The following methods for fitting a given set of data are available in the literature.

- Minimal Energy Method and its Extensions;
- Continuous and Discrete Least Squares Method;
- Penalized Least Squares Spline Method;
- $L_1$ Spline Method and its variants;
- Least Absolute Deviation Method;
- $L_1$ Smoothing Spline Method;
- Possible Research Projects.

We shall give a review of these methods. We need to explain several fundamental questions concerning each method: if a method has a solution or not (i.e., the existence and uniqueness), how to compute that solution (i.e., numerical algorithms), whether the solution surface resembles the given data (i.e., approximation properties), and what to do when the amount of data is very large.
1 Minimal Energy Method and Its Extensions

Let $E(f)$ be the thin-plate energy functional

$$E(f) = \int_{\Omega} \left( \left( \frac{\partial^2}{\partial x^2} f \right)^2 + 2 \left( \frac{\partial^2}{\partial x \partial y} f \right)^2 + \left( \frac{\partial^2}{\partial y^2} f \right)^2 \right) \, dx \, dy.$$ 

Let $\Lambda(f) = \{ s \in S^r_d(\Delta), s(x_i, y_i) = f_i, i = 1, \cdots, N \}$. Find $S_f \in \Lambda(f)$ such that

$$E(S_f) = \min \{ E(s), \quad s \in \Lambda(f) \}.$$ 

The following result was proved in [von Golitschek, Lai, and Schumaker’02] and in [Awanou, Lai, Winston’06] by different methods.

**Theorem 1.1** If $\Lambda(f)$ is not empty, there exists a unique interpolatory spline in $S^r_d(\Delta)$.

Once we have an interpolatory surface, we would like to know how the surface resembles the given data. Let $W^2(\Omega)$ be the Sobolev space of all functions whose second derivatives are essentially bounded over $\Omega$. $|f|_{2,\infty,\Omega}$ is the maximal norm of all second order derivatives of $f$ over $\Omega$. The following results can be found in [von Golitschek, Lai, and Schumaker’02].

**Theorem 1.2** Suppose $z_i = f(x_i, y_i), i = 1, \cdots, N$, for $f \in W^2(\Omega)$. Let $d \geq 3r + 2$, and let $\Delta$ be a triangulation of the data sites $\{(x_i, y_i), i = 1, \cdots, N \}$. Then

$$\|s_f - f\|_{L^\infty(\Omega)} \leq C|\Delta|^2|f|_{2,\infty,\Omega}.$$ 

Our next concern is how to compute interpolatory minimal energy splines using a spline space of arbitrary degree $d$ and arbitrary smoothness $r$ with $d \geq 3r + 2$. The following computational scheme was described in [Awanou, Lai and Winston’06].

1. Express each $s \in S^{-1}_d(\Delta)$ in B-form (cf. [de Boor’87]), i.e.,

$$s(x, y)|_t = \sum_{i+j+k=d} c_{ijk} B_{ijk}^{d}(x, y),$$

where $B_{ijk}^{d}$ are Bernstein-Bézier basis functions defined only on $t$. Let $c = (c_{ijk}, i + j + k = d, t \in \Delta)$ be a coefficient vector for $s$. 


(2) When \( s \in S^*_d(\Delta) \), there are smoothness conditions over interior edges of \( \Delta \) (cf. [Farin’86]). The smoothness conditions are linear. Put all smoothness conditions together to write

\[
Hc = 0,
\]

for a matrix \( H \), i.e., \( s \in S^*_d(\Delta) \) iff \( Hc = 0 \).

(3) Compute the energy functional \( E(s) = c^T Ec \) for an energy matrix \( E \) which is a diagonally block matrix.

(4) The interpolatory conditions can be written \( Ic = f \) for a matrix \( I \) and a vector \( f \) containing all data values \( z_i \).

(5) The minimal energy method for interpolatory splines is equivalent to finding \( c \) such that

\[
\min\{c^T Ec, \text{ subject to } Hc = 0, Ic = f\}.
\]

(6) By the Lagrange multiplier method, we solve

\[
\begin{bmatrix}
E & H^T & I^T \\
H & 0 & 0 \\
I & 0 & 0
\end{bmatrix}
\begin{bmatrix}
c \\
\alpha \\
\beta
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
f
\end{bmatrix}.
\]

(7) To solve this system, we use the following iterative method introduced in [Awanou, Lai, Wenston’06]:

\[
\left( E + \frac{1}{\epsilon} \begin{bmatrix} H^T & I^T \\ I & I \end{bmatrix} \right) \begin{bmatrix} c^{(1)} \end{bmatrix} = \frac{1}{\epsilon} I^T f,
\]

\[
\left( E + \frac{1}{\epsilon} \begin{bmatrix} H^T & I^T \\ I & I \end{bmatrix} \right) \begin{bmatrix} c^{(k+1)} \end{bmatrix} = E c^{(k)} + \frac{1}{\epsilon} I^T f,
\]

for \( k = 1, 2, \cdots \) and \( \epsilon > 0 \), e.g., \( \epsilon = 10^{-6} \).

We need to show that the iterative method above is convergent. To this end, recall that a matrix \( A \) is positive definite with respect to \( B \) if \( c^T Ac \geq 0 \) and if \( Ac = 0 \) and \( Bc = 0 \) for some \( c \), then \( c = 0 \). In [Awanou and Lai’05], we proved the following (cf. [Awanou, Lai, and Wenston’06] for a similar result).
Theorem 1.3 Suppose that $E$ is positive definite with respect to $[H, I]^T$. Then the above iteration converges, and

$$\|c^{(k+1)} - c\| \leq C\epsilon^k, \quad \forall k \geq 1.$$ 

When the number of data sites is large, e.g., $N > 1000$, a computer may not be powerful enough to solve the linear system. A domain decomposition technique for computing an approximation of the minimal energy spline interpolation was proposed in [Lai and Schumaker’09]. The ideas of domain decomposition for scattered data fitting can be explained as follows.

Let $D_1(t)$ be the union of all triangles in $\Delta$ which share a vertex or edge with $t$, and $D_{k+1}(t)$ the union of all triangles sharing a vertex or edge with triangles in $D_k(t)$. For $k \geq 1$, we compute a minimal energy interpolatory spline $S_{f,t,k} \in \Lambda(f)$ such that

$$E_{D_k(t)}(S_{f,t,k}) = \min \{ E_{D_h(t)}(s), s \in \Lambda(f|_t) \},$$

$$E_{D_h(t)}(s) = \int_{D_h(t)} \left( \left( \frac{\partial^2}{\partial x^2} f \right)^2 + 2 \left( \frac{\partial^2}{\partial x \partial y} f \right)^2 + \left( \frac{\partial^2}{\partial y^2} f \right)^2 \right).$$

The following result was established in [Lai and Schumaker’03].

Theorem 1.4 Suppose that $f \in C^2(\Omega)$. For $d \geq 3r+2$, there is a $0 < \rho < 1$ such that

$$\|S_f - S_{f,t,k}\|_{L_{\infty}(t)} \leq C\rho^k |f|_{2,\infty,\Omega}$$

for $k \geq 1$, where $C$ is a constant dependent on $d, \beta$.

This result shows that a (global) minimal energy spline interpolation $S_f$ can be approximated by local minimal energy spline interpolations $S_{f,t,k}$ for all $t \in \Delta$. That is, for each triangle $t$, one can use a local minimal energy spline interpolation $S_{f,t,k}$ to replace the global one $S_f|_t$ within some tolerance. In the following we give a numerical example.

Example 1.1 We are given a set of data shaped like a cone in Fig. 1. There are about 900 points in 3D Euclidean space. A Delaunay triangulation of the given data locations is shown in Fig. 2. A piecewise linear interpolation is given in Fig. 3. We use $C^1$ quintic spline functions and find the minimal energy interpolatory spline surface as shown in Fig. 4. It is clear that the surface is smooth although there are a few bumpy spots which indicate imperfect data values.
Figure 1: A set of scattered data (courtesy Tom Grandine).

Figure 2: A triangulation of the given data locations.
Figure 3: A piecewise linear interpolation.

Figure 4: A $C^1$ quintic spline interpolation.
2 Extensions of Minimal Energy Method

We now outline some extensions to incomplete data interpolation, Hermite data interpolation, hole filling, and spherical scattered data interpolation.

Example 2.1 When a given data set is incomplete, i.e., values at some grid locations are not given as shown in Fig. 5, we can still use the minimal energy method with the assumption that the spline coefficients at those vertices which have no given data values are free. The computation is exactly as above. Indeed, the interpolation conditions $\mathcal{I}c = f$ have fewer entries than the standard one.

We use $C^1$ quintic spline to find an interpolatory surface using the minimal energy method. It is clear from Fig. 6 that the surface is smooth.

Example 2.2 When a given data set contains Hermite data values

\[ \{(x_i, y_i, D^\alpha f(x_i, y_i), |\alpha| \leq r, i = 1, \cdots, N)\}, \]

we can use the minimal energy method to find a spline function $H_f$ in $S^r(\Delta)$ to interpolate all the given data values including derivatives, i.e.,

\[ D^\alpha H_f(x_i, y_i) = D^\alpha f(x_i, y_i), \quad |\alpha| \leq r, \quad i = 1, \cdots, N. \]

The existence, uniqueness, and approximation properties of $H_f$ have been discussed in [Zhou, Han and Lai’07].
Example 2.3 When the given data values as well as normal derivative values are all on the boundary of a surface hole, we can use the minimal energy method to find a $C^1$ spline surface patch to mend the hole (cf. [Chui and Lai’00]). See the example in Fig. 7.

Example 2.4 When the given data values are over the spherical domain, we can use the spherical splines [Alfeld, Neamtu, Schumaker’96] and the minimal energy method to find a $C^r$ interpolatory spline surface. The framework of minimal energy interpolatory splines in the bivariate setting has been generalized to the spherical setting (cf. [Baramidze, Lai and Shum’06]). The computational algorithm is similar to the one for bivariate polynomial splines. In Fig. 8, we present a set of normalized scattered data values over the surface of the earth. They are simulated measurements from a German satellite CHAMP launched on 2000. We use $C^1$ quintic spherical splines to find an interpolant. More detail will be given in Dr. Baramidze’s lecture.
Figure 7: Hole filling using $C^1$ quintic splines.
Figure 8: Normalized simulated geopotential measurements (top) and $C^1$ quintic spherical spline interpolation (bottom)
3 Continuous and Discrete Least Squares Fitting

The discrete least squares method is one of the classical methods for data fitting. Instead of polynomial fitting, we use multivariate splines. Let \( \ell(f) = \sum_{i=1}^{N} |f(x_i, y_i)|^2 \). We look for \( S_f \in S_d^r(\Delta) \) such that

\[
\ell(S_f - f) = \min \{ \ell(s - f), s \in S_d^r(\Delta) \}.
\]

\( S_f \) is called the discrete least squares fit of the given data \( \{(x_i, y_i, f_i), i = 1, \cdots, N \} \) with \( f_i = f(x_i, y_i) \).

To show the existence and uniqueness of the solution \( S_f \), we need to assume

\[
A_1 \|s\|_{L_\infty(T)} \leq \sqrt{\sum_{(x_i, y_i) \in T} |s(x_i, y_i)|^2}
\]

for all \( s \in S_d^r(\Delta) \) and all triangle \( T \in \Delta \) (cf. [von Golitschek and Schumaker’02a]).

**Theorem 3.1** Suppose that the above constant \( A_1 \) is strictly positive. Then there exists a unique spline fit \( S_f \in S_d^r(\Delta) \).

Let

\[
\sqrt{\sum_{(x_i, y_i) \in T} |s(x_i, y_i)|^2} \leq A_2 \|s\|_{L_\infty(T)}
\]

for all \( T \in \Delta \) and \( s \in S_d^r(\Delta) \). It is easy to see that \( A_2 \) must be less than or equal to the maximal number of points per triangle. The following result was established in [von Golitschek and Schumaker’02a].

**Theorem 3.2** Assume that \( f \in W^{m+1}_{\infty}(\Omega) \). Then

\[
\|S_f - f\|_{L_\infty(\Omega)} \leq C \frac{A_2}{A_1} |\Delta|^{m+1} |f|_{m+1, \infty, \Omega}
\]

for a constant \( C \) dependent on \( \beta, d \).

Furthermore, we can show the following
Corollary of Theorem 6. Under the same assumptions above, for $|\alpha| \leq m + 1$, 
\[ \|D^\alpha(S_f - f)\|_{L_\infty(\Omega)} \leq C \frac{A_2}{A_1} |\Delta|^{m+1-|\alpha|} |f|_{m+1,\infty,\Omega} \]
for a constant $C$ dependent only on $\beta$ and $d$.

This can be proved by using a polynomial approximation property and Markov’s inequality. Details are omitted here.

Our next question is how to compute discrete least squares fits. Recall that we write each $s \in S_{d-1}(\Delta)$ in the B-form
\[ s(x, y)|_t = \sum_{i+j+k=d} c_{ijk}^t B_{ijk}(x, y) \]
with coefficient vector $\mathbf{c} = (c_{ijk}^t, i+j+k=d, t \in \Delta)$.

We put all smoothness conditions of $S_d^r(\Delta)$ together as 
\[ \mathcal{H} \mathbf{c} = 0. \]

Let $\mathcal{L}$ be an observation matrix. It is easy to see 
\[ \ell(s - f) = \mathbf{c}^T \mathcal{L} \mathcal{L}^T \mathbf{c} - 2 \mathbf{c}^T \mathcal{L} \mathbf{f} + \mathbf{f}^T \mathbf{f}. \]
The discrete least squares spline is the solution of 
\[ \min \{ \mathbf{c}^T \mathcal{L} \mathcal{L}^T \mathbf{c} - 2 \mathbf{c}^T \mathcal{L} \mathbf{f}, \text{ subject to } \mathcal{H} \mathbf{c} = 0 \}. \]

By the Lagrange multipliers method, we solve 
\[ \begin{bmatrix} \mathcal{L} \mathcal{L}^T & \mathcal{H}^T \\ \mathcal{H} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathcal{L} \mathbf{f} \\ 0 \end{bmatrix}. \]

The ALW iteration introduced in the previous subsection can be applied to solve the above linear system. As before the iterative solutions converge the exact solution.

When the number of data sites is large, especially when the number of triangles is large, a computer may not be powerful enough to solve the associated linear system. We again propose a domain decomposition technique for computing an approximation of the discrete least squares spline (cf. [Lai and Schumaker’03]). That is, for $k \geq 1$, we compute $S_{f,t,k}$ such that 
\[ \ell_{D_k(t)}(S_{f,t,k} - f) = \min \{ \ell_{D_k(t)}(s - f), s \in S_d^r(\Delta) \}, \]
\[ \ell_{D_k(t)}(s - f) = \sum_{(x_i, y_i) \in D_k(t)} |s(x_i, y_i) - f(x_i, y_i)|^2. \]

We have the following (cf. [Lai and Schumaker’09])

**Theorem 3.3** Suppose that \( S_r^d(\Delta) \) with \( d \geq 3r + 2 \) over a \( \beta \) quasi-uniform triangulation \( \Delta \). Suppose that data values are obtained from a continuously differentiable function \( f \in C^{m+1}(\Omega) \). Suppose that \( A_1 > 0 \) and \( A_2 < \infty \) are constants such that \( A_2/A_1 \) is independent of \( \Delta \). Then there is a positive \( \rho < 1 \) such that

\[
\|s_f - S_{f,k}\|_{L^\infty(t)} \leq C \rho^k (k + 2) |\Delta|^{m+1} |f|_{m+1, \infty, \Omega}
\]

for \( k \geq 1 \), where \( C \) is a constant dependent only on \( d, \beta \) and \( A_2/A_1 \).

Continuous least squares fitting is to compute the least squares approximation of any given function. Let \( f \in C(\Omega) \) be a given function. We look for spline function \( S_f \in S_r^d(\Delta) \) such that

\[
\int_{\Omega} |f(x, y) - S_f(x, y)|^2 dxdy = \min_{s \in S_r^d(\Delta)} \int_{\Omega} |f(x, y) - s(x, y)|^2 dxdy.
\]

The existence and uniqueness of continuous least squares spline \( S_f \) are well known. We now explain how to compute \( S_f \).

We write each \( s \in S_r^{-1}(\Delta) \) in the B-form

\[
s(x, y) |_{t} = \sum_{i+j+k=d} c_{ijk} B_{ijk}^{d,t}(x, y)
\]

with coefficient vector \( c = (c_{ijk}, i + j + k = d, t \in \Delta) \).

We put all smoothness conditions of \( S_r^d(\Delta) \) together as

\[
\mathcal{H} c = 0.
\]

Let \( M \) be the mass matrix, i.e., \( M = [m_{ij}]_{1 \leq i, j \leq n} \) with

\[
m_{ij} = \int_{\Omega} \phi_i \phi_j dxdy
\]

where \( \{\phi_1, \cdots, \phi_n\} = \{B_{ijk}^{d,t}, i + j + k = d, t \in \Delta\} \). The continuous least squares spline is the solution of

\[
\min \{c^T M c - 2c^T M f + f^T M f, \text{ subject to } \mathcal{H} c = 0\},
\]

13
where we approximate $f$ by $\sum_{i=1}^n f_i \phi_i$ and denote $f = (f_1, \cdots, f_n)^T$.

By the Lagrange multipliers method, we solve

$$\begin{bmatrix}
M & \mathcal{H}^T \\
\mathcal{H} & 0
\end{bmatrix}
\begin{bmatrix}
c \\
\alpha
\end{bmatrix}
= \begin{bmatrix}
Mf \\
0
\end{bmatrix}.$$ 

Thus, this computation can be done easily.
3.1 Penalized Least Squares Spline Method

Recall that $E(f)$ denotes a thin-plate energy functional of $f$ and $\ell(s) = \sum_{i=1}^{N} (s(x_i, y_i) - f_i)^2$ as before. Fix $\lambda > 0$. Define $P(s) = \ell(s) + \lambda E(s)$. The PLS spline is the minimization solution $S_{f,\lambda} \in S_d(\Delta)$ such that

$$P(S_{f,\lambda}) = \min\{P(s), s \in S_d(\Delta)\}.$$

We refer to [Awanou, Lai, and Wenston'06] for a proof of the following.

**Theorem 3.4** Suppose that $N \geq 3$, and there exist three data sites, say $(x_i, y_i), i = 1, 2, 3$, which are not colinear. Then there exists a unique $S_{f,\lambda}$ in $S_d(\Delta)$ solving the above minimization problem.

We certainly want to know if the penalized least squares fitting surface resembles the given data or not. Since $f - S_{f,\lambda} = f - S_{f,0} + S_{f,0} - S_{f,\lambda}$, we need to estimate $S_{f,0} - S_{f,\lambda}$. To do so, we introduce the following two quantities: (cf. [von Golitschek and Schumaker'02b])

$$K_1 = \sup\{\frac{E(s)^{1/2}}{\ell(s)^{1/2}}, s \in S_d(\Delta), s \neq 0\}$$

and

$$K_2 = \sup\{\frac{\|s\|_{L_\infty(\Omega)}}{\ell(s)^{1/2}}, s \in S_d(\Delta), s \neq 0\}.$$

Then in [von Golitschek and Schumaker'02b], von Golitschek and Schumaker proved the following.

**Theorem 3.5** Let $S_{f,\lambda}$ be the Penalized Least Squares spline in $S_d(\Delta)$ with $d \geq 3r + 2$. Assume that $K_1$ and $K_2$ are finite. Then

$$\|S_{f,\lambda} - S_{f,0}\|_{L_\infty(\Omega)} \leq K_2 \sqrt{\lambda E(S_{f,0})} \min\{1, K_1 \sqrt{\lambda}\}.$$

We now work on estimating $K_1$ and $K_2$. It is easy to get

$$E(s) \leq \sum_{T \in \Delta} A_T \|s\|_{2,\infty,T}^2 \leq \sum_{T \in \Delta} A_T \frac{\ell(s)}{\rho_T^2} \|s\|_{L_\infty(T)}^2 \leq \frac{\beta^2}{\rho^2 (\rho \Delta)^2} \frac{\ell(s)}{A_1^2}.$$

It follows that $K_1 \leq \frac{\beta}{A_1 \rho \Delta}$. 


15
Since \( \|s\|_{L_\infty(\Omega)} = \|s\|_{L_\infty(T)} \) for a triangle \( T \),
\[
\|s\|_{L_\infty(\Omega)} \leq \frac{1}{A_1} \sqrt{\sum_{(x_i,y_i) \in T} |s(x_i,y_i)|^2} \leq \frac{1}{A_1} \ell(s)^{1/2}.
\]
It follows that
\[
K_2 \leq \frac{1}{A_1}.
\]

**Theorem 3.6** Let \( S_{f,\lambda} \) be the PLS spline in \( S_r^*(\Delta) \) with \( d \geq 3r + 2 \). Suppose that \( f \in W^{m+1}_\infty(\Omega) \) with \( 1 \leq m \leq d \). Then
\[
\|S_{f,\lambda} - f\|_{L_\infty(\Omega)} \leq C_1 |\Delta|^{m+1} |f|_{m+1,\infty,\Omega} + \lambda \frac{|\partial f|_{2,\infty,\Omega}}{A_2^2 (\rho \Delta)^2},
\]
where \( C_1 > 0, C_2 > 0 \) are constants dependent on \( A_2/A_1, \beta \) and \( d \).

To see that the convergence is linear in \( \lambda \), we present some numerical experiments: For \( \lambda_i = 1/2^{10+i} \), the maximum errors of \( S_{f,\lambda_i} \) to \( f \) are

<table>
<thead>
<tr>
<th>( \lambda_i )</th>
<th>( S_{1/2}(\Delta) )</th>
<th>( S_{1/4}(\Delta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_2 )</td>
<td>5.466e-4</td>
<td>5.451e-4</td>
</tr>
<tr>
<td>( \lambda_3 )</td>
<td>2.800e-4</td>
<td>2.762e-4</td>
</tr>
<tr>
<td>( \lambda_4 )</td>
<td>1.421e-4</td>
<td>1.408e-4</td>
</tr>
<tr>
<td>( \lambda_5 )</td>
<td>7.819e-5</td>
<td>7.318e-5</td>
</tr>
</tbody>
</table>

As we see the condition for the existence of penalized least squares spline fits is much weaker than that for the existence of the discrete least squares spline fits. However, the approximation result on penalized least squares spline fits is dependent on a very strong condition on the data sites, i.e., \( A_1 > 0 \). It is interesting to see if one can remove this condition while proving that the penalized least squares fits resemble the shape of the data.

Recall \( c \) is the coefficient vector of a spline \( s \in S_{r-1}^*(\Delta) \), \( H \) is the smoothness matrix such that \( Hc = 0 \) if and only if \( s \in S_{r-1}^*(\Delta) \), \( E \) is the energy matrix, and \( L \) is the observation matrix. Then the PLS spline is the minimization solution
\[
\min \{ c^T L L^T c - 2 c^T L f + \lambda c^T E c, \ \text{subject to} \ H c = 0 \}.
\]

By the Lagrange multipliers method, we solve
\[
\begin{bmatrix}
L L^T + \lambda E & H^T \\
H & 0
\end{bmatrix}
\begin{bmatrix}
c \\
\alpha
\end{bmatrix}
= 
\begin{bmatrix}
L f \\
0
\end{bmatrix}.
\]
We apply the ALW iteration introduced before.

When the number of triangles is large, a computer may not be powerful enough to find the PLS splines. We use a domain decomposition technique for computing an approximation of the PLS spline (cf. [Lai and Schumaker’03]). For \( k \geq 1 \), we compute a PLS spline \( S_{f,t,k} \) such that

\[
P_{D_k(t)}(S_{f,t,k}) = \min \{ P_{D_k(t)}(s), s \in S^r_d(\Delta) \},
\]

where

\[
P_{D_k(t)}(s) = \sum_{(x_i, y_i) \in D_k(t)} |s(x_i, y_i) - f(x_i, y_i)|^2 + \lambda E(s|D_k(t)).
\]

Here \( D_k(t) = \text{star}^k(t) \) for each triangle \( t \in \Delta \). We have the following result (cf. [Lai and Schumaker’03]).

**Theorem 3.7** Suppose that \( S^r_d(\Delta) \) with \( d \geq 3r + 2 \) over a \( \beta \) quasi-uniform triangulation \( \Delta \). Suppose that data values are obtained from a continuously differentiable function \( f \in C^{m+1}(\Omega) \). Suppose that \( A_1 > 0 \) and \( A_2 < \infty \) are constants such that \( A_2/A_1 \) is independent of \( \Delta \). Then there is a positive \( \rho < 1 \) such that

\[
\|s_f - S_{f,k}\|_{L_\infty(t)} \leq C\rho^k((k + 2)^{3/2}|\Delta|^{m+1}|f|_{m+1,\infty,\Omega} + \lambda|f|_{2,\infty,\Omega})
\]

for \( k \geq 1 \), where \( C \) is a constant dependent only on \( d, \beta \) and \( A_2/A_1 \).
3.2 $L_1$ Spline Methods

$L_1$ spline methods for data fitting were proposed in [Lavery’2000]. He used $C^1$ cubic spline curves and bivariate $C^1$ cubic Sibson’s elements for scattered data in 1D and grid data in 2D, respectively. Lai and Wenston in 2004 generalized the study to the scattered data in the bivariate setting. Recall that

$$\Lambda(f) = \{ s \in S^2_d(\Delta), s(x_i, y_i) = f(x_i, y_i), i = 1, \cdots, N \}.$$

Let $E_1(s)$ be the $L_1$ energy functional, i.e.,

$$E_1(f) = \int_{\Omega} \left( \left| \frac{\partial^2}{\partial x^2} f \right| + 2 \left| \frac{\partial^2}{\partial x \partial y} f \right| + \left| \frac{\partial^2}{\partial y^2} f \right| \right) dxdy.$$

Find $S_f \in \Lambda(f)$ such that

$$E_1(S_f) = \min \{ E_1(s), s \in \Lambda(f) \}.$$

$S_f$ is called the $L_1$ interpolatory spline of the given data $\{(x_i, y_i, f(x_i, y_i)), i = 1, \cdots, N\}$. A proof of the following theorem can be found in [Lai and Wenston’04]. This can be seen from the fact that the minimization functional is convex. However, the functional is not strictly convex and hence, the solution may not be unique.

**Theorem 3.8** Suppose that $\Lambda(f)$ is not empty. Then there exists at least one $S_f$ solving the above minimization problem.

The interpolatory surfaces which minimize the $L_1$ energy functional are indeed different from the usual $L_2$ minimal energy splines. Figures 9 and 10 show their differences. (These figures are borrowed from [Lai and Wenston’04].)

It is necessary to show that $L_1$ interpolatory splines resembles the shape of the given data. Lai in [Lai’07] proved the following

**Theorem 3.9** Suppose that $f \in C^2(\Omega)$. Let $S_f$ be the $L_1$ interpolatory spline of the data $(x_i, y_i, f(x_i, y_i)), i = 1, \cdots, N$. Then

$$\| S_f - f \|_{L_1(\Omega)} \leq C |\Delta|^2 |f|_{2, \infty, \Omega},$$

for a constant $C$ dependent only on $\beta$ and $d$. 

18
Figure 9: $L_1$ interpolatory spline (the top row) and minimal energy interpolatory spline (the bottom row)
Figure 10: $L_1$ interpolatory spline (the top row) and minimal energy interpolatory spline (the bottom row)
3.3 Least Absolute Deviation

For a given data set \( \{(x_i, y_i, f(x_i, y_i)), i = 1, \ldots, N\} \), let

\[
\ell_1(s) = \sum_{i=1}^{N} |s(x_i, y_i)|.
\]

We find \( S_f \in S_d^*(\Delta) \) such that

\[
\ell_1(S_f - f) = \min\{\ell_1(s - f), s \in S_d^*(\Delta)\}.
\]

\( S_f \) is the least absolute deviation (LAD) from the given data (cf. [Bloomfield and Steiger’83]).

Since the minimization functional is convex, there always exist a minimizer \( S_f \) (cf. [Lai and Wenston’04]). Next we would like to know how well the LAD surface resembles the given data. Let \( F_1 \) and \( F_2 \) be positive numbers such that

\[
F_1 \|s\|_{L_\infty(T)} \leq \sum_{(x_i, y_i) \in T} |s(x_i, y_i)| \leq F_2 \|s\|_{L_\infty(T)}
\]

for all \( s \in S_d^*(\Delta) \) and for all \( T \in \Delta \). We have the following (cf. [Lai’07]).

**Theorem 3.10** Suppose that two constants \( F_1 > 0 \) and \( F_2 < \infty \) such that \( F_2/F_1 \) independent of \( \Delta \). Suppose that \( f \in W_{m+1}^{m+1}(\Omega) \) for \( 0 \leq m \leq d \). Then

\[
\|S_f - f\|_{L_1(\Omega)} \leq C|\Delta|^{m+1}|f|_{m+1, \infty, \Omega}
\]

for a positive constant \( C \) dependent on \( F_2/F_1, \beta \) and \( d \).

3.4 \( L_1 \) Smoothing Splines

\( L_1 \) smoothing splines are \( S_f \in S_d^*(\Delta) \) which minimizes

\[
\ell_1(S_f - f) + \lambda E_1(S_f) = \min\{\ell_1(s - f) + \lambda E_1(s), s \in S_d^*(\Delta)\}.
\]

Since the minimization functional is convex, there exists at least one \( S_f \) solving the above minimization problem. We next need to show that \( S_f \) approximates \( f \) as the size of the triangulations goes to zero (cf. [Lai’07]).
Theorem 3.11 Under the same assumptions as Theorem 3.10,

\[ \|S_f - f\|_{L_1(\Omega)} \leq C|\Delta|^{m+1}|f|_{m+1,\infty,\Omega} + \lambda \frac{C}{F_1} |\Delta|^2 \]

for a positive constant \( C \) dependent on \( F_2/F_1, \beta \) and \( d \).

Algorithms computing these three \( L_1 \) spline methods were discussed in [Lai and Wenston’04]. The main ideas are

1) use discontinuous piecewise polynomial functions and set the smoothness conditions as side constraints;
2) convert \( L_1 \) norm minimization to a linear programming problem;
3) use Karmarkar’s algorithm to solve the linear programming problem.

4 Possible Research Projects

Let me give a list of possible projects:

• 1. Instead of minimal energy method for scattered data interpolation, we can solve the following unconstrained minimization problem:

\[
\min_{s \in S_f(\Delta)} \left\{ E(s) + \frac{1}{2\lambda} \sum_{j=1}^{N} |f(x_i, y_i) - s(x_i, y_i)|^2 \right\},
\]

where \( \lambda > 0 \) is a small parameter and \( E(f) \) is the thin-plate energy functional. If \( \lambda \) is very small, the solution \( S_f \) satisfies the interpolation conditions approximately. We should study its existence, uniqueness, characteristic conditions, stability, extremal value, computational method, ...

• 2. Instead of \( E(f) \), we may use \( E_1(f) \) in the previous problem. The \( L_1 \) norm of the second order derivatives of \( s \). That is, we study

\[
\min_{s \in S_f(\Delta)} \left\{ E_1(s) + \frac{1}{2\lambda} \sum_{j=1}^{N} |f(x_i, y_i) - s(x_i, y_i)|^2 \right\}.
\]
3. Again we replace $E_1(f)$ by the $L_1$ norm of the first order derivatives of spline functions $s$

$$
\min_{s \in S_r^{d}(\Delta)} \left\{ \| \nabla s \|_1 + \frac{1}{2\lambda} \sum_{j=1}^{N} |f(x_i, y_i) - s(x_i, y_i)|^2 \right\}.
$$

Then we study its properties of existence, uniqueness, resemblance, characterization, stability, extremal value, and computational method, and etc..

4. Generalize all the scattered data fitting/interpolation to 4D data or data values over 3D points using trivariate splines. I will explain more and show you my matlab programs for 4D data interpolation and fitting on next Monday morning.

5. Use 2D and 3D splines for statistical applications. See Bree Ettinger’s lecture in the afternoon.

References


