}^2,$$

where $n_\tau := \#\{\mathcal{V}\cap\tau\}$. Therefore

$$\|s\|_{\mathcal{X}} \leq \sqrt{\#\mathcal{V}}\|s\|_{\infty,\mathbb{S}^2}.$$

Then by Theorem 3.21

$$\|S_f - S_{\lambda,f}\|_{\infty,\mathbb{S}^2} \leq \lambda\ k_{\mathcal{S}}K_{\mathcal{S}}\|S_f\|_{\mathcal{Y}}.$$

By the definition of $K_{\mathcal{S}}$

$$\|S_f - S_{\lambda,f}\|_{\infty,\mathbb{S}^2} \leq \lambda\ k_{\mathcal{S}}K_{\mathcal{S}}^2\|S_f\|_{\mathcal{X}},$$

and finally

$$\|S_f - S_{\lambda,f}\|_{\infty,\mathbb{S}^2} \leq \lambda\ k_{\mathcal{S}}K_{\mathcal{S}}^2\sqrt{\#\mathcal{V}}\|S_f\|_{\infty,\mathbb{S}^2}.$$

By the triangle inequality

$$\|S_f\|_{\infty,\mathbb{S}^2} \le \|S_f - f\|_{\infty,\mathbb{S}^2} + \|f\|_{\infty,\mathbb{S}^2}$$

and therefore

$$\|S_f - S_{\lambda,f}\|_{\infty,\mathbb{S}^2} \le \lambda\ K_{17}(\|S_f - f\|_{\infty,\mathbb{S}^2} + \|f\|_{\infty,\mathbb{S}^2}),$$

with $K_{17} = k_{\mathcal{S}} K_{\mathcal{S}}^2 \sqrt{\#\mathcal{V}}$. By Theorem 3.20 there exists a constant K_{18} depending only on d and the smallest angle in Δ, such that for every function f in $W^{m+1,\infty}(\mathbb{S}^2)$

$$\|S_f - f\|_{\infty,\mathbb{S}^2} \le K_{18} \left(\tan\frac{|\Delta|}{2}\right)^{m+1} |f|_{m+1,\infty,\mathbb{S}^2}$$

for any m between 0 and d with $(d-m) \bmod 2 = 0$. Since

$$\|f - S_{\lambda,f}\|_{\infty,\mathbb{S}^2} \le \|S_f - S_{\lambda,f}\|_{\infty,\mathbb{S}^2} + \|S_f - f\|_{\infty,\mathbb{S}^2},$$

we obtain

$$\begin{aligned}\|f - S_{\lambda,f}\|_{\infty,\mathbb{S}^2} &\le (\lambda\ K_{17}+1)\|S_f - f\|_{\infty,\mathbb{S}^2} + \lambda\ K_{17}\|f\|_{\infty,\mathbb{S}^2}\\ &\le (\lambda\ K_{17}+1)K_{18}\left(\tan\frac{|\Delta|}{2}\right)^{m+1} |f|_{m+1,\infty,\mathbb{S}^2} + \lambda\ K_{17}\|f\|_{\infty,\mathbb{S}^2}.\end{aligned}$$

Therefore we have (3.55) with $F_2 = (\lambda\ K_{17}+1)K_{18}$ and $F_3 = K_{17}$.

Next we prove a similar theorem for the penalized least squares functional P'_λ.

THEOREM 3.23. Let Δ be a β-quasi-uniform triangulation of the sphere $\mathbb{S}^2$ whose vertices form a subset of the data sites $\mathcal{V}$ and $|\Delta| \le 1$. Suppose that the data set $\mathcal{V}$ has a property that for every $s \in S_d^r(\Delta)$ and every $\tau \in \Delta$

$$F_1\|s\|_{\infty,\tau} \le \left(\sum_{\mathcal{V}\cap\tau} s(v)^2\right)^{1/2} = \|s\cdot\chi_\tau\|_{\mathcal{X}} \tag{3.57}$$

for some positive constant F_1. Let $S_{\lambda,f}$ be the spline minimizing P'_λ. Then

$$\|f - S_{\lambda,f}\|_{\infty,\mathbb{S}^2} \le F'_2\left(\tan\frac{|\Delta|}{2}\right)^{m+1} |f|_{m+1,\infty,\mathbb{S}^2} + F'_3\lambda\|f\|_{\infty,\mathbb{S}^2}. \tag{3.58}$$

for every function f in $W^{m+1,\infty}(\mathbb{S}^2)$. Here m is taken between 0 and d with $(d-m) \bmod 2 = 0$. The constants F_2', F_3' depend on d, λ, F_1, ρ_Δ, β and cardinality of $\mathcal{V}$.

PROOF. The proof is similar to the one of Theorem 3.22. For any $s \in \mathcal{S}$

$$\|s\|_{\mathcal{Y}}^2 = \mathcal{E}'(s)$$

and then by Lemma 3.9

$$\mathcal{E}_\tau'(s) \leq (|s|_{2,2,\tau}')^2.$$

Since s is a homogeneous polynomial of degree d By Lemma 2.29

$$|s|_{2,2,\tau}' \leq \frac{K_{16}'}{(\tan\frac{\rho_\tau}{2})^2}\|s\|_{2,\tau}$$

for some constant K_{16}' depending on d. From this point the proof continues as in the previous theorem.

Chapter 4

Multiple Star Technique for Minimal Energy Interpolation

4.1 Linear Extensions

The minimal energy method described in Section 3.1 is not very practicable for large data sets. We propose to use an analog of a domain decomposition technique studied in [8]. We proceed as follows.

Divide the spherical domain Ω into several smaller non-overlapping sub-domains $\Omega_i, i = 1, ..., n$ along the edges of an existing triangulation Δ of Ω. Fix $k \geq 1$ and let $q = 2(k+1)\ell + 1$. Here ℓ is the parameter reflecting local support of the basis functions B_ξ discussed in Section 2.8. Let $\mathrm{star}^q(\Omega_i)$ be an enlarged sub-domain Ω_i defined recursively as

$$\mathrm{star}^q(\Omega_i) := \cup\{T \in \Delta, T \cap \mathrm{star}^{q-1}(\Omega_i) \neq \emptyset\} \tag{4.1}$$

and $\mathrm{star}^0(\Omega_i) := \Omega_i$.

We solve the scattered data interpolation problem over each $\mathrm{star}^q(\Omega_i)$ for each i. Let $s_{f,i,k}$ be the minimal energy solution over $\mathrm{star}^q(\Omega_i)$. That is, let $\mathcal{S}_{i,k}$ be the collection of splines in $\mathcal{S}$ restricted to $\mathrm{star}^q(\Omega_i)$ and

$$\Gamma(f, i, k) := \{s \in \mathcal{S}_{i,k}, s(v) = f(v), \forall v \in \mathrm{star}^q(\Omega_i) \cap \mathcal{V}\}.$$

Then $s_{f,i,k} \in \mathcal{S}_{i,k}$ is the spline satisfying

$$\mathcal{E}_{i,k}(s_{f,i,k}) = \min\{\mathcal{E}_{i,k}(s), s \in \Gamma(f, i, k)\}, \tag{4.2}$$

where

$$\mathcal{E}_{i,k}(s) = \mathcal{E}_{\text{star}^q(\Omega_i)}(s) \tag{4.3}$$

defined as in (3.8). We are to show that $s_{f,i,k}|_{\Omega_i}$ approximates the global minimal energy spline (3.4) $S_f|_{\Omega_i}$ very well. The following lemma plays crucial role in the proof of the main result.

LEMMA 4.1. Let $\mathcal{W}$ be a Hilbert space of spline functions defined on a triangulation Δ of a spherical domain Ω associated with the inner product $\langle \cdot, \cdot \rangle_\Omega$ and the norm $\| \cdot \|_\Omega$ defined in Section 3.1. Let $\mathcal{B}$ be the local stable basis discussed in Section 2.8. Let ω be a cluster of triangles in Δ. Suppose s is a function in $\mathcal{W}$ satisfying

$$\langle s, u \rangle = 0, \forall u \in \mathcal{W}, \text{supp}(u) \subset \Omega \backslash \omega. \tag{4.4}$$

Then for any triangle τ outside of $\text{star}^q(\omega)$, there exist constants $0 < \sigma < 1$ and H_1 depending only ℓ, d and β such that

$$\|s\chi_\tau\|_\Omega \leq H_1 \sigma^k \|s\|_\Omega. \tag{4.5}$$

PROOF. Note that $s \in \mathcal{W}$ implies $\mathcal{P}s = s$. Then the hypothesis is equivalent to the one in Theorem 3.6 with the only difference that we work with the cluster of triangles ω and not with a single triangle. The proof however holds word to word.

In the proof of the main result we also need to use a concept of a natural extension. For a spline function s defined on some cluster of triangles ω in Δ we can write

$$s = \sum_{\xi \in \mathcal{M}: B_\xi|_\omega \neq 0} c_\xi B_\xi|_\omega.$$

If we replace $B_\xi|_\omega$ in the above by B_ξ we obtain the natural extension $\widetilde{s}$ of s.

LEMMA 4.2. Let Δ be a β-quasi-uniform triangulation of $\mathbb{S}^2$, and let ω be a cluster of triangles in Δ. For a spline function s defined on ω let $\widetilde{s}$ be its natural extension to all of $\mathbb{S}^2$. There exists a constant H_2 depending on d and the minimal angle in Δ,

Θ_Δ, such that

$$|\widetilde{s}|_{k,\infty,\omega} \leq H_2 \left(\tan \frac{\rho_\Delta}{2}\right)^{-k} \|s\|_{\infty,\mathbb{S}^2} \tag{4.6}$$

for $k \leq d$.

Proof. By Lemma 2.29, taking supremum over triangles in ω we get

$$|\widetilde{s}|_{k,\infty,\omega} \leq K_{19} \left(\tan \frac{\rho_\Delta}{2}\right)^{-k} \|\widetilde{s}\|_{\infty,\omega},$$

for some K_{19} depending on d and Θ_Δ. By Proposition 2.31 2) and definition of $\widetilde{s}$

$$\|\widetilde{s}\|_{\infty,\omega} \leq \|c\|_{\infty,\mathbb{S}^2} \sum_{\xi\in\mathcal{M}:B_\xi|_\omega\neq 0} \|B_\xi\|_{\infty,\mathbb{S}^2} \leq K_{20}K_{21}\|c\|_{\infty,\mathbb{S}^2}$$

for some K_{20} depending on d and Θ_Δ and $K_{21} = \#\{\xi \in \mathcal{M} : B_\xi|\omega \neq 0\}$. Since $\widetilde{s}$ and s have the same coefficients and by Proposition 2.31 3)

$$\|\widetilde{s}\|_{\infty,\omega} \leq K_{20}K_{21}\|c\|_{\infty,\mathbb{S}^2} \leq K_{20}K_{21}K_{22}\|s\|_{\infty,\mathbb{S}^2}.$$

for some K_{22} depending on d and Θ_Δ. We obtain (4.6) with $H_2 = K_{19}K_{20}K_{21}K_{22}$.

We are ready to prove our main result.

Theorem 4.3. Suppose we are given data values $f(v), v \in \mathcal{V}$. Let S_f be the global minimal energy interpolating spline satisfying (3.2). Let $s_{f,i,k}$ be the minimal energy interpolating spline over $\mathrm{star}^q(\Omega_i)$ satisfying (4.3). Then there exists a constant $\sigma \in (0,1)$ such that for $q = 2(k+1)\ell + 1,\ k \geq 1$

$$\|S_f - s_{f,i,k}\|_{2,\Omega_i} \leq H_3\sigma^k \left(\tan \frac{|\Delta|}{2}\right)^2 \left(H_4|f|_{2,\infty,\mathbb{S}^2} + H_5\|f\|_{\infty,\mathbb{S}^2}\right), \tag{4.7}$$

if $f \in C^2(\mathbb{S}^2)$ and d is odd. Here H_3, H_4 are constants depending on d and β, and H_5 in addition depends on ρ_Δ. If $f \in C^3(\mathbb{S}^2)$ and d is even then

$$\|S_f - s_{f,i,k}\|_{2,\Omega_i} \leq H_3\sigma^k \left(\tan \frac{|\Delta|}{2}\right)^2 \left(H_6|f|_{2,\infty,\mathbb{S}^2} + H_7|f|_{3,\infty,\mathbb{S}^2} + H_5\|f\|_{\infty,\mathbb{S}^2}\right), \tag{4.8}$$

for positive constants H_6 depending on d and β, and H_7 in addition depending on $|\Delta|$.

PROOF. To simplify our notation let us denote $\Omega_{q,i} := \text{star}^q(\Omega_i)$. Recall from Section 3.1 that

$$\mathcal{W} := \{s \in \mathcal{S} : s(v) = 0, v \in \mathcal{V}\}.$$

is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathbb{S}^2}$. Note that

$$\langle S_f, u \rangle_{\mathbb{S}^2} = 0$$

for all $u \in \mathcal{W}$. Indeed, $\mathcal{E}_{\mathbb{S}^2}(S_f + \alpha u)$ achieves its minimum when $\alpha = 0$, the derivative with respect to α at $\alpha = 0$ implies the result. Let

$$\mathcal{W}_{i,k} := \{s|_{\Omega_{q,i}} : s \in \mathcal{S}, s(v) = 0, \forall v \in \Omega_{q,i} \cap \mathcal{V}\},$$

be equipped with the inner product $\langle \cdot, \cdot \rangle_{\Omega_{q,i}}$. Then

$$\langle s_{f,i,k}, u \rangle_{\Omega_{q,i}} = 0$$

for all $u \in \mathcal{W}_{i,k}$. In addition,

$$\langle S_f - s_{f,i,k}, u \rangle_{\mathbb{S}^2} = 0$$

for all $u \in \mathcal{W}$ such that $\text{supp}(u) \subset \Omega_{q,i}$. With $\omega = \mathbb{S}^2 \backslash \Omega_{q,i}$, for any $\tau \in \Omega_i$, by Lemma 4.1,

$$\|(S_f - s_{f,i,k})\chi_\tau\|_{\mathbb{S}^2} \leq H_1 \sigma^k \|S_f - \widetilde{s_{f,i,k}}\|_{\mathbb{S}^2}.$$

Here $\widetilde{s_{f,i,k}}$ is the natural extension of $s_{f,i,k}$ on $\mathbb{S}^2$. Then

$$\|S_f - s_{f,i,k}\|_{\Omega_i} = \sum_{\tau \in \Omega_i} \|(S_f - s_{f,i,k})\chi_\tau\|_{\mathbb{S}^2} \leq H_1 \sum_{\tau \in \Omega_i} \sigma^k \|S_f - \widetilde{s_{f,i,k}}\|_{\mathbb{S}^2}.$$

Let m_i denote the number of triangles in Ω_i. Then

$$\|S_f - s_{f,i,k}\|_{\Omega_i} \leq H_1 \sigma^k m_i \|S_f - \widetilde{s_{f,i,k}}\|_{\mathbb{S}^2}.$$

To have a better estimate of $\|S_f - s_{f,i,k}\|_{\Omega_i}$ we may extend $s_{f,i,k}$ in a more convenient way. Let $\widetilde{S_{f,i,k}}$ be the natural extension of the spline $S_{f|_{\mathbb{S}^2 \backslash \Omega_{q+\ell,i}}}$ to $\mathbb{S}^2$. Let $\widehat{s_{f,i,k}} =$

$\widetilde{s_{f,i,k}} + \widetilde{S_{f,i,k}}$. Note that $\text{supp}(\widetilde{s_{f,i,k}})$ and $\text{supp}(\widetilde{S_{f,i,k}})$ are disjoint and then we can replace $\widetilde{s_{f,i,k}}$ above by $\widehat{s_{f,i,k}}$ to get

$$\|S_f - s_{f,i,k}\|_{\Omega_i} \le H_1 \sigma^k m_i \|S_f - \widehat{s_{f,i,k}}\|_{\mathbb{S}^2}. \tag{4.9}$$

Consider the usual L_2 norm

$$\begin{aligned} \|S_f - \widehat{s_{f,i,k}}\|_{2,\mathbb{S}^2} &= \|S_f - \widehat{s_{f,i,k}}\|_{2,\Omega_{q+\ell,i}} \le \|S_f - \widehat{s_{f,i,k}}\|_{2,\Omega_{q,i}} \\ &\quad + \|S_f - \widehat{s_{f,i,k}}\|_{2,\Omega_{q+\ell,i}\setminus\Omega_{q,i}} \le \|S_f - f\|_{2,\Omega_{q,i}} \\ &\quad + \|f - s_{f,i,k}\|_{2,\Omega_{q,i}} + \|S_f - \widehat{s_{f,i,k}}\|_{2,\Omega_{q+\ell,i}\setminus\Omega_{q,i}}. \end{aligned}$$

Since

$$\|S_f - f\|_{2,\Omega_{q,i}} \le A^{1/2}_{\Omega_{q,i}} \|S_f - f\|_{\infty,\Omega_{q,i}}$$

and both $S_f - f$ and $s_{f,i,k} - f$ satisfy the hypothesis of Theorem 3.8, we have

$$\|S_f - f\|_{2,\Omega_{q,i}} + \|f - s_{f,i,k}\|_{2,\Omega_{q,i}} \le 2A^{1/2}_{\Omega_{q,i}} K_{23} \left(\tan \frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2}$$

for $f \in C^2(\mathbb{S}^2)$ and d odd. If $f \in C^3(\mathbb{S}^2)$ and d is even then

$$\begin{aligned} &\|S_f - f\|_{2,\Omega_{q,i}} + \|f - s_{f,i,k}\|_{2,\Omega_{q,i}} \\ \le\ & 2A^{1/2}_{\Omega_{q,i}} \left(K_{24} \left(\tan \frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2} + K_{25} \left(\tan \frac{|\Delta|}{2}\right)^3 |f|_{3,\infty,\mathbb{S}^2} \right). \end{aligned}$$

Consider $\|S_f - \widehat{s_{f,i,k}}\|_{2,\Omega_{q+\ell,i}\setminus\Omega_{q,i}}$. By Lemma 2.30 since both S_f and $\widehat{s_{f,i,k}}$ are in $\Gamma(f)$,

$$\begin{aligned} &\|S_f - \widehat{s_{f,i,k}}\|_{2,\Omega_{q+\ell,i}\setminus\Omega_{q,i}} \\ \le\ & A^{1/2}_{\Omega_{q+\ell,i}} \|S_f - \widehat{s_{f,i,k}}\|_{\infty,\Omega_{q+\ell,i}\setminus\Omega_{q,i}} \\ \le\ & A^{1/2}_{\Omega_{q+\ell,i}} K_{26} \left(\tan \frac{|\Delta|}{2}\right)^2 |S_f - \widehat{s_{f,i,k}}|_{2,\infty,\Omega_{q+\ell,i}\setminus\Omega_{q,i}}, \end{aligned}$$

for some K_{26}. Using the definition of $\widehat{s_{f,i,k}}$ we get

$$\begin{aligned} &\|S_f - \widehat{s_{f,i,k}}\|_{2,\Omega_{q+\ell,i}\setminus\Omega_{q,i}} \\ \le\ & A^{1/2}_{\Omega_{q+\ell,i}} K_{26} \left(\tan \frac{|\Delta|}{2}\right)^2 \left(|S_f|_{2,\infty,\mathbb{S}^2} + |\widetilde{S_{f,i,k}}|_{2,\infty,\Omega_{q+\ell,i}\setminus\Omega_{q,i}} + |\widetilde{s_{f,i,k}}|_{2,\infty,\Omega_{q+\ell,i}\setminus\Omega_{q,i}} \right). \end{aligned}$$

By Lemma 2.29 and Lemma 4.2

$$\begin{aligned} & |S_f|_{2,\infty,\mathbb{S}^2} + |\widetilde{S_{f,i,k}}|_{2,\infty,\Omega_{q+\ell,i}\setminus\Omega_{q,i}} + |\widetilde{s_{f,i,k}}|_{2,\infty,\Omega_{q+\ell,i}\setminus\Omega_{q,i}} \\ \leq\quad & (\tan\frac{\rho_\Delta}{2})^{-2}(K_{27}\|S_f\|_{\infty,\mathbb{S}^2} + H_2\|S_f\|_{\infty,\mathbb{S}^2} + H_2\|s_{f,i,k}\|_{\infty,\mathbb{S}^2}), \end{aligned}$$

for some K_{27} depending on d and Θ_Δ. By Theorem 3.8, if $f \in C^2(\mathbb{S}^2)$ and d is odd

$$\|S_f\|_{\infty,\mathbb{S}^2} \leq \|f\|_{\infty,\mathbb{S}^2} + K_{23}\left(\tan\frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2}$$

and

$$\|s_{f,i,k}\|_{\infty,\mathbb{S}^2} \leq \|f\|_{\infty,\mathbb{S}^2} + K_{23}(\tan\frac{|\Delta|}{2})^2|f|_{2,\infty,\mathbb{S}^2}.$$

If $f \in C^3(\mathbb{S}^2)$ and d is even

$$\|S_f\|_{\infty,\mathbb{S}^2} \leq \|f\|_{\infty,\mathbb{S}^2} + K_{24}\left(\tan\frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2} + K_{25}\left(\tan\frac{|\Delta|}{2}\right)^3 |f|_{3,\infty,\mathbb{S}^2}$$

and

$$\|s_{f,i,k}\|_{\infty,\mathbb{S}^2} \leq \|f\|_{\infty,\mathbb{S}^2} + K_{24}(\tan\frac{|\Delta|}{2})^2|f|_{2,\infty,\mathbb{S}^2} + K_{25}\left(\tan\frac{|\Delta|}{2}\right)^3 |f|_{3,\infty,\mathbb{S}^2}.$$

Hence

$$\begin{aligned} & \|S_f - \widehat{s_{f,i,k}}\|_{2,\Omega_{q+\ell,i}\setminus\Omega_{q,i}} \\ \leq\quad & (K_{27}+2H_2)K_{26}A^{1/2}_{\Omega_{q+\ell,i}}(\frac{\tan|\Delta|/2}{\tan\rho_\Delta/2})^2\left(K_{23}\left(\tan\frac{|\Delta|}{2}\right)^2|f|_{2,\infty,\mathbb{S}^2} + \|f\|_{\infty,\mathbb{S}^2}\right) \end{aligned}$$

and therefore

$$\|S_f - \widehat{s_{f,i,k}}\|_{2,\mathbb{S}^2} \leq A^{1/2}_{\Omega_{q+\ell,i}}\left(\tan\frac{|\Delta|}{2}\right)^2 (K_{23}K_{28}|f|_{2,\infty,\mathbb{S}^2} + K_{29}\|f\|_{\infty,\mathbb{S}^2})$$

for $f \in C^2(\mathbb{S}^2)$ and d odd. Here for convenience we denote $K_{28} = 2 + K_{26}(K_{27} + 2H_2)(\frac{\tan|\Delta|/2}{\tan\rho_\Delta/2})^2$ and $K_{29} = \frac{K_{26}(K_{27}+2H_2)}{(\tan\frac{\rho_\Delta}{2})^2}$. Similarly for $f \in C^3(\mathbb{S}^2)$ and d even we get

$$\begin{aligned} & \|S_f - \widehat{s_{f,i,k}}\|_{2,\mathbb{S}^2} \leq A^{1/2}_{\Omega_{q+\ell,i}}\left(\tan\frac{|\Delta|}{2}\right)^2 \\ & \left(K_{24}K_{28}|f|_{2,\infty,\mathbb{S}^2} + K_{25}K_{28}\left(\tan\frac{|\Delta|}{2}\right)|f|_{3,\infty,\mathbb{S}^2} + K_{29}\|f\|_{\infty,\mathbb{S}^2}\right). \end{aligned}$$

Apply Theorem 3.3 to (4.9) to see that

$$\begin{aligned}\|S_f - s_{f,i,k}\|_{2,\Omega_i} &\leq K_{30}\left(\tan\frac{|\Delta|}{2}\right)^2 \|S_f - s_{f,i,k}\|_{\Omega_i} \\ &\leq K_{30}H_1 m_i\sigma^k\left(\tan\frac{|\Delta|}{2}\right)^2 \|S_f - \widehat{s_{f,i,k}}\|_{\mathbb{S}^2} \\ &\leq K_{30}K_{31}H_1 m_i\sigma^k \|S_f - \widehat{s_{f,i,k}}\|_{2,\mathbb{S}^2},\end{aligned}$$

for some K_{30}, K_{31} depending on d and β. Then for $f \in C^2(\mathbb{S}^2)$ and d odd

$$\|S_f - s_{f,i,k}\|_{2,\Omega_i} \leq H_3\sigma^k\left(\tan\frac{|\Delta|}{2}\right)^2 (H_4|f|_{2,\infty,\mathbb{S}^2} + H_5\|f\|_{\infty,\mathbb{S}^2}).$$

with $H_3 = K_{30}K_{31}H_1 m_i A^{1/2}_{\Omega_{q+\ell,i}}$, $H_4 = K_{23}K_{28}$and $H_5 = K_{29}$. Similarly for $f \in C^3(\mathbb{S}^2)$ and d even we obtain

$$\|S_f - s_{f,i,k}\|_{2,\Omega_i} \leq H_3\sigma^k\left(\tan\frac{|\Delta|}{2}\right)^2 (H_6|f|_{2,\infty,\mathbb{S}^2} + H_7|f|_{3,\infty,\mathbb{S}^2} + H_5\|f\|_{\infty,\mathbb{S}^2}).$$

with $H_6 = K_{24}K_{28}$ and $H_7 = K_{25}K_{28}$. This completes the proof of Theorem 4.3.

4.2 Constant Extensions

We complete this chapter by deriving similar error bounds for the multiple star technique applied to the global interpolating splines minimizing $\mathcal{E}'$ defined in Section 3.2. Similar to Lemma 4.1 we have as in Theorem 3.13

Lemma 4.4. Let $\mathcal{W}$ be a Hilbert space of spline functions defined on a triangulation Δ of a spherical domain Ω associated with the inner product $\langle\cdot,\cdot\rangle_\Omega$ and the norm $\|\cdot\|_\Omega$ defined in Section 3.1. Let $\mathcal{B}$ be the local stable basis discussed in Section 2.8. Let ω be a cluster of triangles in Δ. Suppose s is a function in $\mathcal{W}$ satisfying

$$\langle s, u\rangle = 0, \forall u \in \mathcal{W}, \mathrm{supp}(u) \subset \Omega\backslash\omega. \tag{4.10}$$

Then for any triangle τ outside of $\mathrm{star}^q(\omega)$, there exist constants $0 < \sigma' < 1$ and H_1' depending only on ℓ, d and β such that

$$\|s\chi_\tau\|'_\Omega \leq H_1'(\sigma')^k\|s\|'_\Omega. \tag{4.11}$$

Recall the concept of a natural extension. For a spline function s defined on some cluster of triangles ω in Δ we can write

$$s = \sum_{\xi \in \mathcal{M}: B_\xi|_\omega \neq 0} c_\xi B_\xi|_\omega.$$

If we replace $B_\xi|_\omega$ in the above by B_ξ we obtain the natural extension $\widetilde{s}$ of s. We have, similar to Lemma 4.2,

LEMMA 4.5. Let Δ be a β-quasi-uniform triangulation of $\mathbb{S}^2$, and let ω be a cluster of triangles in Δ. For a spline function s defined on ω let $\widetilde{s}$ be its natural extension to all of $\mathbb{S}^2$. There exists a constant H_2' depending on d and the minimal angle in Δ, Θ_Δ, such that

$$|\widetilde{s}|'_{k,\infty,\omega} \leq H_2' \left(\tan \frac{\rho_\Delta}{2}\right)^{-k} \|s\|_{\infty,\mathbb{S}^2} \tag{4.12}$$

for $k \leq d$.

PROOF. Similar to the proof of Lemma 4.2.

In the proof of the main theorem of this section we use the same ideas as in the previous section. Naturally, the results are different since we are minimizing a different energy functional.

THEOREM 4.6. Suppose we are given data values $f(v), v \in \mathcal{V}$. Let S_f be the global interpolating spline minimizing (3.23). Let $s_{f,i,k}$ be the minimal energy interpolating spline over $\mathrm{star}^q(\Omega_i)$ satisfying

$$\mathcal{E}'_{i,k}(s_{f,i,k}) = \min\{\mathcal{E}'_{i,k}(s), s \in \Gamma(f,i,k)\}. \tag{4.13}$$

Then there exists a constant $\sigma \in (0,1)$ such that for $q = 2(k+1)\ell + 1,\ k \geq 1$

$$\|S_f - s_{f,i,k}\|_{2,\Omega_i} \leq H_3'(\sigma')^k \left(\tan \frac{|\Delta|}{2}\right)^2 (H_4'|f|_{2,\infty,\mathbb{S}^2} + H_5'|f|'_{2,\infty,\mathbb{S}^2} + H_6'\|f\|_{\infty,\mathbb{S}^2}), \tag{4.14}$$

if $f \in C^2(\mathbb{S}^2)$ and d is odd. Here H_3', H_4', H_5' are constants depending on d and β, and H_6' additionally depends on ρ_Δ. If $f \in C^3(\mathbb{S}^2)$ and d is even then

$$\|S_f - s_{f,i,k}\|_{2,\Omega_i} \leq H_3'(\sigma')^k \left(\tan\frac{|\Delta|}{2}\right)^2 (H_5'|f|_{2,\infty,\mathbb{S}^2}' + H_7'|f|_{3,\infty,\mathbb{S}^2} + H_6'\|f\|_{\infty,\mathbb{S}^2}), \tag{4.15}$$

for a positive constant H_7' depending on d, β and the size of the largest triangle in Δ.

Proof. To simplify our notation let us denote $\Omega_{q,i} := \mathrm{star}^q(\Omega_i)$. Recall from Section 3.2 that

$$\mathcal{W} := \{s \in \mathcal{S} : s(v) = 0, v \in \mathcal{V}\}.$$

is a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathbb{S}^2}$. Note that

$$\langle S_f, u\rangle_{\mathbb{S}^2} = 0$$

for all $u \in \mathcal{W}$. Indeed, $\mathcal{E}_{\mathbb{S}^2}(S_f + \alpha u)$ achieves its minimum when $\alpha = 0$, the derivative with respect to α at $\alpha = 0$ implies the result. Let

$$\mathcal{W}_{i,k} := \{s|_{\Omega_{q,i}} : s \in \mathcal{S}, s(v) = 0, \forall v \in \Omega_{q,i} \cap \mathcal{V}\},$$

be equipped with the inner product $\langle\cdot,\cdot\rangle_{\Omega_{q,i}}$. Then

$$\langle s_{f,i,k}, u\rangle_{\Omega_{q,i}} = 0$$

for all $u \in \mathcal{W}_{i,k}$. In addition,

$$\langle S_f - s_{f,i,k}, u\rangle_{\mathbb{S}^2} = 0$$

for all $u \in \mathcal{W}$ such that $\mathrm{supp}(u) \subset \Omega_{q,i}$. With $\omega = \mathbb{S}^2 \backslash \Omega_{q,i}$, for any $\tau \in \Omega_i$, by Lemma 4.4,

$$\|(S_f - s_{f,i,k})\chi_\tau\|_{\mathbb{S}^2}' \leq H_1'(\sigma')^k \|S_f - \widetilde{s_{f,i,k}}\|_{\mathbb{S}^2}'.$$

Here $\widetilde{s_{f,i,k}}$ is the natural extension of $s_{f,i,k}$ on $\mathbb{S}^2$. Then

$$\|S_f - s_{f,i,k}\|_{\Omega_i}' = \sum_{\tau\in\Omega_i} \|(S_f - s_{f,i,k})\chi_\tau\|_{\mathbb{S}^2}' \leq H_1' \sum_{\tau\in\Omega_i} (\sigma')^k \|S_f - \widetilde{s_{f,i,k}}\|_{\mathbb{S}^2}'.$$

Let m_i denote the number of triangles in Ω_i. Then

$$\|S_f - s_{f,i,k}\|'_{\Omega_i} \leq H_1'(\sigma')^k m_i \|S_f - \widetilde{s_{f,i,k}}\|'_{\mathbb{S}^2}.$$

To have a better estimate of $\|S_f - s_{f,i,k}\|'_{\Omega_i}$ we extend $s_{f,i,k}$ in a more convenient way. Let $\widetilde{S_{f,i,k}}$ be the natural extension of the spline $S_{f|_{\mathbb{S}^2\setminus\Omega_{q+\ell,i}}}$ to $\mathbb{S}^2$. Let $\widehat{s_{f,i,k}} = \widetilde{s_{f,i,k}} + \widetilde{S_{f,i,k}}$. Note that $\mathrm{supp}(\widetilde{s_{f,i,k}})$ and $\mathrm{supp}(\widetilde{S_{f,i,k}})$ are disjoint and then we can replace $\widetilde{s_{f,i,k}}$ above by $\widehat{s_{f,i,k}}$ to get

$$\|S_f - s_{f,i,k}\|'_{\Omega_i} \leq H_1'(\sigma')^k m_i \|S_f - \widehat{s_{f,i,k}}\|'_{\mathbb{S}^2}. \tag{4.16}$$

Consider the usual L_2 norm

$$\begin{aligned}\|S_f - \widehat{s_{f,i,k}}\|_{2,\mathbb{S}^2} &= \|S_f - \widehat{s_{f,i,k}}\|_{2,\Omega_{q+\ell,i}} \leq \|S_f - \widehat{s_{f,i,k}}\|_{2,\Omega_{q,i}} \\ &\quad + \|S_f - \widehat{s_{f,i,k}}\|_{2,\Omega_{q+\ell,i}\setminus\Omega_{q,i}} \leq \|S_f - f\|_{2,\Omega_{q,i}} \\ &\quad + \|f - s_{f,i,k}\|_{2,\Omega_{q,i}} + \|S_f - \widehat{s_{f,i,k}}\|_{2,\Omega_{q+\ell,i}\setminus\Omega_{q,i}}.\end{aligned}$$

Since

$$\|S_f - f\|_{2,\Omega_{q,i}} \leq A^{1/2}_{\Omega_{q,i}} \|S_f - f\|_{\infty,\Omega_{q,i}}$$

and both $S_f - f$ and $s_{f,i,k} - f$ satisfy the hypothesis of Theorem 3.15, we have

$$\begin{aligned}&\|S_f - f\|_{2,\Omega_{q,i}} + \|f - s_{f,i,k}\|_{2,\Omega_{q,i}} \leq \\ 2\ \ &A^{1/2}_{\Omega_{q,i}} \left(K'_{22} \left(\tan\frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2} + K'_{23} \left(\tan\frac{|\Delta|}{2}\right)^2 |f|'_{2,\infty,\mathbb{S}^2} \right),\end{aligned}$$

for $f \in C^2(\mathbb{S}^2)$ and d odd. If $f \in C^3(\mathbb{S}^2)$ and d is even then

$$\begin{aligned}&\|S_f - f\|_{2,\Omega_{q,i}} + \|f - s_{f,i,k}\|_{2,\Omega_{q,i}} \leq \\ 2\ \ &A^{1/2}_{\Omega_{q,i}} \left(K'_{24} \left(\tan\frac{|\Delta|}{2}\right)^3 |f|_{3,\infty,\mathbb{S}^2} + K'_{23} \left(\tan\frac{|\Delta|}{2}\right)^2 |f|'_{2,\infty,\mathbb{S}^2} \right).\end{aligned}$$

Consider $\|S_f - \widehat{s_{f,i,k}}\|_{2,\Omega_{q+\ell,i}\setminus\Omega_{q,i}}$. By Lemma 2.30 in since both S_f and $\widehat{s_{f,i,k}}$ are in $\Gamma(f)$,

$$\|S_f - \widehat{s_{f,i,k}}\|_{2,\Omega_{q+\ell,i}\setminus\Omega_{q,i}}$$

$$
\begin{aligned}
&\le \quad A^{1/2}_{\Omega_{q+\ell,i}} \|S_f - \widehat{s_{f,i,k}}\|_{\infty,\Omega_{q+\ell,i}\setminus\Omega_{q,i}} \\
&\le \quad A^{1/2}_{\Omega_{q+\ell,i}} K'_{25} \left(\tan\frac{|\Delta|}{2}\right)^2 |S_f - \widehat{s_{f,i,k}}|'_{2,\infty,\Omega_{q+\ell,i}\setminus\Omega_{q,i}},
\end{aligned}
$$

for some K'_{25}. Using the definition of $\widehat{s_{f,i,k}}$ we get

$$
\begin{aligned}
&\|S_f - \widehat{s_{f,i,k}}\|_{2,\Omega_{q+\ell,i}\setminus\Omega_{q,i}} \le A^{1/2}_{\Omega_{q+\ell,i}} \\
&K'_{25}\left(\tan\frac{|\Delta|}{2}\right)^2 \left(|S_f|'_{2,\infty,\mathbb{S}^2} + |\widetilde{S_{f,i,k}}|'_{2,\infty,\Omega_{q+\ell,i}\setminus\Omega_{q,i}} + |\widetilde{s_{f,i,k}}|'_{2,\infty,\Omega_{q+\ell,i}\setminus\Omega_{q,i}}\right).
\end{aligned}
$$

By Lemma 2.29 and Lemma 4.5

$$
\begin{aligned}
&|S_f|'_{2,\infty,\mathbb{S}^2} + |\widetilde{S_{f,i,k}}|'_{2,\infty,\Omega_{q+\ell,i}\setminus\Omega_{q,i}} + |\widetilde{s_{f,i,k}}|'_{2,\infty,\Omega_{q+\ell,i}\setminus\Omega_{q,i}} \\
\le\quad &\left(\tan\frac{\rho_\Delta}{2}\right)^{-2} (K'_{26}\|S_f\|_{\infty,\mathbb{S}^2} + H'_2\|S_f\|_{\infty,\mathbb{S}^2} + H'_2\|s_{f,i,k}\|_{\infty,\mathbb{S}^2}),
\end{aligned}
$$

for some K'_{26} depending on d and Θ_Δ. By Theorem 3.15, if $f \in C^2(\mathbb{S}^2)$ and d is odd

$$
\|S_f\|_{\infty,\mathbb{S}^2} \le \|f\|_{\infty,\mathbb{S}^2} + K'_{22}\left(\tan\frac{|\Delta|}{2}\right)^2 |f|_{2,\infty,\mathbb{S}^2} + K'_{23}\left(\tan\frac{|\Delta|}{2}\right)^2 |f|'_{2,\infty,\mathbb{S}^2}
$$

and

$$
\|s_{f,i,k}\|_{\infty,\mathbb{S}^2} \le \|f\|_{\infty,\mathbb{S}^2} + K'_{22}(\tan\frac{|\Delta|}{2})^2 |f|_{2,\infty,\mathbb{S}^2} + K'_{23}\left(\tan\frac{|\Delta|}{2}\right)^2 |f|'_{2,\infty,\mathbb{S}^2}.
$$

If $f \in C^3(\mathbb{S}^2)$ and d is even

$$
\|S_f\|_{\infty,\mathbb{S}^2} \le \|f\|_{\infty,\mathbb{S}^2} + K'_{23}\left(\tan\frac{|\Delta|}{2}\right)^2 |f|'_{2,\infty,\mathbb{S}^2} + K'_{24}\left(\tan\frac{|\Delta|}{2}\right)^3 |f|_{3,\infty,\mathbb{S}^2}
$$

and

$$
\|s_{f,i,k}\|_{\infty,\mathbb{S}^2} \le \|f\|_{\infty,\mathbb{S}^2} + K'_{23}\left(\tan\frac{|\Delta|}{2}\right)^2 |f|'_{2,\infty,\mathbb{S}^2} + K'_{24}\left(\tan\frac{|\Delta|}{2}\right)^3 |f|_{3,\infty,\mathbb{S}^2}.
$$

Hence

$$
\begin{aligned}
&\|S_f - \widehat{s_{f,i,k}}\|_{2,\Omega_{q+\ell,i}\setminus\Omega_{q,i}} \le (K'_{26} + 2H'_2)K'_{25}A^{1/2}_{\Omega_{q+\ell,i}} \left(\frac{\tan|\Delta|/2}{\tan\rho_\Delta/2}\right)^2 \\
&\left(K'_{22}(\tan\frac{|\Delta|}{2})^2|f|_{2,\infty,\mathbb{S}^2} + K'_{23}\left(\tan\frac{|\Delta|}{2}\right)^2 |f|'_{2,\infty,\mathbb{S}^2} + \|f\|_{\infty,\mathbb{S}^2}\right)
\end{aligned}
$$

and therefore

$$\begin{aligned}\|S_f - \widehat{s_{f,i,k}}\|_{2,\mathbb{S}^2} &\le A^{1/2}_{\Omega_{q+\ell,i}} \left(\tan\frac{|\Delta|}{2}\right)^2 \\ &\quad (K'_{22}K'_{27}|f|_{2,\infty,\mathbb{S}^2} + K'_{23}K'_{27}|f|'_{2,\infty,\mathbb{S}^2} + K'_{28}\|f\|_{\infty,\mathbb{S}^2}),\end{aligned}$$

for $f \in C^2(\mathbb{S}^2)$ and d odd. Here for convenience we denote $K'_{27} := 2 + K'_{25}(K'_{26} + 2H'_2)(\frac{\tan|\Delta|/2}{\tan\rho_\Delta/2})^2$, $K'_{28} := \frac{K'_{25}(K'_{26}+2H'_2)}{(\tan\frac{\rho_\Delta}{2})^2}$. Similarly for $f \in C^3(\mathbb{S}^2)$ and d even we get

$$\begin{aligned}&\|S_f - \widehat{s_{f,i,k}}\|_{2,\mathbb{S}^2} \le A^{1/2}_{\Omega_{q+\ell,i}} \left(\tan\frac{|\Delta|}{2}\right)^2 \\ &\left(K'_{23}K'_{27}|f|'_{2,\infty,\mathbb{S}^2} + K'_{24}K'_{27}\left(\tan\frac{|\Delta|}{2}\right)|f|_{3,\infty,\mathbb{S}^2} + K'_{28}\|f\|_{\infty,\mathbb{S}^2}\right).\end{aligned}$$

Apply Theorem 3.3 to both sides of (4.16) to see that

$$\begin{aligned}\|S_f - s_{f,i,k}\|_{2,\Omega_i} &\le K'_{29}\left(\tan\frac{|\Delta|}{2}\right)^2 \|S_f - s_{f,i,k}\|'_{\Omega_i} \\ &\le K'_{29}H'_1 m_i(\sigma')^k \left(\tan\frac{|\Delta|}{2}\right)^2 \|S_f - \widehat{s_{f,i,k}}\|'_{\mathbb{S}^2} \\ &\le K'_{30}K'_{29}H'_1 m_i(\sigma')^k \|S_f - \widehat{s_{f,i,k}}\|_{2,\mathbb{S}^2},\end{aligned}$$

for some K'_{29}, K'_{30} depending on d and β. Then for $f \in C^2(\mathbb{S}^2)$ and d odd

$$\|S_f - s_{f,i,k}\|_{2,\Omega_i} \le H'_3(\sigma')^k\left(\tan\frac{|\Delta|}{2}\right)^2 (H'_4|f|_{2,\infty,\mathbb{S}^2} + H'_5|f|'_{2,\infty,\mathbb{S}^2} + H'_6\|f\|_{\infty,\mathbb{S}^2}).$$

with $H'_3 = K'_{29}K'_{30}H'_1 m_i A^{1/2}_{\Omega_{q+\ell,i}}$, $H'_4 = K'_{22}K'_{27}$, $H'_5 = K'_{23}K'_{27}$ and $H'_6 = K'_{28}$. Similarly for $f \in C^3(\mathbb{S}^2)$ and d even we obtain

$$\|S_f - s_{f,i,k}\|_{2,\Omega_i} \le H'_3(\sigma')^k\left(\tan\frac{|\Delta|}{2}\right)^2 (H'_5|f|'_{2,\infty,\mathbb{S}^2} + H'_7|f|_{3,\infty,\mathbb{S}^2} + H'_6\|f\|_{\infty,\mathbb{S}^2}).$$

with $H'_7 = K'_{24}K'_{27}\tan\frac{|\Delta|}{2}$. This completes the proof of Theorem 4.6.

Chapter 5

Computational Details

In this chapter we describe in detail how the global approximation methods are implemented in practice.

5.1 Minimal Energy Interpolation

Given $\mathcal{V} := \{v \in \mathbb{S}^2\}$ a set of points on the unit sphere with real numbers $\{f(v),\ v \in \mathcal{V}\}$ construct a regular spherical triangulation Δ. For $d \geq 1$ and $r \geq 0$, two integers with $3r+2 \leq d$, define $S_d^{-1}(\Delta)$ to be the space of homogeneous splines of degree d and smoothness -1, i.e.

$$S_d^{-1}(\Delta) := \{s : s|_\tau \in \mathcal{H}_d, \forall \tau \in \Delta\}.$$

Then let

$$S_d^r(\Delta) := S_d^{-1}(\Delta) \cap C^r(\mathbb{S}^2).$$

It is shown in [4] that for $d \geq 3r+2$ there is more than one interpolating spline. A typical way to use the extra degrees of freedom is to minimize a functional $\mathcal{E}(s)$ measuring smoothness of s. Let

$$\mathcal{E}_\delta(s) = \int_{\mathbb{S}^2} (\diamondsuit s_\delta)^T (\diamondsuit s_\delta) d\sigma, \tag{5.1}$$

where $\diamondsuit$ is a vector of second order differential operators defined for a trivariate function h by

$$\diamondsuit h = \begin{pmatrix} D^2_{xx}h \\ D^2_{yy}h \\ D^2_{zz}h \\ \sqrt{2}D^2_{xy}h \\ \sqrt{2}D^2_{xz}h \\ \sqrt{2}D^2_{yz}h \end{pmatrix}.$$

In (5.1) s_δ is the unique homogeneous extension of s of degree δ to $\mathbb{R}^3\backslash\{0\}$ defined by $s_\delta = |v|^\delta s(\frac{v}{|v|})$. As we discussed in Chapter 3 we use $\delta = 0$ or $\delta = 1$. After evaluation $\diamondsuit s_\delta$ is restricted to the unit sphere and then integrated.

To establish existence and uniqueness of an interpolating spherical spline in $S^r_d(\Delta)$ which minimizes (5.1), we need the following

LEMMA 5.1. Let Δ be a spherical triangulation and suppose $f \neq 0$. Then

1) $\mathcal{E}_1(f) = 0$ if and only if f is a trivariate homogeneous linear polynomial on $\mathbb{S}^2$,

2) $\mathcal{E}_0(f) = 0$ if and only if f is a constant polynomial.

PROOF. Lemmas 3.2 and 3.10.

Recall that

$$\Gamma(f) := \{s \in S^r_d(\Delta) : s(v) = f(v),\ \forall v \in \mathcal{V}\}$$

is the set of all splines in $S^r_d(\Delta)$ interpolating f at the vertices of triangulation Δ. Let $S_f \in \Gamma(f)$ denote a spherical spline minimizing (5.1) over $\Gamma(f)$.

LEMMA 5.2. There exists a unique spline $s_0 \in S^r_d(\Delta)$ interpolating $f = 0$ and minimizing (5.1) with $\delta = d \bmod(2)$.

PROOF. Since $\mathcal{E}_\delta(s) \geq 0$ for all $s \in S^r_d(\Delta)$, $\mathcal{E}_\delta(s_0) = 0$ is the absolute minimum of $\mathcal{E}_\delta$ at $s_0 = 0$. To show the uniqueness, assume there is another $s \in \Gamma(0)$ with $\mathcal{E}_\delta(s) = 0$. We need to prove that $s = s_0$. By our assumption, $\mathcal{E}_\delta(s) = 0$ on every triangle $\tau \in \Delta$. By Lemma 5.1 s is either a linear homogeneous function (if d is odd) or s is a constant (if d is even) on every triangle $\tau \in \Delta$. Since s interpolates 0 at the vertices of each triangle, $s = 0$ on each triangle. Therefore $s = s_0$.

THEOREM 5.3. Let Δ be a regular triangulation of the unit sphere with vertices $\mathcal{V}$ and $\{f(v),\ v \in \mathcal{V}\}$ be given for some spherical function f. Then for any two positive integers $d,\ r$ with $d \geq 3r + 2$, there exists a unique spline $S_f \in S_d^r(\Delta)$ interpolating the values of f and minimizing $\mathcal{E}_\delta$.

PROOF. Since $d \geq 3r + 2$, $\Gamma(f)$ is not empty (cf. [4]). There exists a minimal energy spline S_f interpolating f since $\Gamma(f)$ is a nonempty closed convex set. To prove uniqueness suppose that there exists another spline $Q_f \in \Gamma(f)$ minimizing $\mathcal{E}_\delta$, i.e. $\mathcal{E}_\delta(S_f) = \mathcal{E}_\delta(Q_f)$. Since $\mathcal{E}_\delta(S_f + \nu s)$ achieves its minimal value at $\nu = 0$ over $s \in \Gamma(0)$, we have

$$\frac{d}{d\nu}\mathcal{E}_\delta(S_f + \nu s)|_{\nu=0} = 0,$$

which leads to

$$\int_{\mathbb{S}^2} (\diamondsuit S_{f,\delta})^T (\diamondsuit s) d\sigma = 0$$

for $s \in \Gamma(0)$. Using $s = S_f - Q_f$ we get

$$\int_{\mathbb{S}^2} (\diamondsuit S_{f,\delta})^T (\diamondsuit S_{f,\delta}) d\sigma = \int_{\mathbb{S}^2} (\diamondsuit S_{f,\delta})^T (\diamondsuit Q_{f,\delta}) d\sigma.$$

Therefore,

$$\mathcal{E}_\delta(S_f - Q_f) = 0,$$

and by Lemma 5.2 $S_f - Q_f \equiv 0$. This completes the proof.

Now we explain how to compute minimal energy interpolating spherical splines. We use a coefficient vector c to represent each spline function in $S_d^r(\Delta)$

$$s|_\tau = \sum_{i+j+k=d} c_{ijk}^\tau B_{ijk}^{d,\tau}, \tau \in \Delta$$

$$c := (c_{ijk}^\tau), i + j + k = d, \tau \in \Delta.$$

To simplify the data management we linearize the triple indices of BB-coefficients c_{ijk} and correspondingly the indices of BB-basis functions B^d_{ijk}. From the properties of SBB-polynomials, we have

$$c_{d00} = f(v_1), c_{0d0} = f(v_2), c_{00d} = f(v_3)$$

on each triangle $\tau \in \Delta$. We can then assemble interpolation conditions into a matrix K, according to the order in which the coefficient vector c is organized. Then $Kc = F$ is the linear system of equations such that a coefficient vector c solving it corresponds to a spline s interpolating f at the data sites $\mathcal{V}$.

To ensure the C^r continuity across each edge of Δ, we impose smoothness conditions, i.e., the conditions in Theorem 2.18, for every edge of Δ. Let M denote the smoothness matrix such that

$$Mc = 0$$

if and only if $s \in S^r_d(\Delta)$.

Next fix $\delta = d \bmod(2)$. The problem of minimizing (5.1) over $S^r_d(\Delta)$ can be formulated as follows:

$$\text{minimize } c^T E c, \text{ subject to } Mc = 0 \text{ and } Kc = F.$$

Here the energy matrix E is defined as follows. $E = \operatorname{diag}(E^\tau, \tau \in \Delta)$ is a diagonally block matrix. Each block E^τ is associated with a triangle τ and contains the following entries

$$E^\tau_{ij} := \int_\tau \diamondsuit(B_i)^T_\delta \diamondsuit(B_j)_\delta d\sigma, \tag{5.2}$$

where B_i denotes a BB-polynomial basis function (2.19) of degree d corresponding to the order of the linearized triple indices $(i, j, k), i + j + k = d$.

By the method of Lagrange multipliers, we need to solve the linear system

$$\begin{bmatrix} E & K^T & M^T \\ K & 0 & 0 \\ M & 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ \eta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}.$$

Here γ and η are vectors of Lagrange multiplier coefficients. Note that E is a singular matrix. We obtain the least squares solution of the linear system by using the iterative method discussed in Chapter 1 and studied in [6]. Lemma 5.2 implies that E is symmetric and nonnegative definite with respect to $L = (K; M)$. Thus, the iterative method converges to the vector c, which is the coefficient vector of the unique interpolating spline minimizing (5.1). This furnishes a computational algorithm.

For interpolating non-homogeneous splines we choose integers d and $r \geq 0$ and define

$$N_d^r(\Delta) = S_d^r(\Delta) \oplus S_{d-1}^r(\Delta).$$

Then

$$N_d^r(\Delta) = \{s : s|_\tau \in \mathcal{P}_d, \forall \tau \in \Delta\} \cap C^r(\mathbb{S}^2).$$

To simplify out notation, we assume that d is an odd integer and write $s = s_1 + s_0$ for a nonhomogeneous spline s. The subscript 1 indicates that s_1 is a spline of odd degree and that we use its extension of degree 1 to compute derivatives. Similarly, the subscript 0 reminds us that s_0 is a spline of even degree, and that we use its constant extension to compute derivatives.

Define a new energy functional which annihilates non-homogeneous linear polynomials as well as constants and homogeneous linear polynomials:

$$\mathcal{E}(s) = \lambda \int_{\mathbb{S}^2} (\diamondsuit s_1)^T (\diamondsuit s_1) d\sigma + (1 - \lambda) \int_{\mathbb{S}^2} (\diamondsuit s_0)^T (\diamondsuit s_0) d\sigma, \tag{5.3}$$

with $0 < \lambda < 1$.

Lemma 5.4. Choose degree d and smoothness r for a spline space $N_d^r(\Delta)$ as above. Given a spherical function f let

$$\widetilde{\Gamma}(f) := \{s \in N_d^r(\Delta) : s(v) = f(v), \ \forall v \in \mathcal{V}\}$$

be the set of all splines in $N_d^r(\Delta)$ interpolating f at the vertices of triangulation Δ. Then there exists a unique spline $s = 0 \in N_d^r(\Delta)$ interpolating $f = 0$ and minimizing (5.3).

PROOF. Suppose that $s = 0$. By the definition $s = s_1 + s_0$, where $s_1|_\tau$ is a homogeneous polynomial of degree d and $s_0|_\tau$ is a homogeneous polynomial of degree $d-1$. Since $S_d^r(\Delta) \cap S_{d-1}^r(\Delta) = 0$, $s_1 = 0$ and $s_0 = 0$. By the definition (5.3), $\mathcal{E}(0) = 0$. Since $\mathcal{E}(s) \geq 0$ for all $s \in N_d^r(\Delta)$, $\mathcal{E}(0) = 0$ is the absolute minimum of $\mathcal{E}$. To show the uniqueness, assume there is $q \in \Gamma(0)$ with $\mathcal{E}(q) = 0$. We need to show that $q = s$. As above we know $q = q_1 + q_0$ for some $q_1 \in S_d^r(\Delta)$ and $q_0 \in S_{d-1}^r(\Delta)$. Then $\mathcal{E}_1(q_1) = 0$ and $\mathcal{E}_0(q_0) = 0$ on every triangle $\tau \in \Delta$. By Lemma 5.1 q_1 is a linear homogeneous function and q_0 is a constant on $\mathbb{R}^3$. Therefore $q_1 + q_0$ is a trivariate linear function satisfying zero interpolation conditions over points $v \in \mathcal{V}$ none of which is the origin. By the linear independence of x, y, z and 1, $q = 0$ on every triangle. Therefore $q = s$.

With the above lemma, we are ready to prove

THEOREM 5.5. Let Δ be a regular triangulation of the unit sphere with vertices $\mathcal{V}$. Let $\{f(v),\ v \in \mathcal{V}\}$ be the given set of data values. Then for any integers $d \geq 1, r \geq 0$ such that with $d \geq 3r + 2$, there exists a unique spline $S_f \in N_d^r(\Delta)$ interpolating values f and minimizing $\mathcal{E}$.

PROOF. First, $\tilde{\Gamma}(f)$ is not empty since $d \geq 3r + 2$ (cf. [4]). Then there exists a minimal energy interpolating spline $S_f \in \tilde{\Gamma}(f)$ since $\tilde{\Gamma}(f)$ is a nonempty convex set. The uniqueness of S_f follows from Lemma 5.4 as in the proof of Theorem 5.3.

To set the linear system for interpolation problem over $N_d^r(\Delta)$, we can do the following. Consider $s = s_1 + s_0$ with splines s_1 and s_0 of degrees d and $d-1$, respectively. Order the coefficients of the splines over each triangle τ as above and denote them by c_1^τ and c_0^τ. Let

$$\mathbf{c} = (c_1, c_0)$$

with $c_1 := (c_1^\tau, \tau \in \Delta)$ and $c_0 := (c_0^\tau, \tau \in \Delta)$. Then we can denote interpolation, smoothness and energy matrices by $K_1,\ K_0,\ M_1,\ M_0,\ E_1,\ E_0$ accordingly. We obtain the following interpolation conditions for s

$$\mathbf{Kc} := \begin{bmatrix} K_1 & K_0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_0 \end{bmatrix} = F.$$

For s to be smooth we require both s_1 and s_0 to be smooth. Thus the requirement of C^r smoothness is expressed in the form of the linear system

$$\mathbf{Mc} := \begin{bmatrix} M_1 & 0 \\ 0 & M_0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_0 \end{bmatrix} = 0.$$

With the definition of (5.3) it is clear that the energy matrix in this case can be defined by

$$\mathbf{E} = \begin{bmatrix} \lambda E_1 & 0 \\ 0 & (1-\lambda)E_0 \end{bmatrix}.$$

Therefore $s \in S_d^r(\Delta) \oplus S_{d-1}^r(\Delta)$ minimizes (5.3), interpolates f at vertices of Δ, and is C^r continuous if and only if the vector $\mathbf{c}$ of its coefficients satisfies the linear system

$$\begin{bmatrix} \mathbf{E} & \mathbf{K}^T & \mathbf{M}^T \\ \mathbf{K} & 0 & 0 \\ \mathbf{M} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \gamma \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}.$$

The linear system is again singular. However the application of the iterative scheme allows us to successfully obtain the least squares solution since

THEOREM 5.6. $\mathbf{E}$ is symmetric and positive definite with respect to $\begin{bmatrix} \mathbf{K} \\ \mathbf{M} \end{bmatrix}$.

PROOF. Since $\mathcal{E}(s) = \mathbf{c}^T\mathbf{E}\mathbf{c} \geq 0$, $\mathbf{c}^T\mathbf{E}\mathbf{c} = 0$ implies that s is a linear polynomial. Zero side conditions force $s = 0$. By the linear independence of basis functions $\mathbf{c} = 0$.

5.2 Discrete Least Squares Fitting

When the given data set is extremely large, e.g., $n \geq 10,000$ and highly redundant, we find a discrete least squares fitting to the given data instead of computing an interpolating spherical spline.

Let $\mathcal{V} = \{v_\ell, \ell = 1, \cdots, n\}$ be the given data sites over the unit sphere and let Δ be a triangulation of $\mathbb{S}^2$ whose vertices may not relate to the data sites. For given degree d, we have to assume that the data sites are rich enough in the following sense.

Definition 5.1. The data sites $v_\ell, \ell = 1, \cdots, n$ are said to be evenly distributed over the triangulation Δ with respect to d if the matrix

$$\left[B^{d,\tau}_{ijk}(v_\ell)\right]_{i+j+k=d, v_\ell \in \tau}$$

is of full rank for each $\tau \in \Delta$.

Suppose that the given data values are from a function f, i.e., $f(v_\ell), \ell = 1, \cdots, n$ are given. Let $\mathcal{L}(s)$ be the least squares functional

$$\mathcal{L}(s) = \sum_{\ell=1}^{n} (s(v_\ell) - f(v_\ell))^2. \tag{5.4}$$

The discrete least squares spherical spline $S_f \in S^r_d(\Delta)$ is the function in $S^r_d(\Delta)$ which minimizes the quantity $\mathcal{L}(s), s \in S^r_d(\Delta)$.

Theorem 5.7 . Suppose that the given data sites $v_\ell, \ell = 1, \cdots, n$ are evenly distributed over Δ with respect to d. Then there exists a unique spline $S_f \in S^r_d(\Delta)$ of degree d and smoothness r approximating given data values $f_\ell = f(v_\ell), \ell = 1, \cdots, n$ in the sense that it minimizing discrete least squares functional (5.4).

Proof. Recall that any $s \in S^r_d(\Delta)$ can be written as

$$s(v)|_\tau = \sum_{i+j+k=d} c^\tau_{ijk} B^\tau_{ijk}(v),$$

on a spherical triangle $\tau \in \Delta$. Let $c = (c^{\tau}_{ijk}, i+j+k=d, \tau \in \Delta)$ be the coefficient vector of s. Note that

$$L(c) := \mathcal{L}(s) = \sum_{\ell=1}^{n} |s(v_\ell) - f_\ell|^2 = \sum_{\tau\in\Delta}\sum_{v_\ell\in\tau}(\sum_{i+j+k=d} c^{\tau}_{ijk}B^{\tau}_{ijk}(v_\ell) - f_\ell)^2$$

is a function of c and that $\mathcal{L}(0) = \|f\|_2^2$, with $f = (f_\ell, \ell = 1, \cdots, n)$ being the data value vector, and $\|f\|_2 := (\sum_{\ell=1}^{n} |f_\ell|^2)^{1/2}$ denoting the standard ℓ_2 norm of the vector f. Consider

$$A = \{c,\ L(c) \le \|f\|_2^2\}.$$

Let us show that A is a bounded and closed set.

Fix any triangle $\tau \in \Delta$. For any $c \in A$ we have

$$|\sum_{i+j+k=d} c^{\tau}_{ijk}B^{\tau}_{ijk}(v_\ell) - f_\ell|^2 \le \|f\|_2^2, \forall v_\ell \in \tau.$$

It follows that

$$|\sum_{i+j+k=d} c^{\tau}_{ijk}B^{\tau}_{ijk}(v_\ell)| \le 2\|f\|_2, \forall v_\ell \in \tau.$$

Since the data sites are evenly distributed, the matrix

$$\left[B^{\tau}_{ijk}(v_\ell)\right]_{i+j+k=d, v_\ell\in\tau}$$

is of full rank and hence, there exists an index set $I_\tau \subset \{1, 2, \cdots, n\}$ such that the square matrix

$$B^{\tau} = [B_{ijk}(v_\ell)]_{i+j+k=d, \ell\in I_\tau}$$

is invertible. Therefore

$$\|(c^{\tau}_{ijk, i+j+k=d})\|_2 \le C_\tau,$$

with C_τ being a positive constant depending only on $\|f\|_2$ and the norm of the inverse matrix of B^{τ}. Hence $\|c\|_2$ is bounded above and A is bounded. It is easy to see that A is closed and that $A_s := \{c : Mc = 0\}$ is also closed where $Mc = 0$ are the linear system representing the smoothness conditions for $S_d^r(\Delta)$. Hence, the set $A \cap A_s$ is compact.

It is clear that $\mathcal{L}(s)$ is a continuous function of c. Therefore, there exists a $c_f \in A \cap A_s$ minimizing $\mathcal{L}(s)$.

To show the uniqueness of the solution c_f, we note that $\mathcal{L}(s)$ is a convex function and assume that there two solutions c_f and $\widehat{c}_f$. Then convexity of $\mathcal{L}(s)$ implies that for any $0 \leq \nu \leq 1$ a convex combination $c_f + \nu(\widehat{c}_f - c_f)$ also minimizes $\mathcal{L}(s)$. Thus

$$\begin{aligned}
1/2 \quad & \frac{d}{d\nu}\mathcal{L}(c_f + \nu(\widehat{c}_f - c_f)) \\
= \quad & \sum_{\tau\in\Delta}\sum_{v_\ell\in\tau}(\sum_{i+j+k=d}(c^\tau_{ijk} + \nu(\widehat{c}^\tau_{ijk} - c^\tau_{ijk}))B^\tau_{ijk}(v_\ell) - f_\ell)(\widehat{c}^\tau_{ijk} - c^\tau_{ijk})B^\tau_{ijk}(v_\ell) \\
= \quad & \nu\sum_{\tau\in\Delta}\sum_{v_\ell\in\tau}(\sum_{i+j+k=d}(\widehat{c}^\tau_{ijk} - c^\tau_{ijk})^2 B^\tau_{ijk}(v_\ell)^2 \\
& +\sum_{\tau\in\Delta}\sum_{v_\ell\in\tau}(\sum_{i+j+k=d}c^\tau_{ijk}(\widehat{c}^\tau_{ijk} - c^\tau_{ijk})B^\tau_{ijk}(v_\ell)^2 \\
& -\sum_{\tau\in\Delta}\sum_{v_\ell\in\tau}(\sum_{i+j+k=d}f_\ell(\widehat{c}^\tau_{ijk} - c^\tau_{ijk})B^\tau_{ijk}(v_\ell) = 0
\end{aligned}$$

for any $0 \leq \nu \leq 1$. Note that

$$\begin{aligned}
& \sum_{\tau\in\Delta}\sum_{v_\ell\in\tau}(\sum_{i+j+k=d}c^\tau_{ijk}(\widehat{c}^\tau_{ijk} - c^\tau_{ijk})B^\tau_{ijk}(v_\ell)^2 \\
- \quad & \sum_{\tau\in\Delta}\sum_{v_\ell\in\tau}(\sum_{i+j+k=d}f_\ell(\widehat{c}^\tau_{ijk} - c^\tau_{ijk})B^\tau_{ijk}(v_\ell) = 0
\end{aligned}$$

at $\nu = 0$. Sice it is independent of ν it must be 0 for all ν. It follows that

$$\sum_{\tau\in\Delta}\sum_{v_\ell\in\tau}(\sum_{i+j+k=d}(\widehat{c}^\tau_{ijk} - c^\tau_{ijk})^2 B^\tau_{ijk}(v_\ell))^2 = 0.$$

Since the data sites are evenly distributed over each $\tau \in \Delta$, $c_f = \widehat{c}_f$. This completes the proof.

We first explain a computational algorithm for the application of the discrete least squares method in $S_d^r(\Delta)$. The Lagrange multipliers method implies the following linear system:

$$\begin{bmatrix} L^T L & M^T \\ M & 0 \end{bmatrix} \begin{bmatrix} c \\ \eta \end{bmatrix} = \begin{bmatrix} L^T F \\ 0 \end{bmatrix},$$

where L is the observation matrix with entries $L_{ij} = B_j(v_i)$, $i = 1, \cdots, n$ and j runs from 1 to $T(d+1)(d+2)/2$ where T denotes the number of triangles in Δ. Here, F is a vector of function values ordered as the spherical locations v_ℓ, $\ell = 1, \cdots, n$, M is smoothness matrix and η is a vector of Lagrange multipliers. The solution of this system is a vector c of coefficients of a homogeneous spline s of degree d and smoothness r defined with respect to the spherical triangulation Δ minimizing (5.4).

For non-homogeneous spherical splines, the discrete least squares approximation problem can be treated similarly. We seek a function $s = s_1 + s_0 \in N_d^r(\Delta)$ minimizing $\mathcal{L}(s)$ (5.4).

Note that the Definition 5.1 applied to the non-homogeneous case has to take into account that the basis functions in $N_d^r(\Delta)$ consist of homogeneous BB-basis polynomials of degrees d and $d-1$.

Definition 5.2. The given data sites $v_\ell, \ell = 1, \cdots, n$ are said to be evenly distributed over the triangulation Δ with respect to d if the matrix

$$\mathbf{L}^T := [B^{d,\tau}_{i^1 j^1 k^1}(v_\ell)\ B^{d-1,\tau}_{i^2 j^2 k^2}(v_\ell)]_{i^1+j^1+k^1=d, i^2+j^2+k^2=d-1, v_\ell \in \tau},$$

is of full rank for every $\tau \in \Delta$.

Theorem 5.8. Suppose that the given data locations $v_\ell, \ell = 1, \cdots, n$ are evenly distributed with respect to d. Then there exists a unique spline $S_f \in N_d^r(\Delta)$ of degree d and smoothness r approximating given data values $f_\ell, \ell = 1, \cdots, n$ and minimizing discrete least squares functional (5.4).

Proof. Similar to that of Theorem 5.7.

To find the discrete least squares spline in $N_d^r(\Delta)$ we construct the observation matrix

$$\mathbf{L} = \begin{bmatrix} L_1 & L_0 \end{bmatrix},$$

and assemble smoothness conditions $\mathbf{M}\mathbf{c} = 0$ with

$$\mathbf{M} = \begin{bmatrix} M_1 & 0 \\ 0 & M_0 \end{bmatrix}$$

as in the previous section. Here L_1 and L_0 are the observation matrices containing values of BB-basis polynomials at the data sites for the spaces $S_d^{-1}(\Delta)$ and $S_{d-1}^{-1}(\Delta)$. We therefore need to solve the linear system

$$\begin{bmatrix} \mathbf{L}^T\mathbf{L} & \mathbf{M}^T \\ \mathbf{M} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \eta \end{bmatrix} = \begin{bmatrix} \mathbf{L}^T F \\ 0 \end{bmatrix},$$

where F contains given data values and η is a vector of Lagrange multiplier coefficients.

THEOREM 5.9. The matrix $\mathbf{L}^T\mathbf{L}$ is positive definite with respect to $\mathbf{M}\mathbf{c} = 0$.

PROOF. It is clear that $\mathbf{c}^T\mathbf{L}^T\mathbf{L}\mathbf{c} \geq 0$ for all $\mathbf{c}$. Suppose $\mathbf{c}^T\mathbf{L}^T\mathbf{L}\mathbf{c} = 0$ for some $\mathbf{c}$. Then $\mathbf{L}\mathbf{c} = 0$. Since $\mathbf{L}$ is of full rank $\mathbf{c} = \mathbf{0}$.

5.3 Penalized Least Squares Approximation

Again we let $\mathcal{V} := \{v_\ell, \ell = 1, \cdots, n\}$ be a set of sites on the unit sphere and $\{f_\ell, \ell = 1, \cdots, n\}$ be the corresponding values for some function f. We need to find a smooth surface resembling f. Another commonly used method in this situation is the penalized least squares fit.

As in the previous section, let Δ be a regular triangulation of the unit sphere $\mathbb{S}^2$ whose vertices may not relate to the data sites in $\mathcal{V}$. Consider the spline space $S_d^r(\Delta)$ of degree d and smoothness r with $r < d$. We look for a spline solution $S_f \in S_d^r(\Delta)$ satisfying

$$\mathcal{P}_\lambda(S_f) = \min\{\mathcal{P}_\lambda(s) : s \in S_d^r(\Delta)\}, \tag{5.5}$$

where λ is a positive weight and

$$\mathcal{P}_\lambda(s) := \mathcal{L}(s) + \lambda\mathcal{E}_\delta(s). \tag{5.6}$$

Here the least squares functional and the energy functional are defined in (5.4) and (5.3), respectively. It is clear that for large $\lambda \gg 1$, S_f is close to minimal energy splines, and for small $\lambda \ll 1$ the solution S_f is close to the discrete least squares spline. One way to choose λ is the cross validation method (cf. [24]). Our recommendation is to choose a small value for λ to get a good approximation, like that of the discrete least squares fitting.

Let us first prove existence and uniqueness of the penalized least squares solution in a homogeneous spherical spline space $S_d^r(\Delta)$.

THEOREM 5.10. Fix $\lambda > 0$. Suppose all vertices $\mathcal{W}$ of Δ are part of the data sites $\mathcal{V}$ and $|\Delta| \leq 1$. There exists a unique spline $S_f \in S_d^r(\Delta)$ minimizing (5.6).

PROOF. Recall that any $s \in S_d^r(\Delta)$ can be written as

$$s(v)|_\tau = \sum_{i+j+k=d} c_{ijk}^\tau B_{ijk}^\tau(v)$$

on a spherical triangle $\tau \in \Delta$. Let $c = (c_{ijk}^\tau, i+j+k = d, \tau \in \Delta)$ be the coefficient vector of s. Recall that the energy functional $\mathcal{E}(s)$ can be expressed in terms of c as

$$\mathcal{E}(s) = c^T E c$$

with the entries of E defined in (5.2). The discrete least squares functional $\mathcal{L}(s)$ is expressed as

$$\begin{aligned}\mathcal{L}(s) &= \sum_{\ell=1}^{n} |s(v_\ell) - f_\ell|^2 = \sum_{\tau\in\Delta}\sum_{v_\ell\in\tau}\Big(\sum_{i+j+k=d} c_{ijk}^\tau B_{ijk}^\tau(v_\ell) - f_\ell\Big)^2 \\ &= c^T L^T L c - 2f^T L c + \|f\|_2^2,\end{aligned}$$

with $f = (f_\ell, \ell = 1, \cdots, n)$ being a data value vector. Thus

$$\mathcal{P}_\lambda(s) = \lambda c^T E c + c^T L^T L c - 2f^T L c + \|f\|_2^2.$$

Note that $\mathcal{P}_\lambda(0) = \|f\|_2^2$. Consider

$$A = \{c, \mathcal{P}_\lambda(s) \leq \|f\|_2^2\}.$$

Let us show that A is a bounded and closed set so that the continuous function $\mathcal{P}_\lambda(s)$ must have a minimum in A.

Fix $c \in A$ and let s be the corresponding spline. Then $\mathcal{P}_\lambda(s) \leq \|\mathbf{f}\|_2^2$. By the definition of $\mathcal{P}_\lambda$ we must have $\lambda\mathcal{E}_\delta(s) \leq \|\mathbf{f}\|_2^2$. By Lemmas 3.1, 3.9 and 2.53 the energy of a spline is equivalent to the square of its second order supremum Sobolev seminorm on every triangle of Δ, i.e. we have

$$|s|_{2,\infty,\tau} \leq \frac{J_1}{\sqrt{\lambda}}\|\mathbf{f}\|_2$$

and

$$|s|'_{2,\infty,\tau} \leq \frac{J_2}{\sqrt{\lambda}}\|\mathbf{f}\|_2$$

with J_1, J_2 depending on degree d of the spline space and the smallest angle in τ. Let r_τ denote the center of the smallest spherical cap containing τ. Let $\mathbf{T}_\tau$ be a plane tangent to τ at r_τ. Define $\bar{\tau}$ in this plane as a set of points $\{w : \frac{w}{|w|} \in \tau\}$. Define $s_\delta(w) = |w|^\delta s(\frac{w}{|w|})$ to be a homogeneous extension of s of degree δ and $\bar{s}_\delta$ to be its restriction to $\bar{\tau}$. Similarly define f_δ and $\bar{f}_\delta$. By Proposition 2.26

$$|\bar{s}_1|_{2,\infty,\bar{\tau}} \leq J_3|s|_{2,\infty,\tau},$$

and

$$|\bar{s}_0|_{2,\infty,\bar{\tau}} \leq J_4|s|'_{2,\infty,\tau}.$$

Therefore

$$|\bar{s}_\delta|_{2,\infty,\bar{\tau}} \leq \frac{J_5}{\sqrt{\lambda}}\|\mathbf{f}\|_2.$$

Since the vertices, say v_1, v_2, v_3, of τ belong to $\mathcal{W}$

$$\begin{aligned} |\bar{s}_\delta(\bar{v}_i)| &\leq |\bar{s}_\delta(\bar{v}_i) - \bar{f}_\delta(\bar{v}_i)| + |\bar{f}_\delta(\bar{v}_i)| \\ &\leq \max\{|v_\ell| : v_\ell \in \tau\}((\sum_{v_\ell \in \tau} |s(v_\ell) - f(v_\ell)|^2)^{1/2} + \|\mathbf{f}\|_2) \\ &\leq J_6((\mathcal{P}_\lambda(s))^{1/2} + \|\mathbf{f}\|_2) \leq 2J_6\|\mathbf{f}\|_2, \end{aligned}$$

for $J_6 = \max\{|v_\ell| : v_\ell \in \tau\}$ and $i = 1, 2, 3$. For any point $\bar{v}$ in $\bar{\tau}$ we use Taylor expansion to get

$$\bar{s}_\delta(\bar{v}_1) = \bar{s}_\delta(\bar{v}) + \nabla\bar{s}_\delta(\bar{v}) \cdot (\bar{v}_1 - \bar{v}) + O(|\bar{s}_\delta|_{2,\infty,\bar{\tau}}|\bar{\tau}|^2)$$

where $|\bar{\tau}|$ denotes the size of $\bar{\tau}$. Using similar expressions for $\bar{v}_2$ and $\bar{v}_3$ we get

$$\bar{s}_\delta(\bar{v}_1) - \bar{s}_\delta(\bar{v}_2) = \nabla\bar{s}_\delta(\bar{v}) \cdot (\bar{v}_1 - \bar{v}_2) + O(|\bar{s}_\delta|_{2,\infty,\bar{\tau}}|\bar{\tau}|^2),$$
$$\bar{s}_\delta(\bar{v}_2) - \bar{s}_\delta(\bar{v}_3) = \nabla\bar{s}_\delta(\bar{v}) \cdot (\bar{v}_2 - \bar{v}_3) + O(|\bar{s}_\delta|_{2,\infty,\bar{\tau}}|\bar{\tau}|^2).$$

Solving this linear system for $\nabla\bar{s}_\delta$, we get

$$D_x\bar{s}_\delta(\bar{v}) = O(|\bar{\tau}|^3|\bar{s}_\delta|_{2,\infty,\bar{\tau}}/A_{\bar{\tau}}) + |\bar{s}_\delta(\bar{v}_1)| + |\bar{s}_\delta(\bar{v}_2)||\bar{\tau}|/A_{\bar{\tau}},$$
$$D_y\bar{s}_\delta(\bar{v}) = O(|\bar{\tau}|^3|\bar{s}_\delta|_{2,\infty,\bar{\tau}}/A_{\bar{\tau}}) + |\bar{s}_\delta(\bar{v}_1)| + |\bar{s}_\delta(\bar{v}_2)||\bar{\tau}|/A_{\bar{\tau}},$$

where $A_{\bar{\tau}}$ denotes the area of $\bar{\tau}$. Using these estimates for $\nabla\bar{s}_\delta$ we get

$$|\bar{s}_\delta(\bar{v})| \le J_7((1 + |\bar{\tau}| + |\bar{\tau}|^2/A_{\bar{\tau}})\|\mathbf{f}\|_2 + |\bar{\tau}|^4|\bar{s}_\delta|_{2,\infty,\bar{\tau}}/A_{\bar{\tau}}).$$

Hence we have

$$|\bar{s}_\delta(\bar{v})| \le J_8\|\mathbf{f}\|_2$$

for J_8 depending on τ. By the definition

$$|s(v)| = |\bar{v}|^{-\delta}|\bar{s}_\delta(\bar{v})| \le J_9\|\mathbf{f}\|_2$$

is bounded since $|\tau|$ is bounded. By the stability of BB basis c is bounded, and A is a bounded set. Since A is closed, it is compact. By the definition $\mathcal{P}_\lambda(s)$ is a continuous function of c. Hence $\mathcal{P}_\lambda$ attains its minimum over A.

To show uniqueness of the minimizer S_f suppose there exists $\hat{S}_f$ with $\mathcal{P}_\lambda(S_f) = \mathcal{P}_\lambda(\hat{S}_f)$. Since $\mathcal{P}_\lambda$ is a convex functional for any $0 \le \nu \le 1$

$$\mathcal{P}_\lambda(\nu S_f + (1-\nu)\hat{S}_f) \le \nu\mathcal{P}_\lambda(S_f) + (1-\nu)\mathcal{P}_\lambda(\hat{S}_f) = \mathcal{P}_\lambda(S_f).$$

On the other hand, since $\mathcal{P}_\lambda$ achieves minimum value at S_f

$$\mathcal{P}_\lambda(S_f) \le \mathcal{P}_\lambda(\nu S_f + (1-\nu)\hat{S}_f).$$

Therefore $\mathcal{P}_\lambda(\hat{S}_f + \nu(S_f - \hat{S}_f))$ is a constant function of ν on $[0,1]$. It follows that $\frac{d}{d\nu}\mathcal{P}_\lambda(\hat{S}_f + \nu(S_f - \hat{S}_f)) = 0$ for all $0 \le \nu \le 1$, i.e.

$$\begin{aligned} 0 &= \frac{d}{d\nu}\mathcal{P}_\lambda(\hat{S}_f + \nu(S_f - \hat{S}_f)) \\ &= 2\lambda(\hat{c}_f + \nu(c_f - \hat{c}_f))^T E(c_f - \hat{c}_f) \\ &+ 2(\hat{c}_f + \nu(c_f - \hat{c}_f))^T L^T L(c_f - \hat{c}_f) - 2\mathbf{f}^T L(c_f - \hat{c}_f). \end{aligned}$$

Note that at $\nu = 0$ we get

$$0 = 2\lambda(\hat{c}_f)^T E(c_f - \hat{c}_f) + 2(\hat{c}_f)^T L^T L(c_f - \hat{c}_f) - 2\mathbf{f}^T L(c_f - \hat{c}_f).$$

Therefore

$$0 = 2\lambda\nu(c_f - \hat{c}_f)^T E(c_f - \hat{c}_f) + 2\nu(c_f - \hat{c}_f)^T L^T L(c_f - \hat{c}_f).$$

Hence, we must have $(c_f - \hat{c}_f)^T E(c_f - \hat{c}_f) = 0$ and $(c_f - \hat{c}_f)^T L^T L(c_f - \hat{c}_f) = 0$ since both E and $L^T L$ are nonnegative definite. Then $\mathcal{E}(S_f - \hat{S}_f) = 0$ and therefore $S_f - \hat{S}_f$ is a linear homogeneous polynomial, and $S_f(v_\ell) - \hat{S}_f(v_\ell) = 0$ at every vertex v_ℓ of Δ. Therefore $S_f = \hat{S}_f$.

To solve the penalized least squares problem using non-homogeneous splines we work in $N_d^r(\Delta) = S_d^r(\Delta) \oplus S_{d-1}^r(\Delta)$. We have to replace the energy functional (5.1) in (5.6) by (5.3). For a spherical spline function $s = s_1 + s_0$ define

$$\mathcal{P}_\lambda(s) = \mathcal{L}(s_1 + s_0) + \lambda_1 \int_{\mathbb{S}^2} (\Diamond s_1)^T (\Diamond s_1) d\sigma + \lambda_2 \int_{\mathbb{S}^2} (\Diamond s_0)^T (\Diamond s_0) d\sigma, \qquad (5.7)$$

with $\lambda_1 > 0, \lambda_2 > 0$.

THEOREM 5.11. Fix $\lambda_1 > 0$ and $\lambda_2 > 0$. Suppose all vertices $\mathcal{W}$ of Δ are part of the data sites $\mathcal{V}$ and $|\Delta| \le 1$. There exists a unique spline $S_f \in N_d^r(\Delta)$ minimizing (5.7).

PROOF. The proof is similar to the proof of Theorem 5.8.

We first consider the penalized least squares linear system in $S_d^r(\Delta)$. By the method of Lagrange multipliers, minimization of (5.6) over $S_d^{-1}(\Delta)$ subject to the smoothness C^r conditions in the matrix form $Mc = 0$ results in a system of linear equations

$$\begin{bmatrix} P & M^T \\ M & 0 \end{bmatrix} \begin{bmatrix} c \\ \eta \end{bmatrix} = \begin{bmatrix} L^T F \\ 0 \end{bmatrix}.$$

Here $P = L^T L + \lambda E$ and F is a vector of function values ordered as the spherical points $v_\ell,\ \ell = 1, \cdots, n$, M is the smoothness matrix and η is a vector of Lagrange multiplier coefficients. The solution of this system is a vector c of coefficients of the homogeneous spline s of degree d and smoothness r defined over the spherical triangulation Δ which minimizes (5.6). Note that the linear system has the same form as the one in Sections 5.1 and 5.2. We use the iterative scheme to compute an approximation of c.

To find the minimal penalized least squares spline in $N_d^r(\Delta)$ we construct the observation matrix

$$\mathbf{L} = \begin{bmatrix} L_1 & L_0 \end{bmatrix},$$

and assemble smoothness conditions $\mathbf{Mc} = 0$ with

$$\mathbf{M} = \begin{bmatrix} M_1 & 0 \\ 0 & M_0 \end{bmatrix}$$

as in the setting of discrete least squares splines. Here L_1 and L_0 are the observation matrices containing values of BB-basis polynomials at the data sites for the spaces $S_d^{-1}(\Delta)$ and $S_{d-1}^{-1}(\Delta)$, respectively. We construct the energy matrices E_1 and E_0 and solve the linear system

$$\begin{bmatrix} \mathbf{P} & \mathbf{M}^T \\ \mathbf{M} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \eta \end{bmatrix} = \begin{bmatrix} \mathbf{L}^T F \\ 0 \end{bmatrix},$$

where $\mathbf{P}$ in this case is

$$\mathbf{L}^T\mathbf{L} + \begin{bmatrix} \lambda_1 E_1 & 0 \\ 0 & \lambda_2 E_0 \end{bmatrix}.$$

THEOREM 5.12. The matrix $\mathbf{P}$ is positive definite with respect to $\mathbf{Mc} = 0$.

PROOF. The proof is a combination of Theorem 5.3 and Theorem 5.6. We omit the details.

5.4 Multiple star technique

Given $\mathcal{V} := \{v \in \mathbb{S}^2\}$ a set of points on the unit sphere with numbers $\{f(v),\ v \in \mathcal{V}\}$, construct a regular triangulation Δ of $\mathbb{S}^2$. For $d \geq 1$ and $r \geq 0$, two integers with $3r + 2 \leq d$, define $S_d^{-1}(\Delta)$ be the space of homogeneous splines of degree d and smoothness -1, i.e.

$$S_d^{-1}(\Delta) := \{s : s|_\tau \in \mathcal{H}_d, \forall \tau \in \Delta\}.$$

Let a sub-domain of $\mathbb{S}^2$ be a single triangle $\tau \in \Delta$. Fix $k \geq 1$ and let $q = 2(k+1)\ell+1$. Here ℓ is the parameter reflecting local support of the basis functions B_ξ discussed in Section 2.8. Recall that $\mathrm{star}^q(\tau)$ is an enlarged sub-domain defined as

$$\mathrm{star}^q(\tau) := \cup\{T \in \Delta, T \cap \mathrm{star}^{q-1}(\tau) \neq \emptyset\} \tag{5.8}$$

and $\mathrm{star}^0(\tau) = \tau$.

To solve the scattered data interpolation problem over each $\mathrm{star}^q(\tau)$, $\tau \in \Delta$, we consider the space $S_d^r(\mathrm{star}^q(\tau)) = S_d^{-1}(\mathrm{star}^q(\tau)) \cap C^r(\mathrm{star}^q(\tau))$. Suppose $E,\ K,\ M$ are the matrices as in Section 5.1 expressing energy, smoothness and interpolation conditions. Let $\mathcal{V}_\tau$ be the subset of $\mathcal{V}$ contained in $\mathrm{star}^q(\tau)$, ϵ_τ the subset of interior edges and Δ_τ triangles of $\mathrm{star}^q(\tau)$. Then the energy matrix for the sub-domain interpolating problem E_τ consists of blocks corresponding to triangles in Δ_τ, i.e. $E_\tau = E(\Delta_\tau)$. The smoothness matrix M_τ for the sub-domain picks up the rows of M corresponding to ϵ_τ and columns corresponding to Δ_τ. Finally the matrix K_τ of the interpolation conditions for the sub-domain consists of rows of K corresponding to $\mathcal{V}_\tau$ and columns corresponding to Δ_τ. Therefore the minimal energy interpolating spline $S_{f,\tau,q}$ over $\text{star}^q(\tau)$ has a coefficient vector c solving the linear system

$$\begin{bmatrix} E_\tau & K_\tau^T & M_\tau^T \\ K_\tau & 0 & 0 \\ M_\tau & 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ \eta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ F_\tau \\ 0 \end{bmatrix}.$$

Here F_τ is a vector consisting of elements of F corresponding to $\mathcal{V}_\tau$ to satisfy the interpolation conditions, η and γ are vectors of Lagrange multiplier coefficients.

To establish existence and uniqueness of a spherical spline in $S_d^r(\text{star}^q(\tau))$ which minimizes

$$\mathcal{E}_{\delta,\tau,q}(s) := \sum_{T \in \text{star}^q(\tau)} \sum_{|\alpha|=2} \|D^\alpha s_\delta\|_{2,T}^2 \tag{5.9}$$

we need the following

LEMMA 5.13. Let Δ be a spherical triangulation and suppose $f \neq 0$. Then

1) $\mathcal{E}_{1,\tau,q}(f) = 0$ if and only if f is a trivariate homogeneous linear polynomial on $\text{star}^q(\tau)$,

2) $\mathcal{E}_{0,\tau,q}(f) = 0$ if and only if f is a constant polynomial.

PROOF. Proof is similar to that of Lemma 5.1.

We also have a result analogous to Lemma 5.2.

LEMMA 5.14. There exists a unique spline $s_0 \in S_d^r(\text{star}^q(\tau))$ interpolating $f = 0$ and minimizing (5.9) with $\delta = d \bmod(2)$.

PROOF. Similar to the proof of Lemma 5.2.

We can therefore conclude by

THEOREM 5.15. Let Δ be a regular triangulation of the unit sphere with vertices $\mathcal{V}$ and $\{f(v),\ v \in \mathcal{V}\}$ be given for some spherical function f. Then for any positive integers $d,\ r,\ k$ with $d \geq 3r+2$, $k \geq 1$, and a triangle $\tau \in \Delta$ there exists a unique spline $S_{f,\tau,q} \in S_d^r(\mathrm{star}^q(\tau))$ interpolating the values of f and minimizing $\mathcal{E}_{\delta,\tau,q}$. Here $q = 2(k+1)\ell + 1$.

Proof. As in Theorem 5.3.

Even though we obtain spline coefficient vector c for a cluster of triangles surrounding τ to assemble our final spline solution we only use the part of this vector corresponding to τ itself. The solution therefore is not smooth, it is not even continuous. The smoothness conditions in case of reasonably high k are usually satisfied with 10^{-4} accuracy. These discontinuities across edges of Δ are not visible to the human eye and might be acceptable.

Chapter 6

Numerical Investigation

6.1 Numerical Experiments for Minimal Energy interpolation

Example 1. The following table contains comparison of the results on the minimal energy spline interpolation in $S_3^1(\Delta)$, $S_4^1(\Delta)$ and $S_3^1(\Delta) \oplus S_4^1(\Delta)$. We interpolate 1, $x+z$ and $z+1$ on 6 points corresponding to the unit directions and their antipodes. Let Δ_1 denote the corresponding triangulation.

$S_d^r(\Delta_1) \ \setminus \ f$	1	$x+z$	$z+1$
$S_3^1(\Delta_1)$	$4.2265e-01$	$1.1016e-15$	$2.1144e-01$
$S_4^1(\Delta_1)$	$4.6629e-15$	$2.5398e-01$	$0.9114e-01$
$N_4^1(\Delta_1)$	$0.6439e-14$	$1.4950e-15$	$1.5551e-15$

Table 6.1: Linear and constant polynomial reproduction on 8 triangles.

We evaluate the splines at 5120 points w almost evenly spaced over $\mathbb{S}^2$ and list the relative errors $\frac{\|s(w)-f(w)\|_\infty}{\|f(w)\|_\infty}$ in Table 6.1.

Not only linear and constant homogeneous polynomials are reproduced in $S_3^1(\Delta) \oplus S_4^1(\Delta)$, a non-homogeneous polynomial $z+1$ is reproduced as well. In Figure 6.1 we present a visualization of the results in the last column of Table 6.1. It was shown in [Alfeld, Neamtu, and Schumaker'96] that spherical linear functions are spheres through the origin. As expected from the table first two surfaces are not spheres.

Example 2. Next we investigate how the choice of λ affects the error of our approximation. We interpolate a general function $f(x,y,z) = 1 + 0.3x^8 + e^{0.2y^3}$. The initial

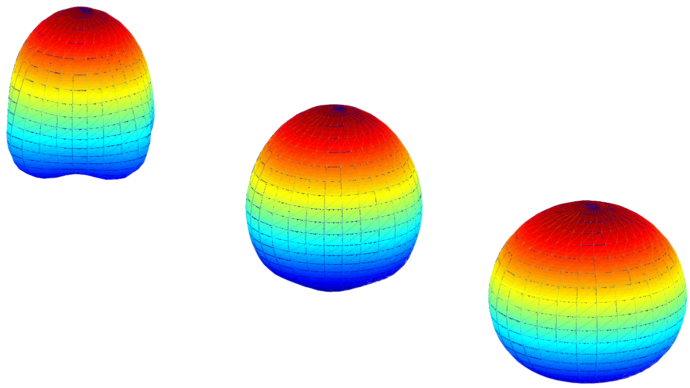

Figure 6.1: Reproduction of $z+1$ in S_3^1, S_4^1 and $S_3^1 \oplus S_4^1$ from left to right.

triangulation is Δ_1. The triangulation Δ_2 is obtained by bisecting the edges of Δ_1 and splitting each triangle into four sub-triangles. Similarly we obtain the uniform refinements Δ_3 and Δ_4. Each time we evaluate the spline interpolant at 5120 evenly spaced points w and list the relative errors $e := \frac{\|s(w)-f(w)\|_\infty}{\|f(w)\|_\infty}$ in Table 6.2.

$\lambda \setminus \Delta$	Δ_1	Δ_2	Δ_3	Δ_4
0.1	$0.9442e-01$	$2.3349e-02$	$2.2270e-03$	$2.1254e-04$
0.2	$0.9436e-01$	$2.2691e-02$	$1.7986e-03$	$2.0737e-04$
0.3	$0.9429e-01$	$2.2085e-02$	$1.7570e-03$	$2.1420e-04$
0.4	$0.9422e-01$	$2.1608e-02$	$1.9870e-03$	$2.2644e-04$
0.5	$0.9414e-01$	$2.1168e-02$	$2.1526e-03$	$2.4197e-04$
0.6	$0.9405e-01$	$2.0780e-02$	$2.4118e-03$	$2.5990e-04$
0.7	$0.9395e-01$	$2.0461e-02$	$2.7717e-03$	$2.7978e-04$
0.8	$0.9383e-01$	$2.0265e-02$	$3.1331e-03$	$3.1210e-04$
0.9	$0.9370e-01$	$2.0109e-02$	$3.5004e-03$	$3.6150e-04$

Table 6.2: Dependence of minimal energy splines on weights in $\mathcal{E}$.

The results suggest that the error depends on the values of λ for a fixed triangulation. However, the same value of λ may not be the best choice for different triangulations.

Example 3. Next we compare the interpolation results for the function $f(x, y, z) = 1 + 0.3x^8 + e^{0.2y^3}$ in non-homogeneous and homogeneous spaces. We use Table 6.2 as a guide for the choice of λ. The results demonstrate that non-homogeneous splines approximate the original function f better than homogeneous splines on finer triangulations.

$S_d^r(\Delta) \ \setminus \ e(\Delta)$	$e(\Delta_1)$	$e(\Delta_2)$	$e(\Delta_3)$	$e(\Delta_4)$
$S_3^1(\Delta)$	$3.7879e-01$	$0.6586e-01$	$3.7846e-03$	$2.9833e-04$
$S_4^1(\Delta)$	$0.8234e-01$	$0.1980e-01$	$3.8708e-03$	$4.1190e-04$
$N_4^1(\Delta)$	$0.9370e-01$	$2.0109e-02$	$1.7570e-03$	$2.0737e-04$

Table 6.3: Convergence of splines interpolating f.

6.2 Numerical Experiments for Discrete Least Squares Method

Example 1. First we conduct experiments similar to the ones for minimal energy splines in $S_3^1(\Delta_1)$, $S_4^1(\Delta_1)$ and $S_3^1(\Delta_1) \oplus S_4^1(\Delta_1)$. Total number of points is 1006, triangulation is based on 6. Evaluation points and computation of errors are the same as in Section 6.1. In addition we test higher degree polynomials. Not only the direct sum space is capable of reproducing both constant and a homogeneous linear function, it reproduces a non-homogeneous linear functions, which was not possible in either one of the homogeneous spline spaces. Moreover, homogeneous polynomials of odd degrees up to 3 can be reproduced by cubic splines minimizing the least squares functional, and homogeneous polynomials of even degrees up to 4 can be reproduced by quartic splines. Since $S_3^1 \cap S_4^1 = 0$, we cannot reproduce non-homogeneous functions or polynomials of degrees different from degree of spline. In the direct sum space however, all polynomials of degrees up to 4 odd or even, homogeneous or non-homogeneous, are reproduced, see Table 6.4.

$S_d^r(\Delta_1) \ \backslash \ f$	1	$x+z$	$z+1$
$S_3^1(\Delta_1)$	$4.1063e-01$	$0.5391e-09$	$2.0543e-01$
$S_4^1(\Delta_1)$	$2.4365e-09$	$0.6325e-01$	$2.8532e-02$
$N_4^1(\Delta_1)$	$0.9419e-13$	$3.3859e-12$	$0.9975e-13$
$S_d^r(\Delta_1) \ \backslash \ f$	y^2+z	y^3+z+1	x^4+z+1
$S_3^1(\Delta_1)$	$1.5315e-01$	$1.8931e-01$	$1.7120e-01$
$S_4^1(\Delta_1)$	$4.5673e-02$	$2.8735e-02$	$2.6810e-02$
$N_4^1(\Delta_1)$	$1.1709e-13$	$1.2950e-13$	$1.5834e-13$

Table 6.4: Polynomial reproduction on 8 triangles.

Example 2. We illustrate convergence of discrete least squares splines approximating $f(x, y, z) = 1 + 0.3x^8 + e^{0.2y^3}$ in Table 6.5.

$S_d^r(\Delta) \ \backslash \ e(\Delta)$	$e(\Delta_1)$	$e(\Delta_2)$	$e(\Delta_3)$
$S_3^1(\Delta)$	$3.4124e-01$	$4.1755e-02$	$3.6864e-03$
$S_4^1(\Delta)$	$2.3321e-02$	$1.8815e-03$	$0.7477e-03$
$N_4^1(\Delta)$	$1.0102e-02$	$1.8007e-03$	$3.6840e-04$

Table 6.5: The relative error of splines approximating f.

Example 3. Our last example in this section is a scattered data approximation problem. The values of geopotential f are measured at scattered locations $\mathcal{V}$ of a sphere-like surface by a satellite at a fixed height above the surface of Earth. We run three similar experiments. In the first one we use the data collected over the period of two days amounting to 5760 values, in the second - four days, total of 11520 values and in the last - six day data of 17280 values. In every experiment we start with a triangulation Δ_1 based on six vertices and consisting of eight triangles as in the examples 1, 2, Section 6.1 and 1, Section 6.2. Then Δ_1 is refined uniformly twice to obtain Δ_2 and Δ_3. For each triangulation we compute the discrete least squares spline solution in the spaces $S_3^1(\Delta_i)$, $S_4^1(\Delta_i)$ and $S_3^1(\Delta_i) \oplus S_4^1(\Delta_i)$, $i = 1, 2, 3$. In Tables 6.6, 6.7, 6.8 we list error values of the form $e := \frac{\max_{v\in\mathcal{V}}|s(v)-f(v)|}{\max_{v\in\mathcal{V}}|f(v)|}$ for each of the computed splines. In Tables 6.9, 6.10, 6.11 we list relative standard deviation values s $:= \frac{\mathrm{std}_{|s(v)-f(v)|}}{\max_{v\in\mathcal{V}}|f(v)|}$ for each of the computed splines.

$S_d^r(\Delta) \ \setminus \ e(\Delta)$	$e(\Delta_1)$	$e(\Delta_2)$	$e(\Delta_3)$
$S_3^1(\Delta)$	$3.7228e-01$	$2.0086e-01$	$0.8692e-01$
$S_4^1(\Delta)$	$2.9349e-01$	$0.9681e-01$	$4.5916e-02$
$N_4^1(\Delta)$	$2.0711e-01$	$0.9531e-01$	$3.1303e-02$

Table 6.6: The relative error for geodata approximating splines, two days data.

$S_d^r(\Delta) \ \setminus \ e(\Delta)$	$e(\Delta_1)$	$e(\Delta_2)$	$e(\Delta_3)$
$S_3^1(\Delta)$	$3.7295e-01$	$2.0141e-01$	$0.8665e-01$
$S_4^1(\Delta)$	$2.9409e-01$	$0.9709e-01$	$4.5073e-02$
$N_4^1(\Delta)$	$2.0840e-01$	$0.9633e-01$	$3.2144e-02$

Table 6.7: The relative error for geodata approximating splines, four days data.

$S_d^r(\Delta) \ \setminus \ e(\Delta)$	$e(\Delta_1)$	$e(\Delta_2)$	$e(\Delta_3)$
$S_3^1(\Delta)$	$3.7309e-01$	$2.0726e-01$	$0.8653e-01$
$S_4^1(\Delta)$	$2.9444e-01$	$0.9726e-01$	$4.6654e-02$
$N_4^1(\Delta)$	$2.0853e-01$	$0.9296e-01$	$3.1735e-02$

Table 6.8: The relative error for geodata approximating splines, six days data.

6.3 Numerical Experiments for Penalized Least Squares Method

Example 1. Our first example in this section is similar to Example 3 of Section 6.2. The values of geopotential f are measured at scattered locations $\mathcal{V}$ of a sphere-like surface by a satellite at a fixed height above the surface of Earth. Note that for the penalized least square fit we require the data at the vertices of a triangulation to be given. To deal with this requirement and to have other conditions of the experiment satisfied closely to the conditions in the discrete least square experiment we start with the triangulation Δ_1 and replace the vertices of this triangulation by the existing points closest to these vertices. We call this new triangulation $\bar{\Delta}_1$. After refining $\bar{\Delta}_1$

$S^r_d(\Delta) \ \backslash \ \mathrm{s}(\Delta)$	$\mathrm{s}(\Delta_1)$	$\mathrm{s}(\Delta_2)$	$\mathrm{s}(\Delta_3)$
$S^1_3(\Delta)$	$0.7020e-01$	$3.5489e-02$	$1.3247e-02$
$S^1_4(\Delta)$	$4.7324e-02$	$1.4640e-02$	$4.4683e-03$
$N^1_4(\Delta)$	$3.5713e-02$	$1.1194e-02$	$3.2933e-03$

Table 6.9: The relative standard deviation for geodata approximating splines, two days data.

$S^r_d(\Delta) \ \backslash \ \mathrm{s}(\Delta)$	$\mathrm{s}(\Delta_1)$	$\mathrm{s}(\Delta_2)$	$\mathrm{s}(\Delta_3)$
$S^1_3(\Delta)$	$0.7017e-01$	$3.5522e-02$	$1.3257e-02$
$S^1_4(\Delta)$	$4.7357e-02$	$1.4591e-02$	$4.4739e-03$
$N^1_4(\Delta)$	$3.5802e-02$	$1.1937e-02$	$3.3873e-03$

Table 6.10: The relative standard deviation for geodata approximating splines, four days data.

$S^r_d(\Delta) \ \backslash \ \mathrm{s}(\Delta)$	$\mathrm{s}(\Delta_1)$	$\mathrm{s}(\Delta_2)$	(Δ_3)
$S^1_3(\Delta)$	$0.7012e-01$	$3.5571e-02$	$1.3248e-02$
$S^1_4(\Delta)$	$4.7390e-02$	$1.4574e-02$	$4.4851e-03$
$N^1_4(\Delta)$	$3.5843e-02$	$1.1902e-02$	$3.2954e-03$

Table 6.11: The relative standard deviation for geodata approximating splines, six days data.

we again may not have the values of geopotential available at the vertices of the refined triangulation. We replace these vertices by the points closest to them where values of geopotential are available. We call this new triangulation $\bar{\Delta}_2$. Similarly we obtain $\bar{\Delta}_3$. Therefore $\bar{\Delta}_{i+1}$ is not exactly a uniform refinement of $\bar{\Delta}_i, i = 1, 2$, but only the closest possible approximation of the uniform refinement available under the circumstances. Again, we run three similar experiments over the periods of two, four and six days. Since the smaller data sets are contained in larger data sets we use the triangulations $\bar{\Delta}_i, i = 1, 2, 3$ in all three experiments. For each data set we compute the penalized least square spline solutions in the spaces $S^1_3(\bar{\Delta}_i)$, $S^1_4(\bar{\Delta}_i)$ and $N^1_3(\bar{\Delta}_i)$, $i = 1, 2, 3$ with $\lambda = \lambda_1 = \lambda_0 = 10^{-6}$. In Tables 6.12, 6.13, 6.14 we list relative error values $e := \frac{\max_{v\in\mathcal{V}}|s(v)-f(v)|}{\max_{v\in\mathcal{V}}|f(v)|}$ for each of the computed splines. In Tables 6.15, 6.16, 6.17 we list relative standard deviation values s := $\frac{\mathrm{std}|s(v)-f(v)|}{\max_{v\in\mathcal{V}}|f(v)|}$ for each of the computed splines.

$S^r_d(\Delta) \ \setminus \ e(\Delta)$	$e(\Delta_1)$	$e(\Delta_2)$	$e(\Delta_3)$
$S^1_3(\Delta)$	$0.7300e-00$	$2.5346e-01$	$1.0829e-01$
$S^1_4(\Delta)$	$3.2211e-01$	$1.1959e-01$	$4.4371e-02$
$N^1_4(\Delta)$	$2.5707e-01$	$0.9400e-01$	$4.2157e-02$

Table 6.12: The relative error for geodata approximating splines, two days data.

$S^r_d(\Delta) \ \setminus \ e(\Delta)$	$e(\Delta_1)$	$e(\Delta_2)$	$e(\Delta_3)$
$S^1_3(\Delta)$	$0.6864e-00$	$2.6187e-01$	$1.0842e-01$
$S^1_4(\Delta)$	$4.7046e-01$	$1.1939e-01$	$4.3390e-02$
$N^1_4(\Delta)$	$2.5714e-01$	$0.9341e-01$	$4.0303e-02$

Table 6.13: The relative error for geodata approximating splines, four days data.

$S^r_d(\Delta) \ \setminus \ e(\Delta)$	$e(\Delta_1)$	$e(\Delta_2)$	$e(\Delta_3)$
$S^1_3(\Delta)$	$0.6415e-00$	$2.6219e-01$	$1.0997e-01$
$S^1_4(\Delta)$	$3.1738e-01$	$1.1977e-01$	$4.3965e-02$
$N^1_4(\Delta)$	$2.5742e-01$	$0.9315e-01$	$4.1752e-02$

Table 6.14: The relative error for geodata approximating splines, six days data.

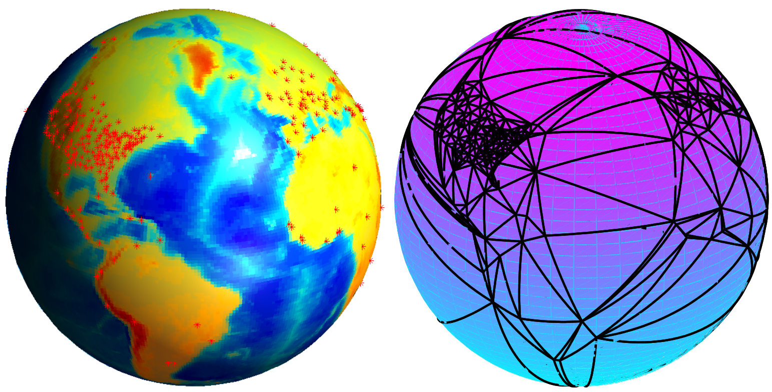

Figure 6.2: A total triangulations based on all data locations.

Example 2. In this example we would like to demonstrate advantages of using penalized least square method in a case of non-uniformly distributed data locations.

$S_d^r(\Delta)\ \setminus$ s(Δ)	s(Δ_1)	s(Δ_2)	s(Δ_3)
$S_3^1(\Delta)$	$1.3024e-01$	$4.4337e-02$	$1.4585e-02$
$S_4^1(\Delta)$	$0.5505e-01$	$1.7656e-02$	$4.5250e-03$
$N_4^1(\Delta)$	$3.9482e-02$	$1.3479e-02$	$4.1267e-03$

Table 6.15: The relative standard deviation for geodata approximating splines, two days data.

$S_d^r(\Delta)\ \setminus$ s(Δ)	s($\tilde{\Delta}_1$)	s($\tilde{\Delta}_2$)	s($\tilde{\Delta}_3$)
$S_3^1(\Delta)$	$1.2101e-01$	$4.4362e-02$	$1.4602e-02$
$S_4^1(\Delta)$	$0.7660e-01$	$1.7673e-02$	$4.5167e-03$
$N_4^1(\Delta)$	$3.9588e-02$	$1.3315e-02$	$3.9346e-03$

Table 6.16: The relative standard deviation for geodata approximating splines, four days data.

$S_d^r(\Delta)\ \setminus$ s(Δ)	s($\tilde{\Delta}_1$)	s($\tilde{\Delta}_2$)	s($\tilde{\Delta}_3$)
$S_3^1(\Delta)$	$1.1283e-01$	$4.4367e-02$	$1.4606e-02$
$S_4^1(\Delta)$	$0.5497e-01$	$1.7672e-02$	$4.5290e-03$
$N_4^1(\Delta)$	$3.9638e-02$	$1.3240e-02$	$3.8623e-03$

Table 6.17: The relative standard deviation for geodata approximating splines, six days data.

In Figure 6.2 we present 302 locations around the globe where average daily temperatures are available on May 29 of 2004 and a triangulation Δ_e based on all these locations. This is the kind of triangulation we have to use for the minimal energy interpolating splines. While such a triangulation is not unique, it is obvious that sharp angles and non-uniform partition size are unavoidable. More precisely, let us test all three methods against the function $f(x, y, z) = 1 + 0.3x^8 + e^{0.2y^3}$ sampled at the given temperature locations.

As mentioned above we can compute the minimal energy interpolating cubic spline on the triangulation in Figure 6.2. The minimal energy spline overshoots and therefore is not satisfactory. We can try to remove some of the data locations to come up with a suitable triangulation. We construct such a triangulation Δ_{me} consisting of 22 triangles. The rest of the data is not used for the minimal energy splines.

However there is no need to ignore any data if we use the penalized least square method on the same triangulation. In the Table 6.18 below we list the relative errors of the form $e := \frac{\max_{v\in\mathcal{V}}|s(v)-f(v)|}{\max_{v\in\mathcal{V}}|f(v)|}$, where the set V is the collection of all data locations, i.e. it is the set used for PLS approximation. We compare the methods in various spline spaces. Results are similar for the higher degrees. These results

Method	$S_8^1(\Delta_{me})$	$S_9^1(\Delta_{me})$	$N_9^1(\Delta_{me})$
ME	$0.7057e-01$	$0.7724e-01$	$0.6868e-01$
PLS	$5.0675e-04$	$5.7879e-04$	$5.0820e-04$

Table 6.18: The relative error for interpolating and approximating splines.

are to be expected since the PLS method minimizes the sum of squared errors at the data locations, therefore to make this experiment complete we evaluate both splines at the arbitrary 5120 points more or less uniformly spread throughout the sphere. The results are presented in Table 6.19. As seen from this table the results

Method	$S_8^1(\Delta_{me})$	$S_9^1(\Delta_{me})$	$N_9^1(\Delta_{me})$
ME	$1.0159e-01$	$1.1798e-01$	$1.0065e-01$
PLS	$0.8882e-01$	$0.9786e-01$	$0.8564e-01$

Table 6.19: The relative error for interpolating and approximating splines.

are better for PLS again. Overall the two tables suggest that we use PLS method for the temperature experiment.

Next let us compare the discrete and penalized least squares methods. For the discrete least square method we need to fulfill the requirement of evenly distributed data, which leads to a lower bound on the number of points inside of every triangle. That is in the ocean areas where data is sparse triangles must be large. On the other hand, splines of lower degree do not have enough flexibility on overcrowded triangles, therefore we need finer triangulation over the regions with dense data. Again, such triangulation will have non-uniform partition size and triangles with thin angles. We tried to construct a suitable triangulation which would balance the uniform data distribution requirement and the comparable triangle sizes. In addition, even though it is not necessary for DLS that the vertices of the triangulation are a subset of the data locations, we need to fulfill this requirement to implement PLS and to compare the two methods.

We have constructed a triangulation Δ_d that works for some lower degree splines as illustrated in Table 6.20. As degree increases we have DLS splines overshooting, and it may not be possible to construct a triangulation suitable for a degree arbitrary high. This is not an issue for the PLS method.

The methods are tested against the same function, and the error values listed in this table are computed over 5120 points.

Method	$S_3^1(\Delta_d)$	$S_4^1(\Delta_d)$	$N_4^1(\Delta_d)$
DLS	$1.4502e-00$	$0.4738e-01$	$5.2904e-02$
PLS	$1.4502e-00$	$0.4766e-01$	$0.4897e-01$
Method	$S_4^1(\Delta_d)$	$S_5^1(\Delta_d)$	$N_5^1(\Delta_d)$
DLS	$0.4738e-01$	$4.0410e-00$	$3.7860e+01$
PLS	$0.4766e-01$	$1.1364e-00$	$0.4492e-01$
Method	$S_5^1(\Delta_d)$	$S_6^1(\Delta_d)$	$N_6^1(\Delta_d)$
DLS	$4.0410e-00$	$1.0184e-00$	$1.5193e+04$
PLS	$1.1364e-00$	$0.7403e-01$	$0.6520e-01$

Table 6.20: The relative error for DLS and PLS approximating splines.

With these results in mind we prepare for the temperature data testing. The experiment is set as follows.

To apply penalized least square method all we have to ensure is that a triangulation to be constructed is based on some subset of locations. In Figure 6.3 we

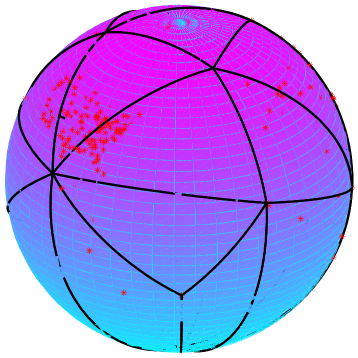

Figure 6.3: A total triangulation based on a subset of data locations.

show such a triangulation, call it Δ_p, based on 13 vertices. This triangulation we use in further experiments. Note that triangles in Δ_p have comparable diameters and angles.

Relative error values $e := \frac{\max_{v\in\mathcal{V}}|s(v)-f(v)|}{\max_{v\in\mathcal{V}}|f(v)|}$ for PLS spline solutions of various degrees are presented in Table 6.21. In all cases we used $\lambda = \lambda_1 = \lambda_0 = 10^{-6}$.

Spline Space	$S^1_7(\Delta_p)$	$S^1_8(\Delta_p)$	$N^1_8(\Delta_p)$
Error	$1.2970e-01$	$1.1002e-01$	$1.0659e-01$
Spline Space	$S^1_8(\Delta_p)$	$S^1_9(\Delta_p)$	$N^1_9(\Delta_p)$
Error	$1.1002e-01$	$0.9679e-01$	$0.9270e-01$
Spline Space	$S^1_9(\Delta_p)$	$S^1_{10}(\Delta_p)$	$N^1_{10}(\Delta_p)$
Error	$0.9679e-01$	$0.8418e-01$	$0.8343e-01$
Spline Space	$S^1_{10}(\Delta_p)$	$S^1_{11}(\Delta_p)$	$N^1_{11}(\Delta_p)$
Error	$0.8418e-01$	$0.8036e-01$	$0.8018e-01$
Spline Space	$S^1_{11}(\Delta_p)$	$S^1_{12}(\Delta_p)$	$N^1_{12}(\Delta_p)$
Error	$0.8036e-01$	$0.7755e-01$	$0.7721e-01$

Table 6.21: The relative error for temperature approximating splines.

6.4 Numerical Experiments for Multiple Star Technique

The first test we conduct illustrates the convergence of the minimal energy interpolating spline to a given smooth function. The following are our test functions.

$$
\begin{aligned}
f_1(x,y,z) &= \sin^7(\theta)\cos(\theta)\sin(7\phi) \\
f_2(x,y,z) &= \sin^8(\theta)\sin(8\phi) \\
f_3(x,y,z) &= \sin^{25}(\theta)\sin(25\theta)
\end{aligned}
$$

All of them are harmonic on the sphere. We start with a triangulation Δ_0 of eight congruent triangles and then uniformly refine it several times to get new triangulations $\Delta_1, \Delta_2, \Delta_3, \cdots,$. That is, Δ_n is the uniform refinement of Δ_{n-1}. Δ_1 contains 66 vertices and 128 triangles. Δ_2 has 258 vertices and 512 triangles. Δ_3 consists of 1026 vertices and 2048 triangles and finally Δ_4 contains 4098 vertices and 8172 triangles.

We use our multiple star method with Ω_i's being triangles in Δ and $\mathcal{S}_{i,k}$ being spaces of quintic C^1 splines. Then we estimate the accuracy of the method by evaluating the spline approximation and the exact functions over 24,000 points almost evenly distributed over the sphere and taking the maximum absolute value of the differences. The maximum errors are listed in Table 6.22.

$f\backslash i$	Δ_1	Δ_2	Δ_3	Δ_4
f_1	$3.7500e-01$	$1.2835e-02$	$1.67039e-03$	$9.7385e-04$
f_2	$1.0368e-00$	$4.5470e-02$	$7.13533e-03$	$3.8240e-03$
f_3	$1.7303e-00$	$2.0809e-00$	$3.54561e-01$	$3.7372e-02$

Table 6.22: Maximum errors of C^1 cubic interpolating splines over various triangulations.

Bibliography

[1] Adams, R., *Sobolev Spaces*, Academic Press, New York, 1975.

[2] Alfeld, P., M. Neamtu, and L. L. Schumaker, Berstein-Bézier polynomials on spheres and sphere-like surfaces, Comp. Aided Geom. Design. **13** (1996), 333–349.

[3] Alfeld, P., M. Neamtu, and L. L. Schumaker, Fitting scattered data on sphere-like surfaces using spherical splines, J. Comp. Appl. Math. **73** (1996), 5-43.

[4] Alfeld, P., M. Neamtu, and L. L. Schumaker, Dimension and local bases of homogeneous spline spaces, SIAM J. Math. Anal. **27** (1996), 1482-1501.

[5] Atkinson K. E., *An Introduction to Numerical Analysis*, John Wiley and Sons, New York,1978.

[6] Awanou, G. M. and M. J. Lai, On Convergence Rate of the Augmented Lagrangian Algorithms for Non symmetric Saddle Point Problems, to appear in Applied Numerical Mathematics, 2005.

[7] Awanou, G. M., M. J. Lai, and P. Wenston, The Multivariate Spline Method for Numerical Solution of Partial Differential Equations and Scattered Data Interpolation, submitted, 2003.

[8] Baramidze, V. and M. J. Lai, Error Bounds for minimal energy interpolatory spherical splines, Approximation Theory XI, edited by C. K. Chui, M. Neamtu, and L. L. Schumaker, Nashboro Press, Brentwood, 2005, pp. 25–50.

[9] de Boor, C., A bound on L_∞ norm of L_2 approximation by splines in terms of a global mesh ratio, Math. Comp. **30**(136) (1976), 765–771.

[10] Davydov, O., and L. L. Schumaker, On stable local bases for bivariate polynomial spline spaces, Constr. Approx. **18** (2001), 87–116.

[11] Cinar, I. D., Mathematical modeling and computer simulation of simultaneous heat and mass transfer during drying of food particles [electronic resource].

[12] Evans, L. C., *Partial Differential Equations*, American Mathematical Society, 1998.

[13] Fasshauer, G. and L. L. Schumaker, Scattered Data Fitting on the Sphere, Mathematical Methods for Curves and Surfaces II, M. Daehlen, T. Lyche, and L. L. Schumaker (eds.), Vanderbilt Univ. Press, 1998, 117–166.

[14] Fasshauer, G. and L. L. Schumaker, Multi-patch parametric surfaces with minimal energy, Comp. Aided. Geom. Design **13** (1996), 45–79.

[15] von Golitschek, M. J. Lai, and L. L. Schumaker, Error bounds for minimal energy interpolatory splines, Numerishe Math., **93** (2002), 315–331.

[16] von Golitchek, M., and L. L. Schumaker, Bounds on projections onto bivariate polynomial spline spaces with stable local bases, Constr. Approx. **18** (2002), 241–254.

[17] von Golitchek, M., and L. L. Schumaker, Penalized Least Square Fitting, Serdica **18** (2002), 1001–1020,

[18] von Golitchek, M., and L. L. Schumaker, Penalized Least Square Fitting, Algorithms for Approximation II, J.Mason and M. G. Cox (eds.), Chapman and Hall, London (1990), 210–227.

[19] Gomide, A., and J. Stolfi, Nonhomogeneous polynomial C^k splines on the sphere S^n, Relatorio Tecnico IC-00-13, 2000.

[20] Lai, M. J., and L. L. Schumaker, On the approximation power of bivariate splines, Adv. in Comp. Math. **9** (1998), 251–279.

[21] Lai, M. J. and P. Wenston, Bivariate Splines for Fluid Flows, Computers and Fluids 33(2004), pp. 1047–1073.

[22] Le Gia, Q.T., Galerkin Approximation for Elliptic PDEs on Spheres, preprint, 2004.

[23] Neamtu, M., and L. L. Schumaker, On the approximation order of splines on spherical triangulations, Adv. in Comp. Math. **21** (2004), 3–20.

[24] Wahba, G., Spline interpolation and smoothing on the sphere, SIAM J. Sci. Statist. Comput. **2** (1981), 5-16.

[25] Wahba, G., Vector Splines on the Sphere with applications to the estimation of vorticity and divergence from discrete noisy data, Technical report no. 674, Department of Statistics, University of Wisconsin-Madison, 1982.