Bivariate Spline Solution of Time Dependent Nonlinear PDE for a Population Density over Irregular Domains

Juan B. Gutierrez, Ming-Jun Lai*, and George Slavov
Department of Mathematics
The University of Georgia
Athens, GA 30602

February 25, 2015

Abstract

We study a time dependent partial differential equation(PDE) which arises from classic models in ecology involving logistic growth with Allee effect by introducing a discrete weak solution. Existence, uniqueness and stability of the discrete weak solutions will be discussed. We use bivariate splines to approximate the discrete weak solution of the nonlinear PDE. A computational algorithm is designed to solve this PDE. A Convergence analysis of the algorithm is presented. Finally, we present some simulations of population development over some irregular domains.

1 Introduction

Empirical evidence shows that the structure of environments and spatial scale can systematically influence population development and interactions in a way that can be described by mathematical models [13, 15]. The first serious attempt to model population dynamics is credited to Malthus in 1798 [26], who hypothesized that human populations grow geometrically while resources grow arithmetically, thus eventually reaching a point in which the population could not be sustained any more; this linear growth model is problematic since it allows unbounded population increase. A major refinement was introduced by Verhulst in 1838 [32] by means of a density-dependent logistic term in Malthus' model, predicting population growth if resources were available or population decay if population surpassed resources; this model takes the form $\dot{p} = r_0 p(1 - p/k)$, where p represents population density, r_0 is the rate of growth, and k represents the carrying capacity. Fisher [11] used in 1937 a diffusion operator to study the propagation of advantageous genes in population; the same year, Kolmogorov and his collaborators [17] studied the following reaction-diffusion equations in the one-dimensional setting:

$$\dot{p} = Dp_{xx} + kp(1 - p) \text{ and } \dot{p} = Dp_{xx} + F(p), \tag{1.1}$$

where $F(p)$ satisfies $F(p) \geq 0$, $F(0) = F(1) = 0, f'(0) > 0, F'(1) < 0$ for $p \in [0, 1]$.

The logistic model has been central to the modern study of population dispersal in spatial domains [24, 6]. Skellam's influential paper [30] in 1951, introduced a variation in Kolmogorov's equation for the study of phytoplankton; the resulting model was $p_t = d\Delta p + c_1(x, y)p - c_2(x, y)p^2$. This basic form of population dispersal is applicable in many notable cases ranging from population dispersal to recent models of information diffusion in online social networks [?]. Nevertheless, Skellam's model is too simplistic in most practical cases; it assumes that lack of interactions with other species, and that populations can grow at the same rate at low and high densities. An important refinement to Kolgomorov's model was introduced by Lewis and Kareiva in 1993 [23]. The correlation hypothesized by Allee in 1938 between population size and

*mjlai@math.uga.edu, the corresponding author of this article.

mean individual fitness [1], was represented in Lewis and Kareiva's model by $p_t = d\Delta p + r_0 p(1-p/k)(p-\sigma)$, where σ represents the population below the carrying capacity below which the population growth is negative. This is the foundation of the model we study in this paper.

More precisely, we are interested in solution of the following nonlinear time dependent partial differential equation: Letting $\Omega \subset \mathbb{R}^2$ be a polygonal domain and $\Omega_T = \Omega \times (0, T]$,

$$\begin{cases} \dfrac{dp(\mathbf{x},t)}{dt} &= \text{ div } (D(p,\mathbf{x})\nabla p(\mathbf{x},t))) + F(p(\mathbf{x},t)), \quad \mathbf{x} = (x,y) \in \Omega, t \in [0,T] \\ p(\mathbf{x},t) &\geq 0, \quad \mathbf{x} \in \Omega, t \in [0,T] \\ p(\mathbf{x},t) &= 0, \quad \mathbf{x} \in \partial\Omega, t \in [0,T] \\ p(\mathbf{x},0) &= p_0, \quad \mathbf{x} \in \Omega, \end{cases} \tag{1.2}$$

where $D(p,\mathbf{x})$ is a diffusive term, e.g. $D(p,\mathbf{x}) = D > 0$ and $F(p)$ is a growth function, e.g. $F(p) = Ap(1-p)$ which is a standard logistic growth function with A being a nonnegative weighted function over Ω. In this paper, we shall mainly study $F(p) = Ap(1-p)(p-\sigma)$, where σ is a positive constant in $[0, 1)$ and $A(x, y)$ are nonnegative functions on $\Omega \times [0, T)$.

Exact solutions to Kolmogorov's equation (1.1) have been found [27]. However, there does not appear to exist an exact solution to the diffusion logistic model with Allee effect; while asymptotics and speed of diffusion waves have been found analytically, the solutions to this problem (1.2) over different domains remain mostly numerical. Lewis and Kareiva [23] used finite differences; The researchers in [28] used a second-order finite elements method; In [29] finite elements were used in a model that incorporated geographic features and population dispersal. In this paper, we would like to present another numerical solution to the diffusion logistic problem with Allee effect. This solution is based on bivariate spline functions over triangulations. The bivariate spline functions have been studied for more than 20 years (see a monograph [21] for theory of bivariate splines) and they are mature enough for numerical solution of various linear and nonlinear PDE, e.g. 2D Navier-Stokes equations. See [4], [22], [18], [19], and [14]. They will enable us to effectively model a population density development over any arbitrary domain. An advantage to use bivariate splines is they are able to generate a smooth density surface over the domain easily. The differentiability can be useful for some applications which involves the rate of changes of population along different direction at any location inside the domain.

Our numerical solution of this PDE is slightly different from the classic approach in a few ways. Instead of defining a weak solution in terms of test functions defined on domain $\Omega_T = \Omega \times (0, T]$, we define a discrete weak solution of the PDE using test functions defined on Ω together with the first order divided difference in time. See Definition 3.1. Another difference from the classic approach is that we use an optimization approach to establish the existence, uniqueness, stability and other properties of this discrete weak solution. We shall use bivariate splines to approximate the discrete weak solution using the discrete weak solution in the finite dimensional spline space. We are able to show that spline discrete weak solution converges to the discrete weak solution in $H^1(\Omega)$ as the size of underlying triangulation goes to zero.

It is clear that there are three nonlinearities in (1.2): the nonlinear diffusive term $D(p,\mathbf{x})$, the nonlinear growth function $F(p)$ and nonlinearity condition $0 \leq p(\mathbf{x},t) \leq 1$ for all x, t which is essential to the theory presented. We have to design a convergent computational algorithm to find bivariate spline solutions and establish how well our bivariate spline solutions are close to the exact discrete weak solution. We implement our computational algorithm in MATLAB. With the numerical solution, we are able to simulate how a population disperses over the area Ω of interest. In particular, we are able to see how the Allee constant σ plays a significant role in the population development.

2 Preliminaries

For the sake of completeness, we list a number of lemmas used in this paper, which are special cases of well-known results.

Lemma 2.1. *For $a, b \geq 0$ and any $\alpha > 0$ we have*

$$ab \leq \frac{\alpha}{2}a^2 + \frac{1}{2\alpha}b^2$$

Lemma 2.2 (Ladyzhenskaya's Inequality)**.** *For any $p \in H_0^1(\Omega)$ for $\Omega \subset \mathbb{R}^2$ we have the following inequality.*

$$\|p\|_{L^4} \leq C\|p\|_{L^2}^{1/2}\|\nabla p\|_{L^2}^{1/2}$$

Theorem 2.1 (Rellich-Kondrachov)**.** *Suppose that Ω is bounded with Lipschitz boundary. Then we have the following compact injection:*

$$H^1(\Omega) \subset L^2(\Omega)$$

That is, any bounded sequence in $H^1(\Omega)$ has a subsequence which converges to an $L^2(\Omega)$ function in L^2 norm.

Theorem 2.2 (General Sobolev Inequality)**.** *If $p \in H^2(\Omega)$, then $p \in C^{0,\gamma}$, the space of Hölder continuous functions with any exponent $0 < \gamma < 1$. Furthermore,*

$$||p||_{C^{0,\gamma}(\Omega)} \leq C\,||p||_{H^2(\Omega)}$$

where C is a constant independent of p.

Preliminary on bivariate splines can be found in the Appendix of this paper. Mainly we use the theory in [21] and computational schemes in [4]. As our PDE (1.2) is nonlinear, we have to extend the MATLAB codes used in [4] to handle this nonlinear PDE discussed in this paper.

3 The Basic Properties of the Discrete Weak Solution

Let us begin with a discrete weak solution of the partial differential equation (1.2). It is a standard calculation from (1.2) to have, for any $q \in H_0^1(\Omega)$,

$$\int_\Omega \frac{dp(\mathbf{x},t)}{dt}q(\mathbf{x})d\mathbf{x} = -\int_\Omega D(\mathbf{x})\nabla p(\mathbf{x},t)\cdot\nabla q(\mathbf{x})d\mathbf{x} + \int_\Omega A(\mathbf{x})F(p(\mathbf{x},t))q(\mathbf{x})d\mathbf{x}. \tag{3.1}$$

Consider $t \in [0,T]$ and $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$. We approximate $\frac{dp(\mathbf{x},t)}{dt}$ by its divided difference, i.e., $\frac{dp(\mathbf{x},t_i)}{dt} \approx (p(\mathbf{x},t_i) - p(\mathbf{x},t_{i-1}))/h$ with $h = t_i - t_{i-1}$. The above equation becomes

$$\begin{aligned}&\int_\Omega p(\mathbf{x},t_i)q(\mathbf{x})d\mathbf{x} + h\int_\Omega D(\mathbf{x})\nabla p(\mathbf{x},t_i)\cdot\nabla q(\mathbf{x})d\mathbf{x}\\ &- h\int_\Omega A(\mathbf{x})F(p(\mathbf{x},t_i))q(\mathbf{x})d\mathbf{x} = \int_\Omega p(\mathbf{x},t_{i-1})q(\mathbf{x})d\mathbf{x}, \quad \forall q \in H_0^1(\Omega).\end{aligned} \tag{3.2}$$

We introduce the following concept of the PDE solution:

Definition 3.1. *Any solution to the above equation (3.2) for a fixed $h > 0$ is called a discrete weak solution of (1.2).*

Let us first show that the discrete weak solution is a good approximation of the exact solution. Indeed, we have

Theorem 3.1. *Let $p(\mathbf{x},t)$ be the classic solution and $p_h(\mathbf{x},t)$ be the discrete weak solution dependent on $h > 0$. Suppose that $p(\mathbf{x},t)$ is twice differentiable with respect to t. Then*

$$\int_\Omega |p(\mathbf{x},t_i) - p_h(\mathbf{x},t_i)|^2 d\mathbf{x} \leq Ch, \quad \forall i = 0,\cdots,m+1, \tag{3.3}$$

as $h = T/(m+1) \to 0$, where $C > 0$ is a constant independent of h.

Proof. By Taylor expansion, we have

$$\frac{p(\mathbf{x},t_i)}{dt} = \frac{p(\mathbf{x},t_i)-p(\mathbf{x},t_{i-1})}{h} + O(h).$$

Using (3.1) and (3.2), we have

$$\int_\Omega \frac{dp(\mathbf{x},t_i)}{dt} q(\mathbf{x})d\mathbf{x} - \int_\Omega \frac{p_h(\mathbf{x},t_i)-p_h(\mathbf{x},t_{i-1})}{h} q(\mathbf{x})d\mathbf{x} = 0.$$

That is,

$$\int_\Omega \frac{p(\mathbf{x},t_i)-p(\mathbf{x},t_{i-1})}{h} q(\mathbf{x})d\mathbf{x} - \int_\Omega \frac{p_h(\mathbf{x},t_i)-p_h(\mathbf{x},t_{i-1})}{h} q(\mathbf{x})d\mathbf{x} = O(h).$$

Letting $q = p(\mathbf{x},t_i) - p_h(\mathbf{x},t_i)$ in the above inequality, we have

$$\begin{aligned}\int_\Omega |p(\mathbf{x},t_i)-p_h(\mathbf{x},t_i)|^2 d\mathbf{x} &= O(h^2) + \int_\Omega (p(\mathbf{x},t_{i-1})-p_h(\mathbf{x},t_{i-1})(p(\mathbf{x},t_i)-p_h(\mathbf{x},t_i)) \\ &\le \frac{1}{2}\int_\Omega |p(\mathbf{x},t_i)-p_h(\mathbf{x},t_i)|^2 d\mathbf{x} + \frac{1}{2}\int_\Omega |p(\mathbf{x},t_{i-1})-p_h(\mathbf{x},t_{i-1})|^2 d\mathbf{x} + O(h^2).\end{aligned}$$

It follows that

$$\int_\Omega |p(\mathbf{x},t_i)-p_h(\mathbf{x},t_i)|^2 d\mathbf{x} \le \int_\Omega |p(\mathbf{x},t_{i-1})-p_h(\mathbf{x},t_{i-1})|^2 d\mathbf{x} + O(2h^2).$$

We add them together for $i = 1, \cdots, k$ to have

$$\int_\Omega |p(\mathbf{x},t_k)-p_h(\mathbf{x},t_k)|^2 d\mathbf{x} \le O(2kh^2).$$

for $k = 1, \cdots, m+1$. Note that $(m+1)h = T$. So we have $\int_\Omega |p(\mathbf{x},t_k)-p_h(\mathbf{x},t_k)|^2 d\mathbf{x} \le O(h)$ for all $0 \le k \le m+1$. This completes the proof. $\square$

Let $\mathcal{A} = \{p \in H_0^1(\Omega), 0 \le p(x,y) \text{ for a.e. } (x,y) \in \Omega\}$ be the set of admissible functions. Here $\Omega \subset \mathbb{R}^2$ is an open, bounded domain with Lipschitz boundary. That is, we look for a population density in the admissible set $p \in \mathcal{A}$ which satisfies the following equation:

$$\int_\Omega pq \, d\mathbf{x} + h\int_\Omega D(\mathbf{x})\nabla p \cdot \nabla q \, d\mathbf{x} = \int_\Omega \hat{p}q \, d\mathbf{x} + h\int_\Omega pF(p)q \, d\mathbf{x} \qquad \forall q \in H_0^1(\Omega) \tag{3.4}$$

where $0 < K \le D(\mathbf{x}) \le K_2$ is a diffusive factor and

$$F(p) = A(\mathbf{x})(1-p)(p-\sigma) \tag{3.5}$$

which models population growth with an Allee effect. Here $A(\mathbf{x})$ is a given nonnegative function bounded by M and $\sigma \in (0,1)$ and $\hat{p} \in \mathcal{A}$ is a given admissible function.

We would like to see that the equation (3.4) has a unique solution. In order to do that, we note that the discrete weak formulation is the Euler-Lagrange equation of the following energy minimization problem.

$$\min_{p\in\mathcal{A}} E(p) = \min_{H_0^1(\Omega), p\ge 0} \int_\Omega p^2 \, d\mathbf{x} + h\int_\Omega D(\mathbf{x})|\nabla p|^2 \, d\mathbf{x} - h\int_\Omega G(p) \, d\mathbf{x} - \int_\Omega \hat{p}p \, d\mathbf{x} \tag{3.6}$$

where

$$G(p) = \int_0^p \xi F(\xi) \, d\xi$$

In order to show that the functional has a minimizer, we need a lower bound for its image.

Lemma 3.1. *Suppose we choose $h < \dfrac{1}{M}$. Then for any function $p \in \mathcal{A}$ the energy functional given in (3.6) satisfies*

$$E(p) \geq C \, ||p||^2_{H^1_0(\Omega)} - ||\hat{p}||^2_2$$

for some constant $C > 0$. In particular, $\inf_{p \in \mathcal{A}} E(p) \geq -\, ||\hat{p}||^2_2 > -\infty$.

Proof. First we will present an upper bound for one of the terms.

$$G(p) = \int_0^p \xi F(\xi) \, d\xi \leq M \int_0^p \xi \, d\xi = \frac{M}{2} p^2$$

$$\int_\Omega G(p) \, d\mathbf{x} \leq \frac{M}{2} \, ||p||^2_2$$

Now we prove the lower bound for the entire functional. We use the Cauchy-Schwarz inequality, the upper bound for $G(p)$ we just established and $D(\mathbf{x}) \geq K$.

$$\begin{aligned} E(p) &\geq ||p||^2_2 + hK \, ||\nabla p||^2_2 - \frac{hM}{2} \, ||p||^2_2 - ||\hat{p}||_2 \, ||p||_2 \\ &= \left(1 - \frac{hM}{2}\right) ||p||^2_2 + hK \, ||\nabla p||^2_2 - ||\hat{p}||_2 \, ||p||_2 \end{aligned}$$

Use our assumption for h in this lemma and Lemma 2.1 on the last term with $\alpha = 2$.

$$\begin{aligned} &\geq \frac{1}{2} \, ||p||^2_2 + hK \, ||\nabla p||^2_2 - ||\hat{p}||^2_2 - \frac{1}{4} \, ||p||^2_2 \\ &\geq \min\left\{\frac{1}{4}, hK\right\} ||p||^2_{H^1_0(\Omega)} - ||\hat{p}||^2_2 \end{aligned}$$

□

Lemma 3.2. *If $h < 1/M$, the energy functional in (3.6) is weakly lower semi-continuous on $H^1(\Omega)$. That is, if $p_k \to p^*$ weakly in $H^1(\Omega)$, then*

$$E(p^*) \leq \liminf_{k \to \infty} E(p_k)$$

Proof. Set $m := \liminf_{k \to \infty} E(p_k)$. By passing to a subsequence we can assume that $E(p_k) - m < 1/k$. That is, $\lim_{k \to \infty} E(p_k) = m$. Any weakly convergent sequence is bounded in $H^1(\Omega)$ norm, so by the Rellich-Kondrachov theorem (Theorem 2.1), we can pass to another subsequence which converges strongly in $L^2(\Omega)$. Taking one last subsequence, we can assume that $p_k \to p^*$ a.e. in Ω.

Fix $\epsilon > 0$. By Egoroff's theorem there exists a measurable set U_ϵ such that $p_k \to p^*$ uniformly on U_ϵ and $|\Omega - U_\epsilon| < \epsilon$. Also write

$$V_\epsilon = \left\{ x \in \Omega \,\middle|\, |p^*(\mathbf{x})| + |\nabla p^*(\mathbf{x})| < \frac{1}{\epsilon} \right\} \tag{3.7}$$

Then $|\Omega - V_\epsilon| \to 0$ as $\epsilon \to 0$. Let $O_\epsilon = U_\epsilon \cap V_\epsilon$ and note that

$$|\Omega - O_\epsilon| = |(\Omega - U_\epsilon) \cup (\Omega - V_\epsilon)| \leq |\Omega - U_\epsilon| + |\Omega - V_\epsilon| \to 0 \quad \text{as } \epsilon \to 0$$

Now

$$E(p_k) + \int_\Omega \hat{p} p_k \, d\mathbf{x} = \int_\Omega p_k^2 + hD(\mathbf{x}) |\nabla p_k|^2 - hG(p_k) \, d\mathbf{x}$$

From the proof of Lemma 3.1 we know that the right-hand side is nonnegative.

$$\geq \int_{O_\epsilon} p_k^2 + hD(\mathbf{x})|\nabla p_k|^2 - hG(p_k)\; d\mathbf{x}$$

Since the function $\eta : \mathbb{R}^n \to \mathbb{R}$ given by $\eta(\mathbf{x}) = |x|^2$ is convex, it follows that

$$\geq \int_{O_\epsilon} p_k^2 + hD(\mathbf{x})\left(|\nabla p^*|^2 + 2\nabla p^* \cdot (\nabla p_k - \nabla p^*)\right) - hG(p_k)\; d\mathbf{x}$$

$$E(p_k) + \int_\Omega \hat{p}p_k\; d\mathbf{x} \geq \int_{O_\epsilon} p_k^2 + hD(\mathbf{x})|\nabla p^*|^2 - hG(p_k)\; d\mathbf{x} + \int_{O_\epsilon} 2hD(\mathbf{x})\nabla p^* \cdot (\nabla p_k - \nabla p^*)\; d\mathbf{x} \tag{3.8}$$

Recall equation (3.7) and note that in the first integral every term is bounded above. In addition, $p_k \to p^*$ uniformly on O_ϵ and G is an absolutely continuous function, so $G(p_k) \to G(p^*)$ uniformly on O_ϵ. Thus,

$$\lim_{k\to\infty} \int_{O_\epsilon} p_k^2 + hD(\mathbf{x})|\nabla p^*|^2 - hG(p_k)\; d\mathbf{x} = \int_{O_\epsilon} (p^*)^2 + hD(\mathbf{x})|\nabla p^*|^2 - hG(p^*)\; d\mathbf{x} \tag{3.9}$$

As for the second integral, note that $\nabla p_k \to \nabla p^*$ weakly in $L^2(\Omega;\mathbb{R}^n)$. Since $hD(\mathbf{x})\nabla p^* \in L^2(\Omega;\mathbb{R}^n)$ it follows that

$$\lim_{k\to\infty} \int_{O_\epsilon} 2hD(\mathbf{x})\nabla p^* \cdot (\nabla p_k - \nabla p^*)\; d\mathbf{x} = 0 \tag{3.10}$$

We then take limits as $k \to \infty$ on both sides of (3.8) and as a result of (3.9) and (3.10), we have

$$\begin{aligned} m + \int_\Omega \hat{p}p^*\; d\mathbf{x} &\geq \int_{O_\epsilon} (p^*)^2 + hD(\mathbf{x})|\nabla p^*|^2 - hG(p^*)\; d\mathbf{x} \\ m &\geq \int_{O_\epsilon} (p^*)^2 + hD(\mathbf{x})|\nabla p^*|^2 - hG(p^*)\; d\mathbf{x} - \int_\Omega \hat{p}p^*\; d\mathbf{x} \end{aligned}$$

Now we take the limit as $\epsilon \to 0$. Since the integrand is nonnegative and $O_\epsilon \uparrow \Omega$, the monotone convergence theorem guarantees that

$$\begin{aligned} m &\geq \int_\Omega (p^*)^2 + hD(\mathbf{x})|\nabla p^*|^2 - hG(p^*) - \hat{p}p^*\; d\mathbf{x} \\ m &\geq E(p^*) \end{aligned}$$

□

Theorem 3.2. *There exists a function $p^* \in \mathcal{A}$ which minimizes the energy functional $E(p)$ defined in (3.6).*

Proof. Set $m := \inf_{p\in\mathcal{A}} E(p)$ and choose a minimizing sequence $\{p_k\}$. Then $E(p_k) \to m$. As a result of Lemma 3.1 we know that

$$||p_k||_{H_0^1(\Omega)} \leq E(p_k) + ||\hat{p}||_2^2$$

$E(p_k) \to m$, so $\sup_k E(p_k) < \infty$. Thus, the minimizing sequence is bounded in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is weakly compact, there exists a subsequence p_k which converges weakly to some function $p^* \in H_0^1(\Omega)$. We'd like to know that p^* is also in the admissible set $\mathcal{A}$. By the Rellich-Kondrachov theorem (Theorem 2.1), we can pass to a subsequence which converges strongly in $L^2(\Omega)$. By taking another subsequence, we can assume that $p_k \to p^*$ a.e., so we conclude that $p^* \geq 0$ a.e. That is, p^* is in the admissible set $\mathcal{A}$.

It remains to show that p^* is a minimizer of $E(p)$. Lemma 3.2 assures us that

$$E(p^*) \leq \liminf_{k\to\infty} E(p_k) = m \tag{3.11}$$

Since $p^* \in \mathcal{A}$, we have $m \leq E(p)$. Together with (3.11), this implies that $E(p^*) = m = \min_{p\in\mathcal{A}} E(p)$. □

Lemma 3.3. *If h is small enough so that*

$$2 - hM - hM'p_{max} > 0$$

then the functional $E(p)$ defined in (3.6) is μ-strongly convex. That is, $\exists \mu > 0$ such that

$$E(y) \geq E(\mathbf{x}) + \langle \nabla E(\mathbf{x}), x - y \rangle + \frac{\mu}{2} ||x - y||_2^2$$

where $\langle \nabla E(\mathbf{x}), x - y \rangle$ is the Gâteaux derivative of E at the point x in the direction $x - y$.

Proof. We use an equivalent formulation of μ-strong convexity. It is enough to show that $\forall q \in O$ we have

$$\partial^2 E(p, q) \geq \mu ||q||_2^2$$

We compute the second Gâteaux derivative. Let $q \in H_0^1(\Omega)$. Then the second derivative is given by $\mathcal{F}''(0)$.

$$\begin{aligned}
\mathcal{F}(t) &= E(p + tq) \\
\mathcal{F}'(t) &= \int_\Omega 2(p + tq)q \, d\mathbf{x} + 2h \int_\Omega D(\mathbf{x}) \nabla (p + tq) \cdot \nabla q \, d\mathbf{x} - h \int_\Omega (p + tq) F(p + tq) q \, d\mathbf{x} - \int_\Omega \hat{p} q \\
\mathcal{F}''(t) &= 2 \int_\Omega q^2 \, d\mathbf{x} + 2h \int_\Omega D(\mathbf{x}) |\nabla q|^2 \, d\mathbf{x} - h \int_\Omega F(p + qt) q^2 + (p + tq) F'(p + tq) q^2 \, d\mathbf{x}
\end{aligned}$$

$$\begin{aligned}
\partial^2 E(p, q) = \mathcal{F}''(0) &= 2 ||q||_2^2 + 2h \int_\Omega D(\mathbf{x}) |\nabla q|^2 \, d\mathbf{x} - h \int_\Omega F(p) q^2 - h \int_\Omega p F'(p) q^2 \, d\mathbf{x} \\
&\geq 2 ||q||_2^2 - hM ||q||_2^2 - hM' p_{\max} ||q||_2^2 \\
&= (2 - hM - hM' p_{\max}) ||q||_2^2
\end{aligned}$$

as desired. □

Theorem 3.3. *The energy functional in (3.6) has a unique minimizer.*

Proof. Suppose p and $\tilde{p}$ are both minimizers of $E(p)$. Then for any $q \in H_0^1(\Omega)$ we have

$$\langle \nabla E(p, q) \rangle = \langle \nabla E(\tilde{p}, q) \rangle = 0$$

By Lemma 3.3 the following two inequalities hold.

$$\begin{aligned}
E(p) &\geq E(\tilde{p}) + \frac{\mu}{2} ||p - \tilde{p}||_2^2 \\
E(\tilde{p}) &\geq E(p) + \frac{\mu}{2} ||p - \tilde{p}||_2^2
\end{aligned}$$

Add the two inequalities.

$$0 \geq \mu ||p - \tilde{p}||_2^2$$

Thus, $p = \tilde{p}$ a.e. □

Theorem 3.4. *A function $p \in \mathcal{A}$ is the minimizer of (3.6) if and only if p is a discrete weak solution to (3.4).*

Remark 3.1. *Theorem 3.4 implies that there exists a unique discrete weak solution to (3.4).*

Lemma 3.4. *The minimizer p^* of the energy functional (3.6), hereby denoted by $E_{\hat{p}}$, is stable with respect to perturbations in $\hat{p}$. In particular, if we let q^* be the minimizer associated with the energy functional*

$$E_{\hat{q}}(q) = \int_\Omega q^2 \, d\mathbf{x} + h \int_\Omega D(\mathbf{x})|\nabla q|^2 \, d\mathbf{x} - h \int_\Omega G(q) \, d\mathbf{x} - \int_\Omega \hat{q}q \, d\mathbf{x}$$

then we are assured that

$$||p^* - q^*||_2 \leq \frac{1}{\mu} ||\hat{p} - \hat{q}||_2$$

Proof. Since p^* is the minimizer, we know $\partial E_{\hat{p}}(p^*, \nu) = 0$ for all ν. Similarly, $\partial E_{\hat{q}}(q^*, \nu) = 0$ for all ν. As a result of Lemma 3.3 we get the following two inequalities.

$$\begin{aligned} E_{\hat{p}}(q^*) &\geq E_{\hat{p}}(p^*) + \frac{\mu}{2}\|p^* - q^*\|_2^2 \\ E_{\hat{q}}(p^*) &\geq E_{\hat{q}}(q^*) + \frac{\mu}{2}\|p^* - q^*\|_2^2 \end{aligned}$$

We add the two inequalities. After some cancellation we obtain the following inequality.

$$\begin{aligned} -\langle \hat{p}, q^* \rangle - \langle \hat{q}, p^* \rangle &\geq -\langle \hat{p}, p^* \rangle - \langle \hat{q}, q^* \rangle + \mu\|p^* - q^*\|_2^2 \\ \langle \hat{p}, p^* - q^* \rangle - \langle \hat{q}, p^* - q^* \rangle &\geq \mu\|p^* - q^*\|_2^2 \\ \langle \hat{p} - \hat{q}, p^* - q^* \rangle &\geq \mu\|p^* - q^*\|_2^2 \end{aligned}$$

We use the Cauchy-Schwarz's inequality to conclude

$$\|p^* - q^*\|_2 \leq \frac{1}{\mu}\|\hat{p} - \hat{q}\|_2$$

which is the desired inequality. □

4 Bivariate Spline Approximation of the Discrete Weak Solution

4.1 The Discrete Weak Solution in Finite Dimensional Space

So far we have established that there exists a unique discrete weak solution to the problem posed in (3.4). Our next goal is to find an approximate solution in a finite-dimensional spline space. That is, we will approximate p and $\hat{p}$ by using the spline space $S_d^r(\triangle)$ defined as follows.

Definition 4.1 (Spline Space)**.** *Let $\triangle$ be a given triangulation of a domain Ω. Then we define the spline space of smoothness r and degree d over $\triangle$ by,*

$$S_d^r(\triangle) = \{s \in C^r(\Omega) \mid s|_T \in \mathcal{P}_d, \ \forall \ T \in \triangle\},$$

where $\mathcal{P}_d$ is the space of polynomials of degree at most d.

We shall denote the basis of this space as $\{\phi_j\}_{1\leq j\leq n}$. We now set out to find $p^* \in S_d^r(\triangle)$ which satisfies the following equation.

$$\int_\Omega pq \, d\mathbf{x} + h \int_\Omega D(\mathbf{x})\nabla p \cdot \nabla q \, d\mathbf{x} = \int_\Omega \hat{p}q \, d\mathbf{x} + h \int_\Omega pF(p)q \, d\mathbf{x} \qquad \forall q \in S_d^r(\triangle) \tag{4.1}$$

Theorem 4.1. *If h is small enough, then there exists $p^* \in S_d^r(\triangle)$ which satisfies (4.1).*

Proof. The proof of this theorem is constructive and we only give an overview of the construction here. The detail is contained in the rest of this subsection and the next subsection. We first devise an iterative computational scheme. Each iteration requires solving a simple linear equation, for which we can guarantee the existence of such iterative solution. We then show that this sequence of iterative solutions actually forms a Cauchy sequence. Thus, the sequence converges to a spline in $S_d^r(\triangle)$ which is a finite dimensional, and hence a complete space. Finally, by simply taking limits as the number of iteration goes to infinity, we demonstrate that we get a discrete weak spline solution satisfying (4.1). □

We shall need the following

Theorem 4.2. *The weak solution of (4.1) is unique.*

Proof. The proof is analogous to the one in Theorem 3.3. Detail is omitted here. □

4.2 Our Computational Scheme

At each time step t_i, we have to solve the nonlinear problem (4.1). Our approach is to linearize the equation using a fixed-point method.

Algorithm 4.1. *Writing $\hat{p} = p(\mathbf{x}, i-1)$ or $\hat{p} = p_0(\mathbf{x})$, the initial value, find $p^{(k)} := p^{(i,k)}, k \geq 1$ such that*

$$\int_\Omega p^{(k)} q + hD \int_\Omega \nabla p^{(k)} \cdot \nabla q = \langle \hat{p}, q \rangle + h \int_\Omega p^{(k)} F\left(p^{(k-1)}\right) q \, d\mathbf{x} \quad \forall q \in S_d^r(\triangle) \tag{4.2}$$

for $k = 1, 2, \cdots$, until a given accuracy for $\|p^{(k)} - p^{(k-1)}\|$ is met.

Remark 4.1. *We stated in the outline of the proof for Theorem 4.1 that we will show the sequence of $p^{(k)}$ is Cauchy and hence converges to a limit $p^* \in S_d^r(\triangle)$. Note that in (4.2), we can take the limit as $k \to \infty$ of both sides and obtain precisely (4.1). This requires the use of the Dominated Convergence Theorem and so we prove boundedness of all the iterates in Theorem 4.3.*

Lemma 4.1. *Given splines $p^{(k-1)}$ and $\hat{p}$, there exists a unique spline solution for $p^{(k)}$ in equation (4.2).*

Proof. Let ϕ_j be any spline basis function. Any spline function in $S_d^r(\triangle)$ can be written as $\sum_{i=1}^n c_i \phi_i$. Let ϕ_j be any spline basis function. Let $\vec{c}$ be the vector of coefficients for $p^{(k)}$ and $\vec{p}$ be the vector of coefficients for $\hat{p}$. Define the following matrices.

$$\begin{aligned} M(i,j) &:= \int_\Omega \phi_i \phi_j \, d\mathbf{x} \\ K_D(i,j) &:= \int_\Omega D(\mathbf{x}) \nabla \phi_i \cdot \nabla \phi_j \, d\mathbf{x} \\ M_{F(p^{(k-1)})}(i,j) &:= \int_\Omega F(p^{(k-1)}) \phi_i \phi_j \, d\mathbf{x} \end{aligned}$$

Note that all these matrices are symmetric. In addition, M is positive-definite.

We have to solve (4.2) for each $q \in S_d^r(\triangle)$, but it's sufficient to solve for each basis spline ϕ_j. Thus, we have n equations and n unknowns in the coefficient vector, which is equivalent to the following linear system.

$$\begin{aligned} M\vec{c} + hK_D\vec{c} &= M\vec{p} + hM_{F(p^{(k-1)})}\vec{c} \\ \left(M + hK_D - hM_{F(p^{(k-1)})}\right)\vec{c} &= M\vec{p} \end{aligned}$$

Let $L = M + hK_D - hM_{F(p^{(k-1)})}$. M is positive-definite and invertible. If h is small enough, L is also invertible. Thus, we can solve for $\vec{c}$, the spline coefficients of $p^{(k)}$. □

Theorem 4.3. *If $h < 1/M$, then the successive solutions $p^{(k)}$ of the equation (4.2) satisfy*

$$\left|\left|p^{(k)}\right|\right|_2 \leq \frac{1}{1-hM} \left|\left|\hat{p}\right|\right|_2 \tag{4.3}$$

$$\left|\left|\nabla p^{(k)}\right|\right|_2 \leq \frac{1}{\sqrt{hK}} \sqrt{\left|\left|p^{(k)}\right|\right|_2 \left(\left|\left|\hat{p}\right|\right|_2 - (1-hM)\left|\left|p^{(k)}\right|\right|_2\right)} \tag{4.4}$$

If we substitute the estimate from (4.3) into (4.4), we obtain a bound which is less sharp but is independent of k.

$$\left|\left|\nabla p^{(k)}\right|\right|_2 \leq \frac{1}{\sqrt{hK}} \sqrt{\left|\left|p^{(k)}\right|\right|_2 \left|\left|\hat{p}\right|\right|_2} \leq \frac{1}{\sqrt{hK(1-hM)}} \left|\left|\hat{p}\right|\right|_2$$

Proof. Substitute $q = p$ into (4.2). Then

$$\left|\left|p^{(k)}\right|\right|_2^2 + h \underbrace{\int_\Omega D(\mathbf{x}) |\nabla p^{(k)}|^2 \, d\mathbf{x}}_{\geq 0} = \langle \hat{p}, p^{(k)} \rangle + h \int_\Omega F(p^{(k-1)})(p^{(k)})^2 \, d\mathbf{x}$$

Use the Cauchy-Schwarz inequality and the fact that $F(p) \leq M$ for any p.

$$\begin{aligned}
\left|\left|p^{(k)}\right|\right|_2^2 &\leq \left|\left|\hat{p}\right|\right|_2 \left|\left|p^{(k)}\right|\right|_2 + hM \left|\left|p^{(k)}\right|\right|_2^2 \\
\left|\left|p^{(k)}\right|\right|_2 &\leq \left|\left|\hat{p}\right|\right|_2 + hM \left|\left|p^{(k)}\right|\right|_2 \\
\left|\left|p^{(k)}\right|\right|_2 &\leq \frac{1}{1-hM} \left|\left|\hat{p}\right|\right|_2
\end{aligned}$$

Now we prove the bound for $\nabla p^{(k)}$ by substituting $q = p$ once more into (4.2).

$$\begin{aligned}
\left|\left|p^{(k)}\right|\right|_2^2 + h \int_\Omega D(\mathbf{x}) |\nabla p^{(k)}|^2 \, d\mathbf{x} &= \langle \hat{p}, p^{(k)} \rangle + h \int_\Omega F(p^{(k-1)})(p^{(k)})^2 \, d\mathbf{x} \\
\left|\left|p^{(k)}\right|\right|_2^2 + hK \left|\left|\nabla p^{(k)}\right|\right|_2^2 &\leq \left|\left|\hat{p}\right|\right|_2 \left|\left|p^{(k)}\right|\right|_2 + hM \left|\left|p^{(k)}\right|\right|_2^2 \\
hK \left|\left|\nabla p^{(k)}\right|\right|_2^2 &\leq \left|\left|\hat{p}\right|\right|_2 \left|\left|p^{(k)}\right|\right|_2 - \left|\left|p^{(k)}\right|\right|_2^2 + hM \left|\left|p^{(k)}\right|\right|_2^2 \\
hK \left|\left|\nabla p^{(k)}\right|\right|_2^2 &\leq \left|\left|p^{(k)}\right|\right|_2 \left(\left|\left|\hat{p}\right|\right|_2 - (1-hM)\left|\left|p^{(k)}\right|\right|_2\right) \\
\left|\left|\nabla p^{(k)}\right|\right|_2 &\leq \frac{1}{\sqrt{hK}} \sqrt{\left|\left|p^{(k)}\right|\right|_2 \left(\left|\left|\hat{p}\right|\right|_2 - (1-hM)\left|\left|p^{(k)}\right|\right|_2\right)}
\end{aligned}$$

□

Remark 4.2. *The constant in the bound for $\nabla p^{(k)}$, which can be found under the square root, is non-negative as a result of the bound for $p^{(k)}$. In fact, it can be very close to zero.*

Remark 4.3. *Since we are now working within a finite-dimensional space, all norms are equivalent. As a result, we have just established that p and its derivatives are bounded functions. That is,*

$$\left|\left|p^{(k)}\right|\right|_\infty \leq \frac{C}{1-hM} \left|\left|\hat{p}\right|\right|_2$$

Theorem 4.4. *If h is small enough so that*

$$hL \frac{C}{(1-hM)^2} \left|\left|\hat{p}\right|\right|_2 < 1$$

where C is the constant from Remark 4.3, then successive iterates of (4.2) are Cauchy in $L^2(\Omega)$. That is,

$$\left|\left|p^{(k)} - p^{(k-1)}\right|\right|_2 \leq \alpha \left|\left|p^{(k-1)} - p^{(k-2)}\right|\right|_2$$

where $0 < \alpha < 1$.

Proof. Take two successive solutions which satisfy the following equations.

$$\int_\Omega p^{(k)} q \, d\mathbf{x} + h \int_\Omega D(\mathbf{x}) \nabla p^{(k)} \cdot \nabla q \, d\mathbf{x} = \int_\Omega \hat{p} q \, d\mathbf{x} + h \int_\Omega p^{(k)} F(p^{(k-1)}) q \, d\mathbf{x}$$
$$\int_\Omega p^{(k-1)} q \, d\mathbf{x} + h \int_\Omega D(\mathbf{x}) \nabla p^{(k-1)} \cdot \nabla q \, d\mathbf{x} = \int_\Omega \hat{p} q \, d\mathbf{x} + h \int_\Omega p^{(k-1)} F(p^{(k-2)}) q \, d\mathbf{x}$$

Subtract the two equations and substitute $q = p^{(k)} - p^{(k-1)}$.

$$\left|\left|p^{(k)} - p^{(k-1)}\right|\right|_2^2 + h \underbrace{\int_\Omega D(\mathbf{x}) |\nabla p^{(k)} - \nabla p^{(k-1)}|^2 \, d\mathbf{x}}_{\geq 0}$$
$$= h \int_\Omega \left(F(p^{(k-1)}) p^{(k)} - F(p^{(k-2)}) p^{(k-1)} \right) (p^{(k)} - p^{(k-1)}) \, d\mathbf{x}$$

Add and subtract $F(p^{(k-1)})$ and rearrange.

$$\left|\left|p^{(k)} - p^{(k-1)}\right|\right|_2^2 \leq h \int_\Omega F(p^{(k-1)}) \left(p^{(k)} - p^{(k-1)} \right)^2$$
$$+ h \int_\Omega \left(F(p^{(k-1)}) - F(p^{(k-2)}) \right) p^{(k-1)} (p^{(k)} - p^{(k-1)}) \, d\mathbf{x}$$

Use remark 4.3 to bound $|p^{(k-1)}|$.

$$\left|\left|p^{(k)} - p^{(k-1)}\right|\right|_2^2 \leq hM \left|\left|p^{(k)} - p^{(k-1)}\right|\right|_2^2$$
$$+ h \frac{C}{1 - hM} ||\hat{p}||_2 \int_\Omega \left| F(p^{(k-1)}) - F(p^{(k-2)}) \right| \left| p^{(k)} - p^{(k-1)} \right| \, d\mathbf{x}$$
$$(1 - hM) \left|\left|p^{(k)} - p^{(k-1)}\right|\right|_2^2 \leq h \frac{C}{1 - hM} ||\hat{p}||_2 \int_\Omega \left| F(p^{(k-1)}) - F(p^{(k-2)}) \right| \left| p^{(k)} - p^{(k-1)} \right| \, d\mathbf{x}$$

$F(p)$ is a differentiable function and by Remark 4.3, it has a bounded derivative on the compact interval $\left[0, \sup_k \left|\left|p^{(k)}\right|\right|_\infty\right]$. Thus, $F(p)$ is Lipschitz continuous with constant L_F.

$$\leq h L_F \frac{C}{1 - hM} ||\hat{p}||_2 \int_\Omega \left| p^{(k-1)} - p^{(k-2)} \right| \left| p^{(k)} - p^{(k-1)} \right| \, d\mathbf{x}$$

Apply the Cauchy-Schwartz inequality.

$$(1 - hM) \left|\left|p^{(k)} - p^{(k-1)}\right|\right|_2^2 \leq h L_F \frac{C}{1 - hM} ||\hat{p}||_2 \left|\left|p^{(k-1)} - p^{(k-2)}\right|\right|_2 \left|\left|p^{(k)} - p^{(k-1)}\right|\right|_2$$
$$\left|\left|p^{(k)} - p^{(k-1)}\right|\right|_2 \leq h L_F \frac{C}{(1 - hM)^2} ||\hat{p}|| \left|\left|p^{(k-1)} - p^{(k-2)}\right|\right|_2$$

We can choose an h small enough so that $\alpha = hL \dfrac{C}{(1 - hM)^2}$ satisfies $0 < \alpha < 1$. $\square$

4.3 Bivariate Spline Approximation to the Discrete Weak Solution in Sobolev Space

In this subsection, we show that the spline solutions obtained above are a good approximation to the weak solution in (3.4). Let p^* be the weak solution of (3.4) and let S^* be the spline solution which is the limit of the iterative solutions from Algorithm 4.1. By using Lemma 3.3 and noting that $\nabla E(p^*, q) = 0$ for any $q \in H_0^1(\Omega)$, we have

$$E(S^*) - E(p^*) \geq \frac{\mu}{2} ||S^* - p^*||_2^2 \tag{4.5}$$

Let S_{p^*} be the quasi-interpolant of p^* in the spline space $S_d^r(\triangle)$ as in the Appendix. Since S^* is the minimizer of (3.6) with respect to all $q \in S_d^r(\triangle)$, we conclude that $E(S_{p^*}) > E(S^*)$. Together with (4.5) we can write

$$\frac{\mu}{2} ||S^* - p^*||_2^2 \leq E(S_{p^*}) - E(p^*) \tag{4.6}$$

Theorem 4.5. *Suppose that $h > 0$ is small enough and p^*, the weak solution of (3.4), is in $H^{m+1}(\Omega)$ with $m \geq 1$. Then S^*, the limit of the iterative solutions from Algorithm 4.1, approximates p^* in the following sense:*

$$||S^* - p^*||_2 \leq C|\triangle|^m |p^*|_{m+1,2,\Omega} \tag{4.7}$$

where C is a constant.

Proof. We rewrite equation (4.6)

$$\begin{aligned}
\frac{\mu}{2} ||S^* - p^*||_2^2 &\leq \int_\Omega S_{p^*}^2 - (p^*)^2 \, d\mathbf{x} + h \int_\Omega D(\mathbf{x}) \left(|\nabla S_{p^*}|^2 - |\nabla p^*|^2\right) \, d\mathbf{x} + h \int_\Omega G(p^*) - G(S_{p^*}) \, d\mathbf{x} \\
&= \int_\Omega (S_{p^*} - p^*)(S_{p^*} + p^*) \, d\mathbf{x} + h \int_\Omega D(\mathbf{x})(\nabla S_{p^*} - \nabla p^*) \cdot (\nabla S_{p^*} + \nabla p^*) \, d\mathbf{x} \\
&+ h \int_\Omega G(p^*) - G(S_{p^*}) \, d\mathbf{x}
\end{aligned}$$

G is a differentiable function by construction. Since $p^* \in H^2(\Omega)$, by Theorem 2.2 we conclude that p^* is Hölder continuous and hence it has some maximal value M^* on the compact set $\overline{\Omega}$. Analogously, we can conclude the same for S_{p^*}. As a result, $G'(p)$ has a maximum value on the compact set $[0, M^*]$ and so G is Lipschitz continuous with some constant L_G. Continuing where we left off above, we use the Cauchy-Schwarz inequality and L_G:

$$\begin{aligned}
&\leq ||S_{p^*} - p^*||_2 \, ||S_{p^*} + p^*||_2 + hK_2 \, ||\nabla S_{p^*} - \nabla p^*||_2 \, ||\nabla S_{p^*} + \nabla p^*||_2 + hL_G \int_\Omega |p^* - S_{p^*}| \, d\mathbf{x} \\
&\leq C_1 \, ||S_{p^*} - p^*||_2 + hK_2C_2 \, ||\nabla S_{p^*} - \nabla p^*||_2 + hL_G|\Omega|^{1/2} \, ||p^* - S_{p^*}||_2
\end{aligned}$$

where $C_1 = ||S_{p^*}||_2 + ||p^*||_2$, $C_2 = ||\nabla S_{p^*}||_2 + ||\nabla p^*||_2$.

By the approximation property of bivariate spline spaces, Theorem 6.1 in the Appendix, we can write

$$\begin{aligned}
||S_{p^*} - p^*||_2 &\leq C_3|\triangle|^2 |p^*|_{2,2,\Omega} \\
||\nabla S_{p^*} - \nabla p^*||_2 &\leq C_4|\triangle| |p^*|_{2,2,\Omega}
\end{aligned}$$

where $|\triangle|$ is the length of the longest edge in the triangulation and C_3 and C_4 are constants independent of p^*. □

As a corollary, we have that $E(S_{p^*}) - E(p^*) \to 0$ as $|\triangle| \to 0$.

5 Numerical Simulation and Computational Results

We have implemented the computational scheme discussed in the previous section in MATLAB. In this section we will show some of our computational results. Since no exact solutions to this PDE were known, we modify the equation by adding an appropriate forcing term. By doing so we can force any twice differentiable function $p(\mathbf{x}, t)$ to be a solution. Then we can make sure our algorithm recovers it for any given just $p(\mathbf{x}, 0)$, the initial condition. In this way, we are able to verify that our MATLAB code works. Then we remove the forcing term and use the resulting MATLAB code to numerically solve (1.2) for various initial conditions, various diffusive factor $D(\mathbf{x})$, and Allee constant σ.

Although in this paper we focused on the theory for Dirichlet zero boundary condition, the theory holds equally well for Neumann boundary conditions. We tested both boundary conditions numerically. Let us present some of our numerical results.

5.1 Dirichlet Boundary with Forcing

In order to make sure that our implementation works, we use a few test functions over a rectangular domain $\Omega = [0, 1] \times [0, 1]$. These test functions are not weak solutions to the PDE in (3.4), but instead they are the exact solutions to the following modified PDE with forcing term.

$$\frac{dp(\mathbf{x}, t)}{dt} = \operatorname{div}\,(D(\mathbf{x})\nabla p(\mathbf{x}, t))) + p(\mathbf{x}, t)F(p(\mathbf{x}, t)) + f(\mathbf{x}, t), \quad \mathbf{x} = (x, y) \in \Omega \subset \mathbb{R}^2, t \geq 0, \tag{5.1}$$

where $f(\mathbf{x}, t)$ can be computed by using Mathematica. The weak solution p satisfies

$$\begin{aligned} &\int_\Omega p(\mathbf{x}, t_i)q(\mathbf{x})d\mathbf{x} + h\int_\Omega D(\mathbf{x})\nabla p(\mathbf{x}, t_i) \cdot \nabla q(\mathbf{x})d\mathbf{x} \\ &\qquad = \int_\Omega p(\mathbf{x}, t_{i-1})q(\mathbf{x})d\mathbf{x} + \int_\Omega p(\mathbf{x}, t_i)F(p(\mathbf{x}, t_i)d\mathbf{x} + \int_\Omega f(\mathbf{x}, t_i)q(\mathbf{x})d\mathbf{x}. \end{aligned} \tag{5.2}$$

We then ran our MATALB code to recover the function p using bivariate splines of degree d and recorded the maximal error in population density at some fixed time on a 100 by 100 grid. The numerical results are given in Tables 1, 2, where d is the spline degree and $|\triangle|$ is the number of triangles in the triangulation, h is the size of the time step, T refers to how far in time we have evolved. In all cases $A(\mathbf{x}) = 1$ and the domain is $\Omega = [0, 1] \times [0, 1]$.

Table 1: $d = 5$, $T = 5$, $p(t, x) = \dfrac{13x(x-1)y(y-1)}{1+t}$,$D(\mathbf{x}) = 1/200$.

$h\backslash\lvert\triangle\rvert$	2	8	32	128	512
5×10^{-2}	0.039429	0.032977	0.034431	0.034433	0.034433
5×10^{-3}	0.054059	0.0041368	0.0033432	0.0033453	0.0033454
5×10^{-4}	0.055708	0.0055911	3.3120e-004	3.3343e-004	3.3353e-004
5×10^{-5}	0.055873	0.0057463	3.1034e-005		3.3341e-005

In Table 1 we see that in order to reduce the error, it is necessary to reduce both h and the size of the triangulation. A refinement in just one of these parameters, usually has diminishing returns. The error decreases roughly like $O(h)$.

In Table 2 we complicate the model further by using diffusion which varies inside Ω.

In Table 3 we use a solution which is not a polynomial and hence is not exactly representable in spline space.

Table 2: $d = 5$, $T = 1$, $p(t, x) = \dfrac{13x(x-1)y(y-1)}{1+t}$,$D(x, y) = \dfrac{1}{200}e^{-(x-.5)^2-(y-.5)^2}$.

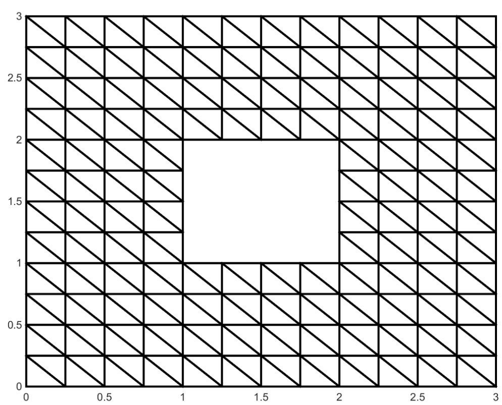
$h\backslash\lvert\triangle\rvert$	2	8	32	128
5×10^{-2}	0.019157	0.016883	0.016599	0.016599
5×10^{-3}	0.0046455	0.0019749	0.0016833	0.0016832
5×10^{-4}	0.0043546	4.7506e-04	1.6861e-04	1.6852e-04

Table 3: $d = 5$, $T = 1$, $p(t, x) = \dfrac{\sin(\pi x)\sin(\pi x)}{1+t}$,$D(\mathbf{x}) = 1/200$.

$h\backslash\lvert\triangle\rvert$	2	8	32	128
5×10^{-2}	0.01982	0.019187	0.018405	0.018398
5×10^{-3}	0.005989	0.0026479	0.0018789	0.0018710
5×10^{-4}	0.005609	9.6351e-04	1.962e-04	1.8732e-04

5.2 Several Simulations of Population Development

From the previous subsection, we have seen that our MATLAB code works well. Thus we removed the forcing terms and ran simulations of the solution of (1.2) for various initial conditions and parameters. We shall use the following two domains shown with a triangulation in Fig. 1



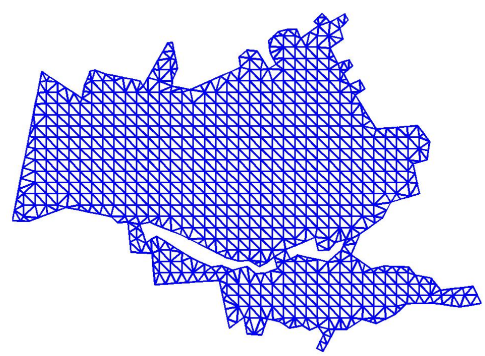


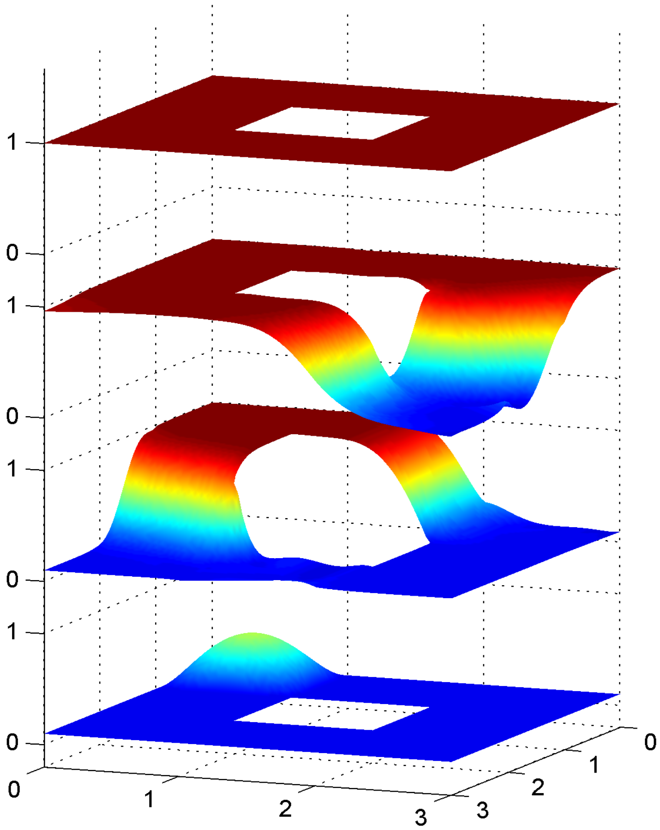
Figure 1: Two Domains with Triangulation for Simulation

We provide several examples to show how various growth functions affect the rate at which the solution reaches the asymptotically stable constant solution of $p(x, y) = 1$ or $p(x, y) = 0$.

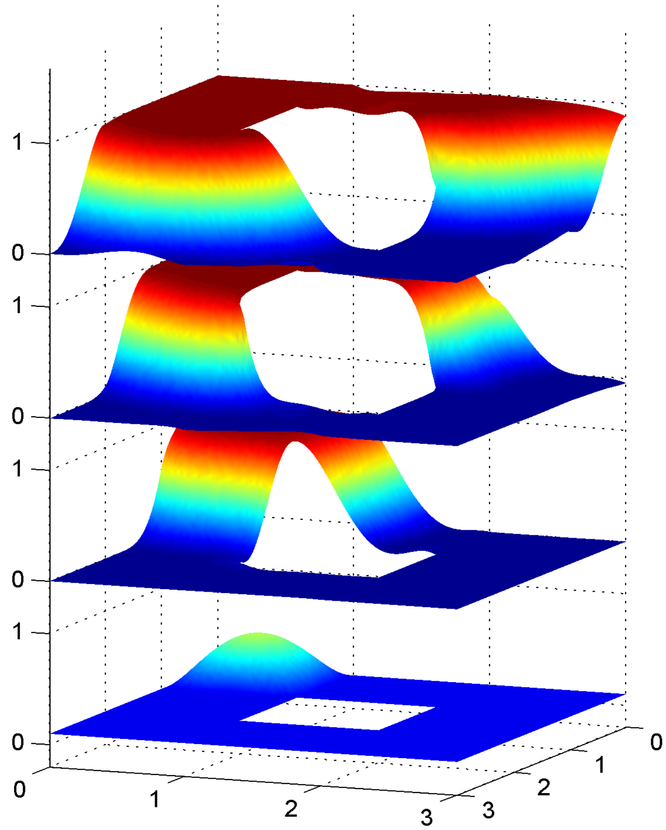
Figures 2 through 5 show several 3D renders of how solutions grow over time over two domains indicated in Fig 1. Each subfigure shows four equally-spaced time slices, plotted on the same xy-axes, one on top of each other, allowing the reader to observe how the solution grows over time. In addition, each figure shows the effect of varying the Allee threshold σ. In order to make the difference in the behavior of the solution clearer, each figure ranges from $t = 0$ to $t = T$, where T is a specified final time.

Figures 6 through show average population over time over a city of Mali. Each subfigure corresponds to a certain set of initial conditions for the PDE, while separating the cases by the choice for σ, emphasizing the effect σ has on the rate at which the population reaches an asymptotically stable solution.

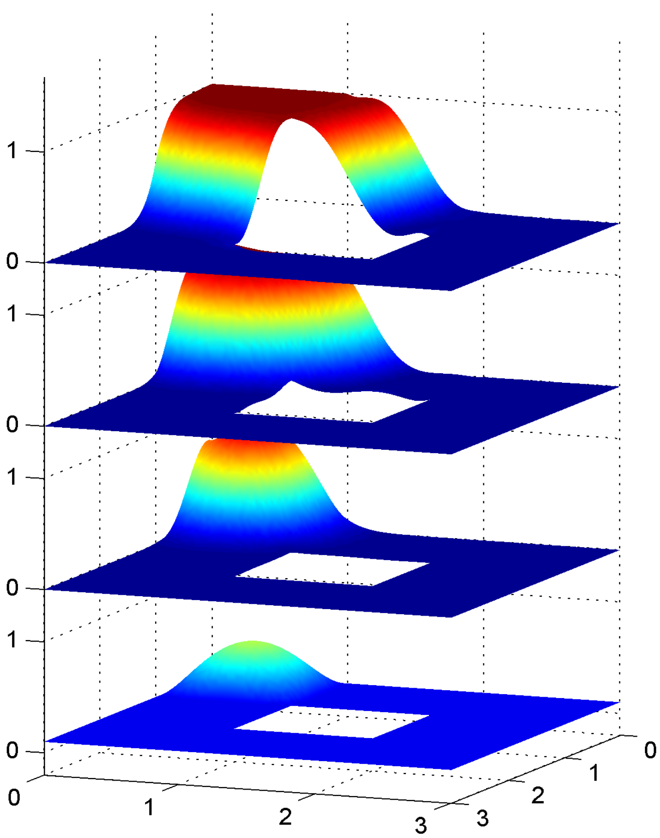
We can observe some expected behavior from the solutions presented in Figure 2. The initial condition



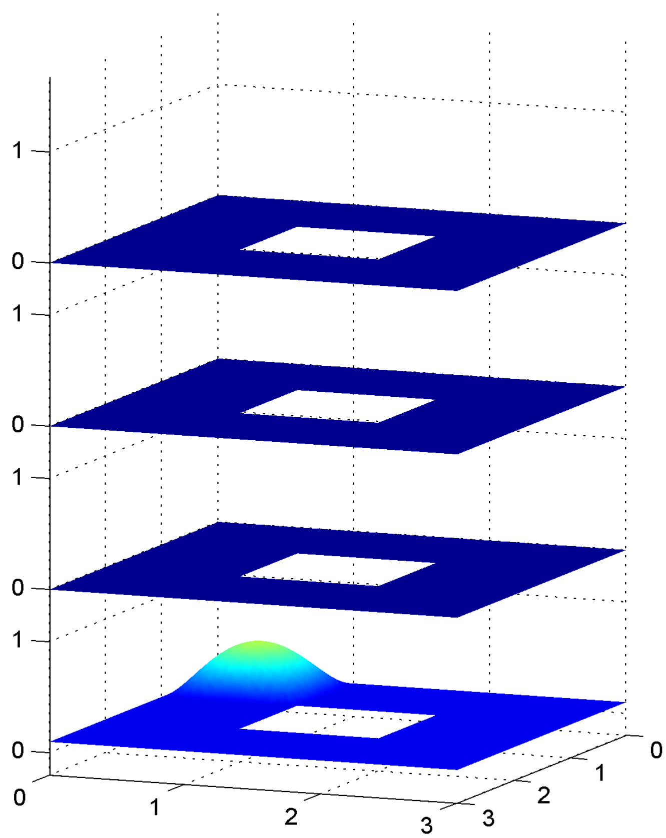
(a) $\sigma = 0.1$



(b) $\sigma = 0.2$

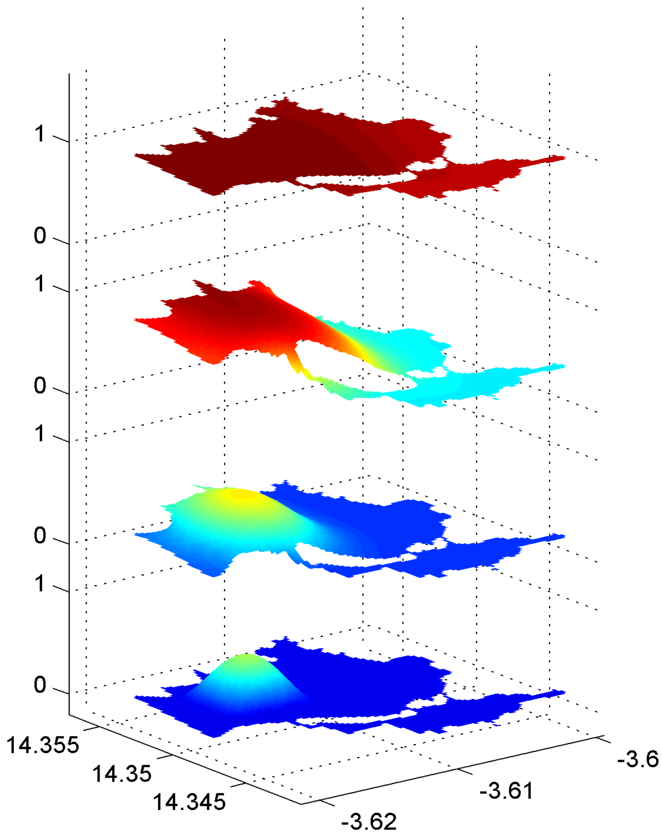


(c) $\sigma = 0.3$

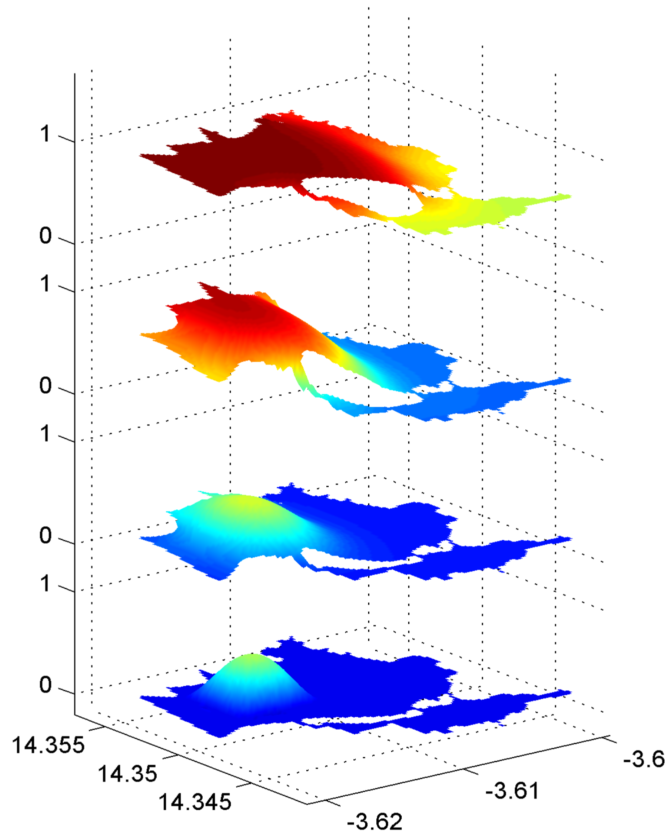


(d) $\sigma = 0.5$

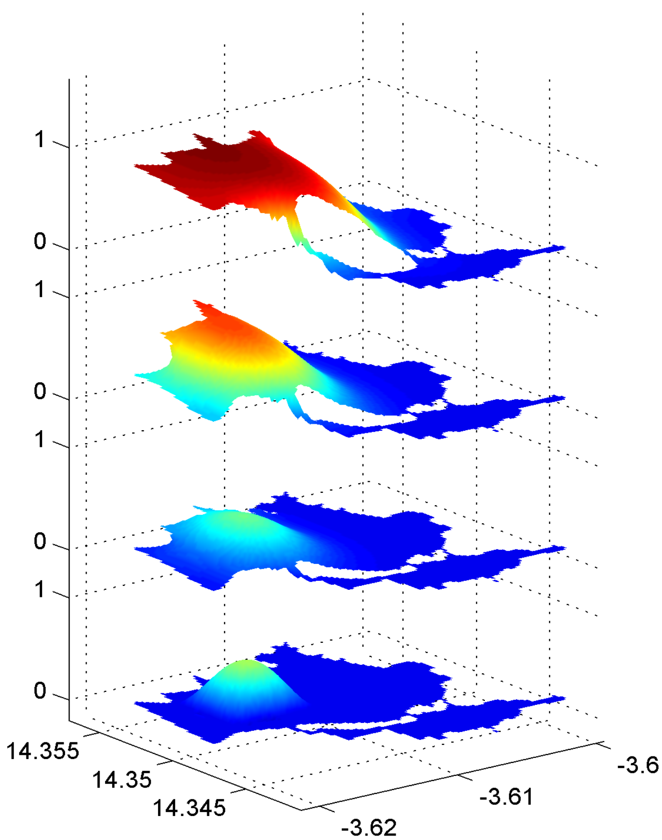
Figure 2: Donut-shape domain. Constant growth and diffusion. Various Allee effect thresholds σ. $T = 90$.



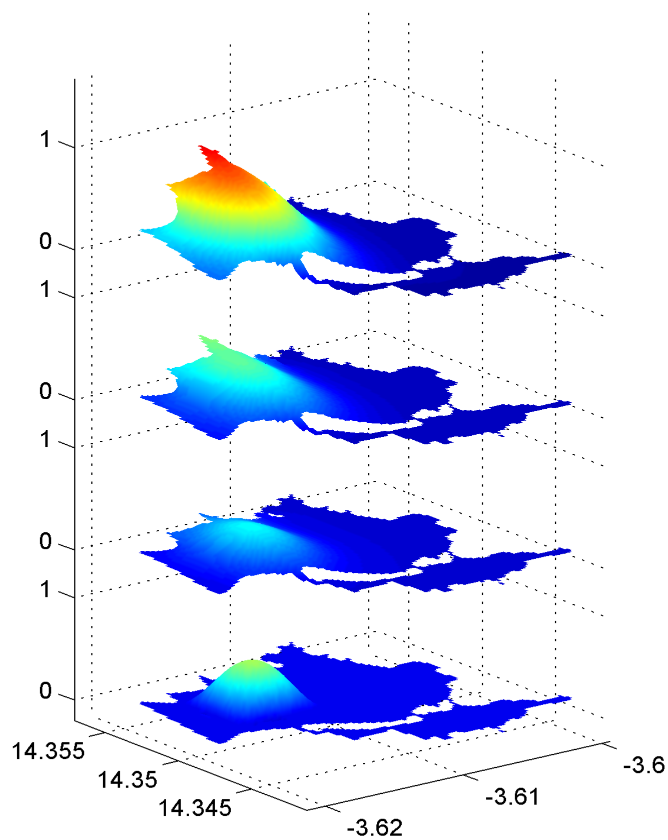
(a) $\sigma = 0.02$



(b) $\sigma = 0.05$

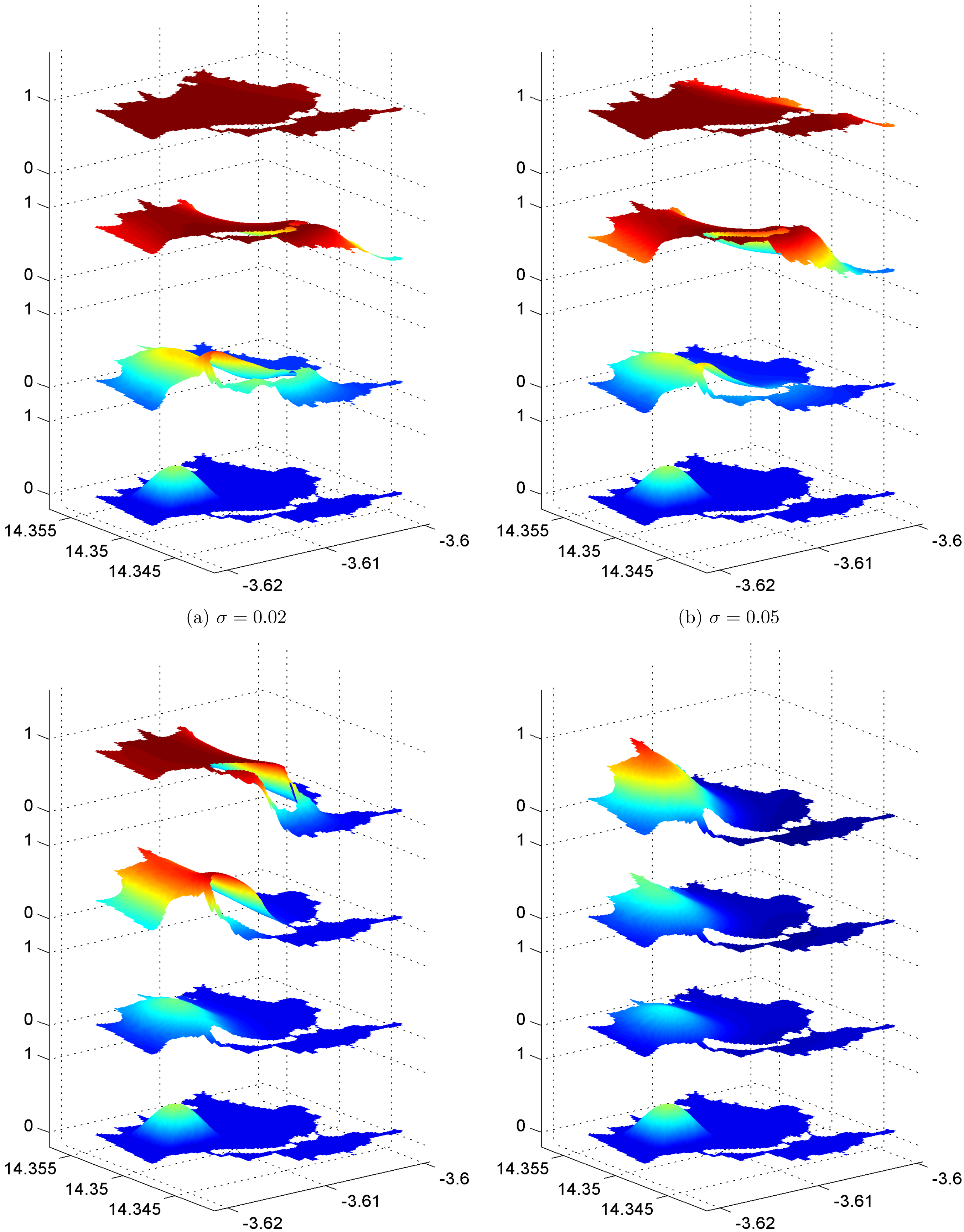


(c) $\sigma = 0.1$



(d) $\sigma = 0.15$

Figure 3: City of Bandiagara, Mali. Constant growth and diffusion. Various Allee effect thresholds σ. Here $T = 20$.



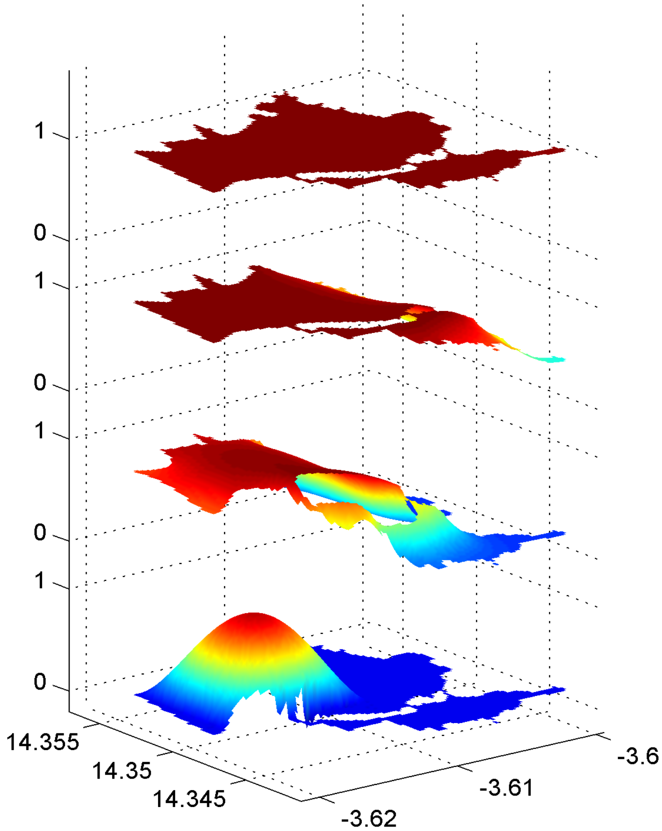
(a) $\sigma = 0.02$

(b) $\sigma = 0.05$

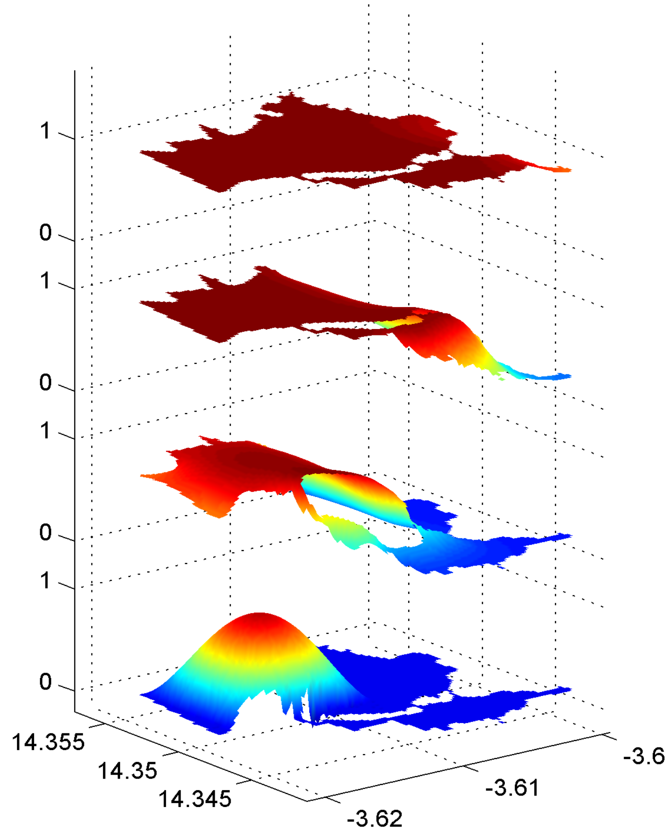
(c) $\sigma = 0.1$

(d) $\sigma = 0.15$

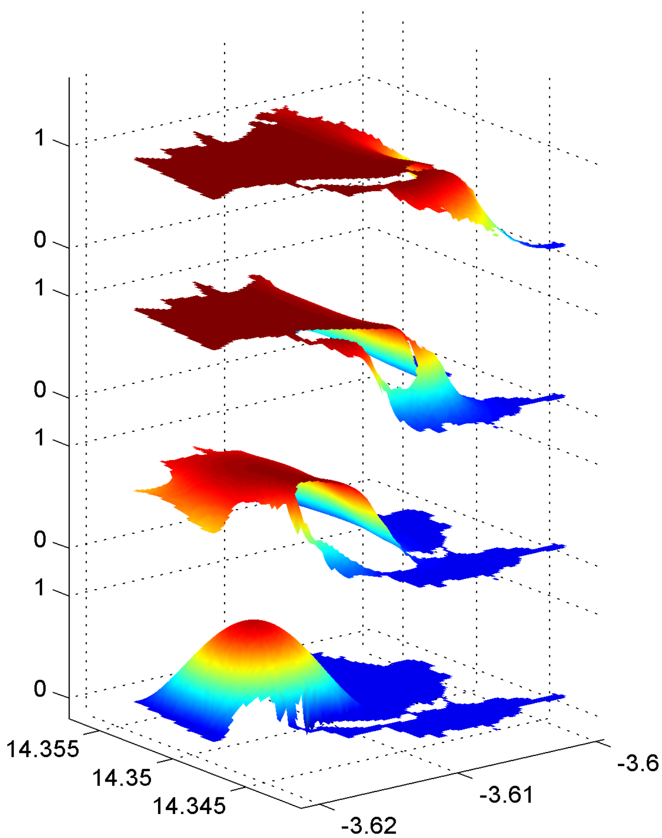
Figure 4: City of Bandiagara, Mali. Constant diffusion. Various Allee effect thresholds. Growth function is piecewise-constant with triple magnitude for patches near the city's river. Here $T = 20$.



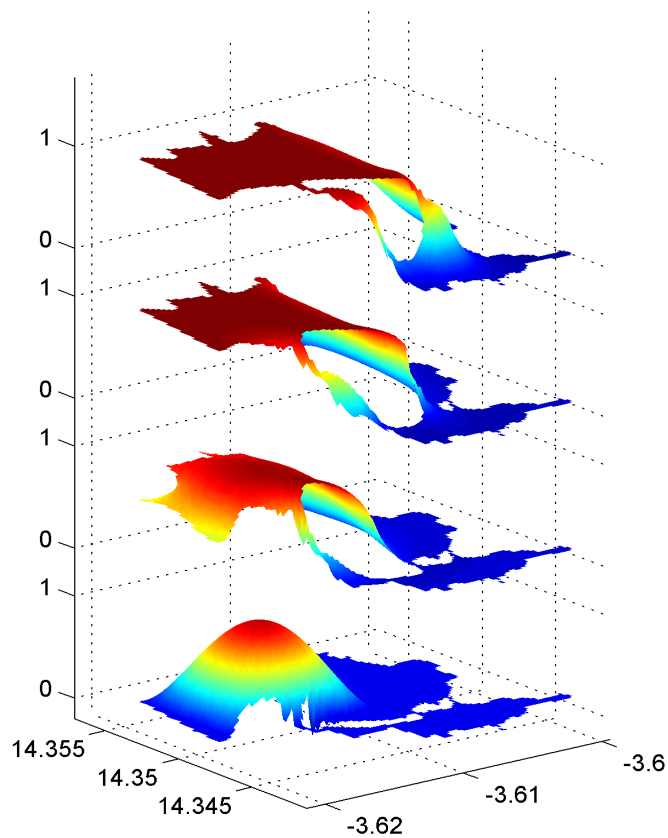
(a) $\sigma = 0.02$



(b) $\sigma = 0.05$

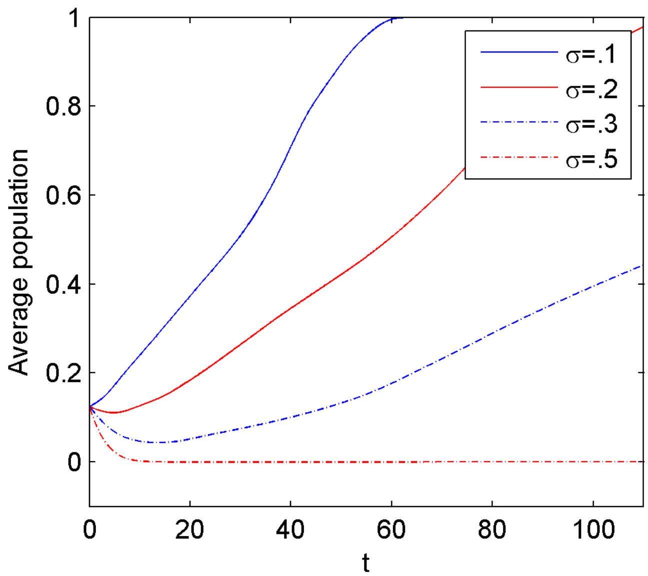


(c) $\sigma = 0.1$

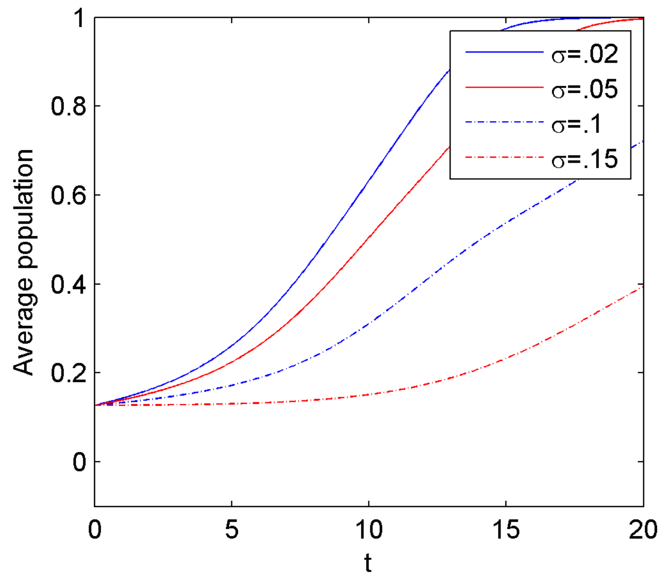


(d) $\sigma = 0.15$

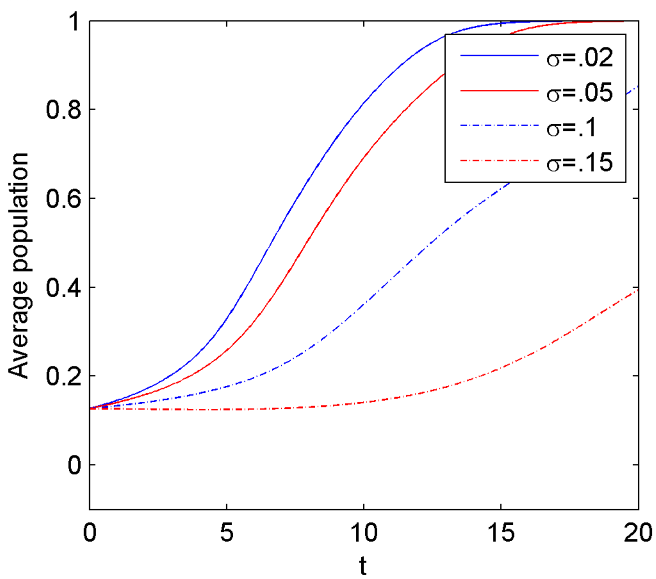
Figure 5: City of Bandiagara, Mali. Same as Figure 4 but the initial condition has a much higher total population. Here $T = 20$.



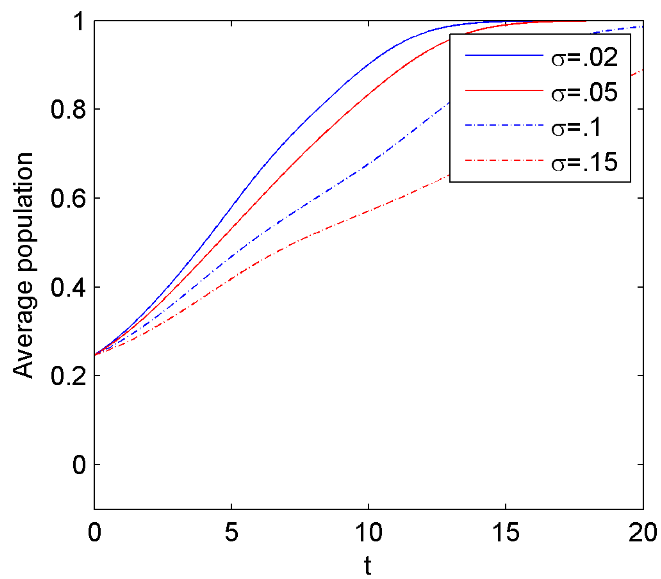
(a) Average population plot for simulations in Figure 2.



(b) Average population plot for simulations in Figure 3.



(c) Average population plot for simulations in Figure 4.



(d) Average population plot for simulations in Figure 5.

Figure 6: Average population density in Ω plotted over time for each of the four preceding figures.

is uniformly $p = 0.1$ on a large portion of Ω with an isolated bump function in one corner. In Figure 2b the second time slice shows the population has become extinct on the area where $p = 0.1$. At the same time the bump grows to population capacity and eventually spreads life into formerly dead areas. We observe similar results in Figure 2c, but the rate at which the population grows has been severely diminished. In Figure 2d, the threshold σ is so high that the population becomes extinct everywhere and very quickly.

References

[1] W. C. Allee, The social life of animals. W. W. Norton and Co. Inc., New York, 1938.

[2] F. Andreu-Vaillo, V. Caselles, and J. M. Mazón.: Parabolic quasilinear equations minimizing linear growth functionals. Birkhäuser, 2004.

[3] F. Andreu, V. Caselles, and J. M. Mazón, A Fisher-Kolmogorov equation with finite speed of propagation, J. Diff. Eq. 248(2010), 2528–2561.

[4] G. Awanou, M. J. Lai, and P. Wenston: The multivariate spline method for numerical solution of partial differential equations and scattered data interpolation. In *Wavelets and Splines* (G. Chen and M. J. Lai, eds.). Nashboro Press, 24–74, (2006).

[5] F. Ciarlet, The Finite Element Method for Elliptic Problems, North- Holland, Amsterdam, New York, 1978.

[6] R. Cantrell and C. Cosner, *Spatial Ecology via Reaction-Diffusion Equations*, John Wiley & Sons, England, 2003.

[7] R. Cantrell and C. Cosner, Implications of model formulation when transitioning from spatial to landscape ecology, Math. Biosciences Engineering, 2011.

[8] I. Ekeland and R. Temam, Convex Analysis and Variational Problem, Cambridge University Press, 1999.

[9] L. C. Evans, Partial Differential Equations, American Mathematical Society, 2002, pp 629–631.

[10] G. Farin, Triangular Bernstein-Bézier patches, Comput. Aided Geom. Design, 3 (1986), 83–127.

[11] R. Fisher, The advance of advantageous genes, Ann. of Eugenics 7 (1937) 355?-369.

[12] E. J. Fuselier and G. B. Wright, A high-order kernel method for diffusion and reaction-diffusion equations on surface, J. Sci. Comput. 56 (2013), 535–565.

[13] G. F. Gause. Experimental demonstration of Volterra's periodic oscillations in the numbers of animals. *Journal of Experimental Biology*, 12(1):44–48, 1935.

[14] X.-L. Hu, D.-F. Han, M.-J. Lai, Bivariate Splines of Various Degrees for Numerical Solution of Partial Differential Equations, SIAM Journal of Scientific Computing, 29(2007) 1338–1354.

[15] C.B. Huffaker. Biological control of weeds with insects. *Annual Review of Entomology*, 4:251–276, 1959.

[16] R. Kobayashi, J. A. Warren, and W. C. Carter, A simple continuum model of grain boundaries. Physica D (2000) 140, 141-150.

[17] A. Kolmogorov, I. Petrovskii, N. Piskunov, Edude de l?equation de la quantite de la matriere et son application a un probleme biologique, Moscow Univ. Math. Bull. 1 (1937) 1?-25.

[18] M. J. Lai, C. Liu, and P. Wenston, Numerical simulations on two nonlinear biharmonic evolution equations, Applicable Analysis, vol. 83 (2004) pp. 563–577

[19] M. J. Lai, C. Liu, and P. Wenston, On two nonlinear biharmonic evolution equations: existence, uniqueness and stability, Applicable Analysis, vol. 83 (2004) pp. 541–562

[20] M. J. Lai and L. L. Schumaker, Approximation power of bivariate splines, Advances in Comput. Math., 9(1998), pp. 251–279.

[21] M. J. Lai and L. L. Schumaker, *Spline Functions on Triangulations*, Cambridge Univ. Press, (2007).

[22] M. J. Lai and P. Wenston, Bivariate splines for fluid flows, Computers and Fluids, vol. 33 (2004) pp. 1047–1073.

[23] Lewis, M. A., and Kareiva, P. (1993). Allee dynamics and the spread of invading organisms. Theoretical Population Biology, 43(2), 141–158.

[24] J. D. Murray, *Mathematical Biology: I. An Introduction*, Springer Verlag, 2001.

[25] J. D. Murray, *Mathematical Biology: II Spatial Models and Bio-Medical Applications*, Springer Verlag, 2003.

[26] T.R. Malthus, *An Essay on the Principle of Population; or a view of its pasts and present effects on human happiness; with an inquiry into our prospects respecting the future removal or mitigation of the evils which it occasions*, J. Murray, London, 6^{th} edition, 1826. (First edition: 1798).

[27] Ma, W. X., and Fuchssteiner, B. (1996). Explicit and exact solutions to a Kolmogorov-Petrovskii-Piskunov equation. International Journal of Non-linear Mechanics, 31(3), 329–338.

[28] Roques, L., Roques, A., Berestycki, H., and Kretzschmar, A. (2008). A population facing climate change: joint influences of Allee effects and environmental boundary geometry. Population ecology, 50(2), 215–225.

[29] Richter, O., Moenickes, S., and Suhling, F. (2012). Modelling the effect of temperature on the range expansion of species by reactiondiffusion equations. Mathematical biosciences, 235(2), 171–181.

[30] J.G. Skellam, Random dispersal in theoretical populations, *Biometrika*, 38:433–435, 1951.

[31] X.-C. Tai et al., eds., *Scale Space and Variational Methods in Computer Vision*, Lecture Notes in Computer Science, Vol. 5567, Springer Berlin, Heidelberg, 2009.

[32] P.F. Verhulst. Notice sur la loi que la population poursuit dans son accroissemen. *Correspondance mathématique et physique*, 10:113–121, 1838.

6 Appendix: Preliminary on Bivariate Splines

In this section, we explain bivariate spline functions of any degree d and smoothness $r \geq 1$ over arbitrary triangulation $\triangle$. Most of the following discussion can be found in [21]. We outline these functions here just for convenience. Let Ω be a polygonal domain in $\mathbb{R}^2$ and $\triangle$ a triangulation of Ω. That is, $\triangle$ is a finite collection of triangles $T \subset \Omega$ such that $\cup_{T\in\triangle} T = \Omega$ and the intersection of any two triangles is either the empty set, a common edge, or a common vertex. For each $T \in \triangle$, let $|T|$ denote the length of the longest edge of T, and let ρ_T be the radius of the inscribed circle of T. The longest edge length in the triangulation $\triangle$ is denoted by $|\triangle|$ and is referred to as the size of the triangulation. For any triangulation $\triangle$ we define its shape parameter by

$$\kappa_\triangle := \frac{|\triangle|}{\rho_\triangle}, \tag{6.1}$$

where $\rho_\triangle$ is the minimum of the radii of the in-circles of the triangles of $\triangle$. The shape parameter for a single triangle, κ_T, satisfies

$$\kappa_T := \frac{|T|}{\rho_T} \le \frac{2}{\tan(\theta_T/2)} \le \frac{2}{\sin(\theta_T/2)}, \tag{6.2}$$

where θ_T is the smallest angle in the triangle T. The shape of a given triangulation affects how well we can approximate a function over the triangulation. Hence we have the following definition of a β-quasi-uniform triangulation.

Definition 6.1 (β-Quasi-Uniform Triangulation)**.** *Let $0 < \beta < \infty$. A triangulation $\triangle$ is a β-quasi-uniform triangulation provided that*

$$\frac{|\triangle|}{\rho_\triangle} \le \beta.$$

Once we have a triangulation, we define the spline space of degree d and smoothness r over that triangulation as follows:

Definition 6.2 (Spline Space)**.** *Let $\triangle$ be a given triangulation of a domain Ω. Then we define the spline space of smoothness r and degree d over $\triangle$ by,*

$$S^r_d(\triangle) = \{s \in C^r(\Omega) \mid s|_T \in \mathcal{P}_d,\ \forall\ T \in \triangle\},$$

where $\mathcal{P}_d$ is the space of polynomials of degree at most d.

We next explain how to represent a spline function in $S^r_d(\triangle)$. Let $T = \langle (x_1, y_1), (x_2, y_2), (x_3, y_3)\rangle$. For any point (x, y), let b_1, b_2, b_3 be the solution of

$$\begin{aligned} x &= b_1x_1 + b_2x_2 + b_3x_3 \\ y &= b_1y_1 + b_2y_2 + b_3y_3 \\ 1 &= b_1 + b_2 + b_3. \end{aligned}$$

(b_1, b_2, b_3) is the so-called barycentric coordinates of (x, y) with respect to T. Note that b_i is a linear polynomial of (x, y) for $i = 1, 2, 3$. Fix a degree $d > 0$. For $i + j + k = d$, let

$$B^T_{ijk}(x, y) = \frac{d!}{i!j!k!} b_1^i b_2^j b_3^k$$

which is called Bernstein-Bézier polynomial. Let

$$S|_T = \sum_{i+j+k=d} c^T_{ijk} B^T_{ijk}(x, y).$$

We use $\mathbf{s} = (c^T_{ijk}, i + j + k = d, T \in \triangle)$ to represent the coefficient vector for spline function $S \in S^{-1}_d(\triangle)$. In order to make $S \in S^0_d(\triangle)$, we have to construct a smoothness matrix H such that $H\mathbf{s} = 0$ ensure that S is a continuous function. Such a smoothness matrix is known and in fact it is known for any smoothness $r \ge 0$ (cf. [10]).

Note that Bernstein-Bézier representation of spline functions is very convenient for basic evaluation, derivatives and integration. We use the de Casteljau algorithm to evaluate a Bernstein-Bézier polynomial at any point inside the triangle. It is a simple and stable computation. See [21]. Let $T = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\rangle$ and $S|_T = \sum_{i+j+k=d} c_{ijk} B_{ijk}(x, y)$. Then the directional derivative $D_{\mathbf{v}_2-\mathbf{v}_1} S|_T$ is

$$D_{\mathbf{v}_2-\mathbf{v}_1} S|_T = d \sum_{i+j+k=d-1} (c_{i,j+1,k} - c_{i+1,j,k}) B_{ijk}(x, y).$$

Similar for $D_{\mathbf{v}_3-\mathbf{v}_1}S|_T$. D_x and D_y are linearly combinations of these two directional derivatives. Let s be a spline in $S^r_d(\triangle)$ with $s|_T = \sum_{i+j+k=d} c^T_{ijk} B_ijk(x,y), T \in \triangle$. Then

$$\int_\Omega s(x,y)dxdy = \sum_{T\in\triangle} \frac{A_T}{\binom{d+2}{2}} \sum_{i+j+k=d} c^T_{ijk}.$$

If $p = \sum_{i+j+k=d} a_{ijk}B_{ijk}(x,y)$ and $q = \sum_{i+j+k=d} b_{ijk}B_{ijk}(x,y)$ over a triangle T, then

$$\int_T p(x,y)q(x,y)dxdy = \mathbf{a}^\top M_d \mathbf{b},$$

where $\mathbf{a} = (a_{ijk}, i+j+k=d)^\top$, $\mathbf{b} = (b_{ijk}, i+j+k=d)^\top$, M_d is a symmetric matrix with known entries (a formula for these entries is known (cf. [21]). These elementary operations have been implemented in MATLAB. See [4]. Many different linear and nonlinear partial differential equations have been solved by using these bivariate spline functions. See [22], [4], [14].

When $d \geq 3r+2$ the spline space $S^r_d(\triangle)$ possesses an optimal approximation order which is achieved by the use of a quasi-interpolation operator. Let $\|f\|_{L_p(\Omega)}$ denote the usual L_p norm of f over Ω, $|f|_{m,p,\Omega}$ denotes the L_p norm of the m^{th} derivatives of f over Ω, and $W^{m+1}_p(\Omega)$ stands for the usual Sobolev space over Ω.

To define the quasi-interpolation operator we need linear functionals $\{\lambda_{ijk,T}\}_{i+j+k=d}, T \in \triangle$ which are based on values of f at the set of domain points over triangles in $\triangle$, that is

$$\lambda_{ijk,T}(f) = \sum_{|\nu|=d} a^{ijk}_\nu f(\xi^T_\nu), \tag{6.3}$$

where $\xi^T_\nu = (i\mathbf{v}^T_1 + j\mathbf{v}^T_2 + k\mathbf{v}^T_3)/d$ for $\nu = (i,j,k)$ with $i+j+k=d$ and $\mathbf{v}_i, i = 1,2,3$ are vertexes of triangle T.

A quasi-interpolation operator of f is defined by

$$Qf := \sum_{T\in\triangle} \sum_{i+j+k=d} \lambda_{ijk,T}(f) B^T_{ijk}. \tag{6.4}$$

Now, we are ready to state a theorem on optimal approximation order (cf. [20] and [21]).

Theorem 6.1 (Optimal Approximation Order)**.** *Assume $d \geq 3r+2$ and let $\triangle$ be a triangulation of Ω. Then there exists a quasi-interpolatory operator $Qf \in S^r_d(\triangle)$ mapping $f \in L_1(\Omega)$ into $S^r_d(\triangle)$ such that Qf achieves the optimal approximation order: if $f \in W^{m+1}_p(\Omega)$,*

$$\|D^\alpha_x D^\beta_y (Qf - f)\|_{L_p(\Omega)} \leq C|\triangle|^{m+1-\alpha-\beta}|f|_{m+1,p,\Omega} \tag{6.5}$$

for all $\alpha+\beta \leq m+1$ with $0 \leq m \leq d$, where D_x and D_y denote the derivatives with respect to the first and second variables and the constant C depends only on the degree d and the smallest angle $\theta_\triangle$ and may be dependent on the Lipschitz condition on the boundary of Ω.

We sometimes need to use the so-called Markov inequality to compare the size of the derivative of a polynomial with the size of the polynomial itself on a given triangle t. As a spline function is a piecewise polynomial function, this inequality can be also applied to any spline function. See [21] for a proof.

Theorem 6.2. *Let $t := \langle v_1, v_2, v_3\rangle$ be a triangle, and fix $1 \leq q \leq \infty$. Then there exists a constant K depending only on d such that for every polynomial $p \in \mathcal{P}_d$, and any nonnegative integers α and β with $0 \leq \alpha+\beta \leq d$,*

$$\|D^\alpha_1 D^\beta_2 p\|_{q,t} \leq \frac{K}{\rho_t^{\alpha+\beta}}\|p\|_{q,t}, \quad 0 \leq \alpha+\beta \leq d, \tag{6.6}$$

where ρ_t denotes the radius of the largest circle inscribed in t.