

A C^r Trivariate Macro-Element Based on the Alfeld Split of Tetrahedra

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Abstract

We construct trivariate macro-elements of class C^r for any $r \geq 1$ over the Alfeld refinement of any tetrahedral partition in \mathbb{R}^3 . In our construction, the degree of polynomials used for these macro-elements is the lowest possible. We also give the dimension formula for the subspace of consisting of these macro-elements.

Keywords: trivariate splines, C^r smoothness, macro-elements, Alfeld refinement

1. Introduction

In the literature, several trivariate C^1 and C^2 macro-elements were constructed over various refinement schemes of underlying tetrahedral partitions, see [1–4, 15, 16]. These results are summarized in [9]. Much less is known about macro-elements with arbitrary smoothness. The only known trivariate C^r macro-elements are based on Worsley-Farin split tetrahedra, see [12], and on non-split tetrahedra with polynomials of degree $8r + 1$, see [10]. In this paper we consider trivariate macro-elements over the Alfeld refinement of any arbitrary tetrahedral partition. The Alfeld refinement scheme consists of splitting each tetrahedron T of a partition into four subtetrahedra at a split point v_T in the interior of T . It was first considered in [1] to construct C^1 macro-elements, where it was called a trivariate Clough-Tocher scheme. In [3], C^2 macro-elements over the Alfeld split were constructed. A natural question is to ask how to extend these results and construct C^r macro-elements over the Alfeld split for $r > 2$. That is, what are the minimal degree one has to use in order to construct them and how to choose degrees of freedom? According to the construction in [3], one ingredient is to use supersmoothness around the edges and vertices of a given tetrahedral partition and around vertices at the split points. So how to choose various supersmoothness around the vertices,

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edges, faces and split pints is a research question which will be answered in this paper. In addition, are these ways of choosing degrees of freedom enough to determine C^r macro-elements over the Alfeld refinement of a tetrahedral partition for $r > 2$? Our study shows that these degrees of freedom are not enough when $r > 2$. We shall explain how to choose additional degrees of freedom in order to construct C^r macro-elements when $r > 2$.

More precisely, in this paper, we shall use piecewise polynomials of degree $12m + 1$ and $12m + 5$ for $r = 2m$ and $r = 2m + 1$, respectively. We also impose several supersmoothness conditions at the edges and vertices of the given tetrahedral partition. These orders of supersmoothness will be explicitly given in Table 4 (cf. section 3). It will be shown that these conditions are minimal in the sense that it is not possible to construct macro-elements on the Alfeld refinement with lower order of supersmoothness. In addition we shall also show that the degrees of polynomials given above are minimal. To construct C^r macro-elements for $r > 2$, we have to introduce two new kinds of degrees of freedom (cf. 4.1.4 and 4.1.5 in Section 4). That is, for $r > 2$ our construction has to be significantly different to the one in [3]. However, the new kinds of degrees of freedom reduce to be an empty set when $r = 1$ and $r = 2$. Thus, our macro-elements are consistent with those for C^1 smoothness and C^2 smoothness constructed over this refinement in [1] and [3]. Our major effort is to demonstrate that with the degree mentioned above, various orders of supersmoothness, and the new kinds of degree of freedom, the C^r macro-elements can be indeed constructed.

The paper is organized as follows. In section 2, we first introduce some notation for the well known Bernstein-Bézier representation for trivariate polynomials and review some useful lemmas for smoothness conditions for two trivariate polynomials sharing a common triangular face. In section 3, we review the minimal degree conditions for bivariate macro-elements on the Clough-Tocher split of a triangle and on non-split triangles. These are used to derive the minimal degree conditions for the macro-elements over the Alfeld split of a tetrahedron. The main results of the paper, a local and stable minimal determining set for our macro-elements on one tetrahedron and a collection of tetrahedra and the corresponding dimension formulae, are considered in Section 4. In the subsequent section, we shall give some examples for the macro-elements for $r = 1, \dots, 4$. We conclude the paper with a nodal determining set for the macro-elements and some remarks.

2. Preliminaries

Let Δ be a given tetrahedral partition of a polyhedral domain $\Omega \subset \mathbb{R}^3$. For each tetrahedron $T \in \Delta$, let v_T be a point strictly inside T (usually the barycenter). Then we define T_A to be the Alfeld split of T which consists of the four subtetrahedra obtained by connecting v_T to each of the vertices of T . We refer to a tetrahedral partition Δ_A where all tetrahedra in Δ are subjected to the Alfeld split. That is, $\Delta_A := \{T_A : \forall T \in \Delta\}$. In this paper, we shall consider the

following spline space

$$\mathcal{S}_d^r(\Delta_A) := \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_d^3, \text{ for all } T \in \Delta_A\},$$

where \mathcal{P}_d^3 is the $\binom{d+3}{3}$ dimensional linear space of trivariate polynomials of degree d , $r \geq -1$ and T stands for a tetrahedron in Δ_A . When $r = -1$, \mathcal{S}_d^{-1} is the space of discontinuous piecewise polynomial functions.

We make use of the well-known Bernstein-Bézier techniques. For any given d , let

$$\mathcal{D}_{d,\Delta_A} := \bigcup_{T \in \Delta_A} \mathcal{D}_{d,T},$$

be the set of domain points, where

$$\mathcal{D}_{d,T} := \{\zeta_{ijkl}^T := \frac{iv_1 + jv_2 + kv_3 + lv_4}{d}, i + j + k + l = d\},$$

and $T := \langle v_1, v_2, v_3, v_4 \rangle$. Then, the *shell* of radius n around v_1 is defined by

$$R_n^T(v_1) := \{\zeta_{ijkl}^T : i = d - n\},$$

and the *ball* of radius n around v_1 is defined by

$$D_n^T(v_1) := \{\zeta_{ijkl}^T : i \geq d - n\}.$$

The definitions are similar for the other vertices of T . If v is a vertex of Δ_A , we define

$$R_n(v) := \bigcup_{\{T \in \Delta_A : v \in T\}} R_n^T(v),$$

$$D_n(v) := \bigcup_{\{T \in \Delta_A : v \in T\}} D_n^T(v).$$

The *tube* of radius n around $e := \langle v_1, v_2 \rangle$ is defined by

$$E_n^T(e) := \{\zeta_{ijkl}^T : k + l \leq n\}.$$

If e is an edge of Δ_A , we define

$$E_n(e) := \bigcup_{\{T \in \Delta_A : e \in T\}} E_n^T(e).$$

Moreover, let

$$F_n^T(F) := \{\zeta_{ijkl}^T : l = n\}$$

be the set of domain points with a distance of n from the face $F := \langle v_1, v_2, v_3 \rangle$ of $T := \langle v_1, v_2, v_3, v_4 \rangle$.

Every spline $s \in \mathcal{S}_d^{-1}(\Delta_A)$ restricted to a tetrahedron $T \in \Delta_A$ can be written as

$$s|_T = \sum_{i+j+k+l=d} c_{ijkl}^T B_{ijkl}^{d,T},$$

where $B_{ijkl}^{d,T} := \frac{d!}{i!j!k!l!} b_1^i b_2^j b_3^k b_4^l$ are the Bernstein polynomials of degree d associated with T and $b_v \in \mathcal{P}_1$, $v = 1, 2, 3, 4$, are the barycentric coordinates on T . Then each spline $s \in \mathcal{S}_d^{-1}(\Delta_A)$ is uniquely determined by its corresponding set of B-coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\Delta_A}}$, with $c_{\xi_{ijkl}}^T := c_{ijkl}^T$, $i + j + k + l = d$.

Now, let $T := \langle v_1, v_2, v_3, v_4 \rangle$ and $\tilde{T} := \langle v_1, v_2, v_3, v_5 \rangle$ be two tetrahedra sharing the common face $F := \langle v_1, v_2, v_3 \rangle$. Let p and \tilde{p} be two polynomials of degree d with B-coefficients c_{ijkl}^T and \tilde{c}_{ijkl}^T relative to T and \tilde{T} , respectively. Then p and \tilde{p} join with C^r continuity across the face F if and only if

$$\tilde{c}_{ijkl}^T = \sum_{\substack{\alpha+\beta+\gamma+\delta=l \\ \alpha,\beta,\gamma,\delta \geq 0}} c_{i+\alpha,j+\beta,k+\gamma,\delta}^T B_{\alpha,\beta,\gamma,\delta}^{l,T}(v_5),$$

for $l = 0, \dots, r$, and $i + j + k = d - l$.

Suppose that the B-coefficients of p are known and that \tilde{p} and p join with C^r continuity. Then, these smoothness conditions can be used to determine some of the B-coefficients of \tilde{p} . They can also be used to determine some B-coefficients of p and \tilde{p} in cases where some of the B-coefficients of both polynomials are known.

Lemma 2.1 ([12], Lemma 1) *Let $T := \langle v_1, v_2, v_3, v_4 \rangle$, $\tilde{T} := \langle v_1, v_2, v_3, v_5 \rangle$, and p and \tilde{p} two polynomials of degree d . Suppose that all B-coefficients of p and \tilde{p} are already known except for the B-coefficients*

$$c_i := c_{2(m+l)-i+1,d-2(m+l)-n-1,n,i}^T, \quad i = j, \dots, m+l+j, \quad (2.1)$$

$$\tilde{c}_i := \tilde{c}_{2(m+l)-i+1,d-2(m+l)-n-1,n,i}^T, \quad i = j, \dots, m+l+j, \quad (2.2)$$

for some $m > 0, l \geq 0$, and $0 \leq n \leq d - 2(m+l) - j - 1$ and $j = 0, 1$. Then these B-coefficients are uniquely and stably determined by the smoothness conditions

$$\begin{aligned} & \tilde{c}_{2(m+l)-i+1,d-2(m+l)-n-1,n,i}^T \\ &= \sum_{\substack{\alpha+\beta+\gamma+\delta=i \\ \alpha,\beta,\gamma,\delta \geq 0}} c_{2(m+l)-i+1+\alpha,d-2(m+l)-n-1+\beta,n+\gamma,\delta}^T B_{\alpha,\beta,\gamma,\delta}^{i,T}(v_5), \quad (2.3) \\ & i = j, \dots, 2(m+l) + j + 1. \end{aligned}$$

Example 2.2 *Let $d = 5, m = 1, l = 0$ and $j = 0$. Then the B-coefficients associated with the domain points marked with \oplus and \otimes in Fig. 1 can be uniquely and stably determined from C^0, C^1, C^2 , and C^3 smoothness conditions. The B-coefficients of p and \tilde{p} associated with the domain points indicated by \blacksquare and \blacktriangle in Fig. 1 are assumed to be known. Let us now consider the B-coefficients of p and \tilde{p} corresponding*

to the domain points marked with \oplus . In each figure, the common edge of the two shown triangles is in fact just the common face $F := \langle v_1, v_2, v_3 \rangle$ of the two tetrahedra T and \tilde{T} , which becomes an edge restricted to $R_i(v_1)$, $i = 2, 3, 4, 5$. There are two undetermined B-coefficients associated with the domain point marked with \oplus in in Fig. 1 (mid right). From the C^0 smoothness condition at F , we know that these two B-coefficients must be equal. Now, we consider the remaining trivariate smoothness conditions at F associated with this B-coefficient. The C^1 smoothness condition connects all B-coefficients corresponding to domain points in Fig. 1 (mid right and right) marked with \oplus or \blacktriangle . The C^2 smoothness condition connects all B-coefficients corresponding to domain points in Fig. 1 (mid left, mid right, and right) marked with \oplus or \blacktriangle . The C^3 smoothness condition connects all B-coefficients corresponding to domain points in Fig. 1 marked with \oplus or \blacktriangle . Thus, we are left with three unknown B-coefficients, whose domain points are marked with \oplus in Fig. 1 (mid right and right), and three smoothness conditions. By Lemma 2.1, these undetermined B-coefficients can be uniquely and stably determined by the C^1, C^2 , and C^3 smoothness conditions. In the same way, the B-coefficients associated with the domain points marked with \otimes can be uniquely determined.

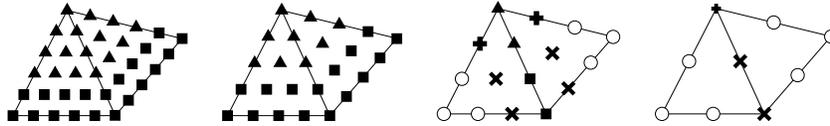


Figure 1: Domain points in the layers $R_5(v_1)$ (left), $R_4(v_1)$ (mid left), $R_3(v_1)$ (mid right), and $R_2(v_1)$ (right).

Example 2.3 Let $d = 5$, $m = 1$, $l = 0$, and $j = 1$. Then the B-coefficients associated with the domain points marked with \oplus and \otimes in Fig. 2 can be uniquely and stably determined from C^1, C^2, C^3 , and C^4 smoothness conditions. The B-coefficients of p and \tilde{p} associated with the domain points indicated by \blacksquare and \blacktriangle in Fig. 2 are already determined. Let us now consider the B-coefficients of p and \tilde{p} corresponding to the domain points marked with \oplus . In each figure, the common edge of the two shown triangles is in fact just the common face $F := \langle v_1, v_2, v_3 \rangle$ of the two tetrahedra T and \tilde{T} , which becomes an edge restricted to $R_i(v_1)$, $i = 2, 3, 4, 5$. Thus, we have trivariate smoothness conditions here. There are two undetermined B-coefficients associated with the domain points marked with \oplus in Fig. 2 (mid right). The C^1 smoothness condition connects these B-coefficients with the B-coefficient associated with the domain point indicated by \blacktriangle in Fig. 2 (right). The C^2 smoothness condition connects all B-coefficients corresponding to domain points in Fig. 2 (middle, mid right, and right) marked with \oplus or \blacktriangle . The C^3 smoothness condition connects all B-coefficients corresponding to domain points in Fig. 2 (mid left, middle, mid right, and right) marked with \oplus or \blacktriangle . The C^4 smoothness condition connects all B-coefficients corresponding to domain points in Fig. 2 marked with \oplus or \blacktriangle .

Thus, we are left with four unknown B-coefficients, whose domain points are marked with \blacktriangle in Fig. 2 (middle and mid right), and four smoothness conditions. By Lemma 2.1, these undetermined B-coefficients can be uniquely and stably determined by the C^1, C^2, C^3 , and C^4 smoothness conditions. In the same way, the B-coefficients associated with the domain points marked with \blacktimes can be uniquely determined.

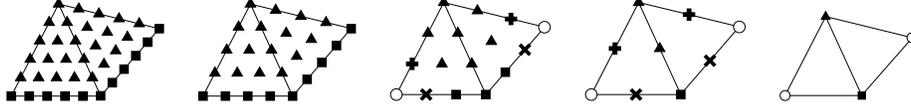


Figure 2: Domain points in the layers $R_5(v_1)$ (left), $R_4(v_1)$ (mid left), $R_3(v_1)$ (middle), $R_2(v_1)$ (mid right), and $R_1(v_1)$ (right).

Remark 2.4 A more general version of Lemma 2.1 and more examples can be found in [11].

The next lemma deals with the determination of some B-coefficients of a tetrahedron T from derivatives at a point in the interior of T .

Lemma 2.5 ([12], Lemma 2) Let p be a polynomial of degree $d \geq n(m+1)$, $m \geq -1$, over a triangle F for $n = 3$, or a tetrahedron T for $n = 4$, respectively. Suppose we already know the B-coefficients of p corresponding to the domain points within a distance of m from the edges of F , or the faces of T . Let v_F be any point in the interior of F , or v_T any point in the interior of T . Then p is uniquely and stably determined by the values of

$$\{D^\alpha p(v_F)\}_{|\alpha| \leq d-n(m+1)} \quad (2.4)$$

with $\alpha = (\alpha_1, \dots, \alpha_{n-1})$.

We also need to solve certain bivariate interpolation problems where some of the B-coefficients are already known, and the rest are to be determined by interpolation at certain domain points.

Lemma 2.6 ([9], Lemma 2.25) Let $F := \langle v_1, v_2, v_3 \rangle$ and $\Gamma := \mathcal{D}_{d,F} \setminus \{\xi_{ijk}^F : i \geq m_1, j \geq m_2, k \geq m_3\}$ for some $m_1, m_2, m_3 \geq 0$ with $m := m_1 + m_2 + m_3 < d$. Then the matrix

$$M := \left[B_v^{d,F}(\xi) \right]_{v, \xi \in \Gamma}$$

is nonsingular.

Let $s \in \mathcal{S}_d^0(\Delta)$ be a spline that satisfies additional smoothness conditions beyond C^0 continuity, for a tetrahedral partition Δ . Then clearly not all B-coefficients of s can be chosen independently. Thus, a *determining set* for a spline space $\mathcal{S} \in \mathcal{S}_d^0(\Delta)$ is a subset \mathcal{M} of the set of domain points $\mathcal{D}_{d,\Delta}$ such that

if $s \in \mathcal{S}$ and $c_{\xi} = 0$, for all $\xi \in \mathcal{M}$, then $c_v = 0$ for all $v \in \mathcal{D}_{d,\Delta}$ and $s \equiv 0$. The set \mathcal{M} is called a *minimal determining set* for \mathcal{S} if there is no smaller determining set. Then a spline $s \in \mathcal{S}$ can be uniquely determined by the B-coefficients of s associated with the domain points in \mathcal{M} .

Suppose \mathcal{N} is a collection of linear functionals λ , where λs is defined by a combination of values or derivatives of $s \in \mathcal{S}$ at points ζ in Ω . Then \mathcal{N} is called a *nodal determining set* for \mathcal{S} provided that if $s \in \mathcal{S}$ and $\lambda s = 0$ for all $\lambda \in \mathcal{N}$, then $s \equiv 0$. It is called a *nodal minimal determining set* for \mathcal{S} provided that for each set of real numbers $\{z_\lambda\}_{\lambda \in \mathcal{N}}$, there exists a unique $s \in \mathcal{S}$ such that $\lambda s = z_\lambda$ for all $\lambda \in \mathcal{N}$.

Let $F := \langle v_1, v_2, v_3 \rangle$ be a triangle and v_F be a point strictly inside F (usually the barycenter). Then we define F_{CT} to be the Clough-Tocher split of F which consists of the three subtriangles obtained by connecting v_F to each of the vertices of F (cf. [6]).

3. Minimal Degrees of Spline Spaces

3.1. Minimal degrees for bivariate macro-elements

We begin with the following basic result (cf. [8]):

Lemma 3.1 *Suppose that $F := \langle v_1, v_2, v_3 \rangle$ is a triangle and that Δ_F is a refinement of F such that there are $n \geq 0$ interior edges connected to the vertex v_1 . Let s be a piecewise polynomial spline of degree d and smoothness r defined on Δ_F . Then the cross derivatives of s up to order r on the edges $e_1 = \langle v_1, v_2 \rangle$ and $e_2 = \langle v_1, v_3 \rangle$ can be specified independently only if we require $s \in C^\rho(v_1)$, with*

$$\rho \geq \left\lceil \frac{(n+2)r - n}{n+1} \right\rceil.$$

For non-split triangles we have $n = 0$. This implies that, in order to ensure a smoothness of order r for the triangle, we must enforce supersmoothness of order $\rho = 2r$ at each vertex of the triangle. With another proof this was also obtained in [17] (see also [13], and [9]).

Lemma 3.2 *Given a smoothness r , a polynomial macro-element of class C^r can be constructed provided that the following minimal conditions on the supersmoothness ρ at the vertices of the triangles and the degree of polynomials d are fulfilled:*

ρ	$2r$
d	$4r + 1$

Table 1: The minimal conditions for polynomial macro-elements on triangles.

For the Clough-Tocher refinement we have $n = 1$. This implies that, in order to ensure a smoothness of order r for the triangle, we must enforce supersmoothness of order

$$\rho \geq \left\lceil \frac{3r-1}{2} \right\rceil = \begin{cases} 3m, & \text{if } r = 2m, \\ 3m+1, & \text{if } r = 2m+1, \end{cases}$$

at each vertex of the triangle.

Based on Lemma 3.1, Lai and Schumaker [7] obtained optimal Clough-Tocher finite elements, in the sense that they involve the lowest possible degree for a given smoothness r :

Lemma 3.3 *Given a smoothness r , a Clough-Tocher macro-element of class C^r can be constructed provided that the following minimal conditions on the supersmoothness μ at the vertices of the triangles and the degree of polynomials d are fulfilled.*

r	$2m$	$2m+1$
μ	$3m$	$3m+1$
d	$6m+1$	$6m+3$

Table 2: The minimal conditions for macro-elements over the Clough-Tocher split of triangles.

Remark 3.4 *Note that Lemma 3.1 can be used to obtain a lower bound for the degree of any polynomial finite element of class C^r , defined on a polygon $K \subset \mathbb{R}^2$ (It is valid not only for triangles, but also for any polygons).*

3.2. Minimal degrees for trivariate macro-elements

We are now able to illustrate the minimal conditions for the supersmoothnesses and the degree of polynomials for macro-elements over the Alfeld split of a tetrahedron.

Theorem 3.5 *Given a smoothness r , in order to define a macro-element based on the Alfeld split of a tetrahedron of class C^r , the minimal conditions on the supersmoothnesses μ and ρ at the edges and vertices of the refined tetrahedron, and the degree d of polynomials for the corresponding superspline space, given in Table 3 have to be fulfilled. Moreover, the supersmoothness η at the interior split point of the tetrahedron can be at most as in Table 3.*

Proof Let $T := \langle v_1, v_2, v_3, v_4 \rangle$ be a tetrahedron and T_A the refinement of T with the Alfeld split at a point v_T strictly inside T .

Now, let F be a plane that intersects the tetrahedron T_A at the points v_T, v_1 and v_2 (or equivalently, any other two vertices of T). The Alfeld refinement of T_A induces a Clough-Tocher refinement of a triangle on F . Since any trivariate spline with C^r smoothness defined on T_A is a bivariate spline of the same degree with the same

r	μ	ρ	η	d
$2m$	$3m$	$6m$	$11m + 1$	$12m + 1$
$2m + 1$	$3m + 1$	$6m + 2$	$11m + 4$	$12m + 5$

Table 3: The minimal and maximal conditions on macro-elements over the Alfeld split of a tetrahedron.

smoothness r , when restricted to F , we can apply Lemma 3.3. It follows that the supersmoothness μ at the three vertices of the triangle on F (since it has a Clough-Tocher refinement) is at least $3m$ for $r = 2m$ and $3m + 1$ for $r = 2m + 1$. Thus, coming back to T_A , it follows that the supersmoothness at the edges of T is equal to μ .

Subsequently, there is another triangle of interest, namely $\tilde{F} = \langle v_1, v_2, v_3 \rangle$ (or equivalently, any other face of T). This triangle is non-split and any trivariate spline function restricted to \tilde{F} is a bivariate spline function with the same degree but the smoothness μ . Hence we can use Lemma 3.2 to get that the smoothness ρ at the vertices of \tilde{F} is at least $6m$ for $r = 2m$, and $6m + 2$ for $r = 2m + 1$. There above, the degree of polynomials d is at least $12m + 1$ for $r = 2m$, and $12m + 5$ for $r = 2m + 1$. Thus, it follows that the supersmoothness at the vertices of T is equal to ρ and the degree of polynomials for the spline functions defined on T has to be at least d .

Next, we consider the supersmoothness η at the inner vertex v_T . Following [14] it must be at least $4m$ for $r = 2m$, and $4m + 2$ for $r = 2m + 1$. It is desirable to have a higher degree of supersmoothness at the inner vertex, to remove unnecessary degrees of freedom. However, we can not choose η to be too large, since this would lead to too many smoothness conditions for the number of coefficients. Thus, we again regard the Clough-Tocher split triangle in a plane F that intersects the tetrahedron T at the points v_T, v_1 and v_2 . From the above consideration, we know that this triangle has supersmoothness μ at the vertex \hat{v} lying on the edge $\langle v_3, v_4 \rangle$. Suppose we choose d, μ , and r as above and that we enforce supersmoothness C^η at the split point \hat{v}_F of the triangle. Now, we consider the domain points on the ring $R_{\mu+1}(\hat{v})$, which lie inside the disk $D_\eta(\hat{v}_F)$. This is a set of $2(\mu + 1 - d + \eta) + 1$ points. Setting the B-coefficients of the domain points within a distance of r from the faces of T , leaves $2(\mu + 1 - d + \eta) + 1 - 2(r - d + \eta + 1) = 2(\mu - r) + 1$ of these domain points with undetermined B-coefficients. These points must satisfy $\mu + 1 - d + \eta$ smoothness conditions across the edge $\langle \hat{v}, \hat{v}_F \rangle$. Then $2(\mu - r) + 1 \geq \mu + 1 - d + \eta$ must be satisfied in order to avoid incompatibilities. Thus, we get that $\eta \leq \mu + d - 2r$. With the values for μ, d and r from above it follows that η can be at most $11m + 1$ for $r = 2m$ and $11m + 4$ for $r = 2m + 1$. \square

Remark 3.6 In [1], the first C^1 macro-element over the Alfeld split of a tetrahedron was constructed using splines of degree 5 and C^2 supersmoothness at the vertices. A noncondensed version can be found in [5]. Moreover, in [9] this macro-element is described using splines of degree 5, with C^2 smoothness at the vertices and C^4 smoothness at the inner vertex v_T , as we do here for $r = 1$.

Remark 3.7 The first C^2 macro-element over the Alfeld split of a tetrahedron was constructed in [3]. There polynomials of degree 13, with C^6 supersmoothness at the vertices, C^3 supersmoothness at the edges and C^{12} supersmoothness at the splitting point v_T of T for each tetrahedron $T \subset \Delta$ were used. We use the same degree of polynomials and the same orders of supersmoothness conditions for the case $r = 2$.

4. Construction of Trivariate Macro-Elements over the Alfeld Refinement

We are now ready to describe how to construct trivariate macro-elements of class C^r over the Alfeld refinement of one tetrahedron $T := \langle v_1, v_2, v_3, v_4 \rangle$, with faces $F_i := \langle v_i, v_{i+1}, v_{i+2} \rangle$, $i = 1, \dots, 4$, where $v_5 := v_1$ and $v_6 := v_2$, which will be used throughout this section. Therefore, let $\mathcal{V}_T, \mathcal{E}_T$, and \mathcal{F}_T be the set of vertices, edges and faces of T , respectively, and T_A the corresponding Alfeld split consisting of the subtetrahedra $T_i := \langle v_i, v_{i+1}, v_{i+2}, v_T \rangle$, $i = 1, \dots, 4$. Moreover, let $\mathcal{F}_{I,T}$ be the set of six interior faces of T_A of the form $\langle u, v, v_T \rangle$, with $\langle u, v \rangle \in \mathcal{E}_T$ and for each face $F := \langle u, v, v_T \rangle \in \mathcal{F}_{I,T}$ let $T_F := \langle u, v, v_T, w \rangle$ be a tetrahedron containing F .

The macro-elements constructed in this section for one tetrahedron T will be in the superspline space

$$\begin{aligned} S_d^{r,\rho,\mu,\eta}(T_A) := \{ & s \in C^r(T) : s|_{\tilde{T}} \in \mathcal{P}_d^3 \text{ for all } \tilde{T} \in T_A, \\ & s \in C^\mu(e) \text{ for all } e \in \mathcal{E}_T, \\ & s \in C^0(v) \text{ for all } v \in \mathcal{V}_T, \\ & s \in C^\eta(v_T)\}, \end{aligned}$$

with r, ρ, μ, η and d as in Table 3.

Theorem 4.1 Let \mathcal{M}_T be the union of the following sets of domain points:

- 4.1.1. $D_\rho^{T_i}(v_i)$, $i = 1, \dots, 4$,
- 4.1.2. $\bigcup_{e:=\langle u,v \rangle \in \mathcal{E}_T} E_\mu^{T_e}(e) \setminus (D_\rho(u) \cup D_\rho(v))$, with $e \in T_e$,
- 4.1.3. $\bigcup_{j=0}^r F_j^{T_i}(F_i) \setminus (\bigcup_{v \in F_i} D_\rho(v) \cup \bigcup_{e \in F_i} E_\mu(e))$, $i = 1, \dots, 4$,
- 4.1.4. $\bigcup_{F \in \mathcal{F}_{I,T}} \{\tilde{\xi}_{\mu-2i+j, d-\rho-j, \mu+2i, 0}^{T_F} : i = 1, \dots, \lfloor \frac{r}{3} \rfloor, j = 1, \dots, \mu + 2i\}$,
- 4.1.5. $\bigcup_{j=1}^{\lfloor \frac{r}{3} \rfloor} \left(F_{r+j}^{T_i}(F_i) \setminus (\bigcup_{v \in F_i} D_\rho(v) \cup \bigcup_{e \in F_i} E_{\mu+2j}(e)) \right)$, $i = 1, \dots, 4$,
- 4.1.6. $D_{\eta-4(m+\lfloor \frac{r}{3} \rfloor)+1}^{T_1}(v_T)$,

where $\lfloor \frac{r}{3} \rfloor$ denotes the largest integer less than or equal to $\frac{r}{3}$ and m is as in Table 3. Then \mathcal{M}_T is a stable minimal determining set for $\mathcal{S}_d^{r,\rho,\mu,\eta}(T_A)$ and

$$\dim \mathcal{S}_d^{r,\rho,\mu,\eta}(T_A) = \begin{cases} \frac{2035m^3+1317m^2+314m+24}{6}, & \text{for } r = 2m, \\ \frac{2035m^3+3582m^2+2057m+390}{6}, & \text{for } r = 2m + 1. \end{cases} \quad (4.1)$$

Proof First, we show that \mathcal{M}_T is a minimal determining set. Therefore, we set the B-coefficients c_ξ of a spline $s \in \mathcal{S}_d^{r,\rho,\mu,\eta}(T_A)$ to arbitrary values for all $\xi \in \mathcal{M}_T$ and show that the remaining B-coefficients of s associated with domain points in T_A are uniquely and stably determined.

The undetermined B-coefficients associated with the domain points in the balls $D_\rho(v_i)$, $i = 1, \dots, 4$, can be uniquely and stably computed from the B-coefficients associated with the domain points in 4.1.1 using the C^ρ supersmoothness conditions at v_i , $i = 1, \dots, 4$.

Using the C^μ supersmoothness at the edges \mathcal{E}_T the undetermined B-coefficients associated with the domain points in the tubes $E_\mu(e)$, $e \in \mathcal{E}_T$, can be uniquely and stably computed from those associated with the domain points in 4.1.2.

Next, we regard the shells $R_d(v_T), \dots, R_{d-r}(v_T)$. We have already determined all B-coefficients associated with the domain points in $D_\rho(v_i)$, $i = 1, \dots, 4$, and those associated with the domain points in $E_\mu(e)$, $e \in \mathcal{E}_T$. Thus, together with the B-coefficients associated with the domain points in 4.1.3, we have already uniquely and stably determined all B-coefficients of s associated with the domain points in $R_d(v_T) \cup \dots \cup R_{d-r}(v_T)$.

To show that the remaining undetermined B-coefficients of s are uniquely determined, note that by the C^η smoothness at v_T , the B-coefficients of s associated with the domain points in $D_\eta(v_T)$ can be regarded as those of a polynomial p of degree η on the tetrahedron \tilde{T} , which is bounded by the domain points $R_\eta(v_T)$. Note that \tilde{T} is also subdivided at v_T . Considering p , we have already uniquely determined the B-coefficients associated with the domain points within a distance of m from the faces of \tilde{T} , those within a distance of $2m$ from the edges of \tilde{T} and those within a distance of $\rho - d + \eta$ from the vertices of \tilde{T} . Now, we show how to determine the remaining B-coefficients of p .

For $i = 1, \dots, \lfloor \frac{r}{3} \rfloor$, we consider the B-coefficients associated with the domain points in the shells $R_{d-r-i}(v_T)$. We repeat the following three steps:

We first use Lemma 2.1 to uniquely determine the remaining B-coefficients associated with domain points within a distance of $2(m+i) - 1$ from the edges of \tilde{T} . These B-coefficients correspond to domain points in $R_{d-r-i-j}(v_T)$, $j = 0, \dots, m+i-1$, with a distance of $m+i-j-1$ from the inner faces of \tilde{T} and are determined by trivariate $C^{2(m+i)-1}$ smoothness conditions at these faces. Secondly, together with the B-coefficients associated with the domain points in 4.1.4 with a distance of $2(m+i)$ from the edges of \tilde{T} , we can uniquely determine the remaining B-coefficients within a distance of $2(m+i)$ from the edges of \tilde{T} using Lemma 2.1. The B-coefficients determined this way correspond to domain points in $R_{d-r-i-j}(v_T)$, $j = 0, \dots, m+i-1$, with a distance of $m+i-j$ from the inner faces of \tilde{T} and are determined by trivariate

$C^{2(m+i)}$ smoothness conditions at these faces. Thirdly, together with the B-coefficients of p associated with the corresponding domain points in 4.1.5, we can uniquely and stably determine all B-coefficients corresponding to domain points in $R_{d-r-i}(v_T)$.

Now, we have determined all B-coefficients of p within a distance of $m + \lfloor \frac{r}{3} \rfloor$ from the faces of \tilde{T} . Setting the B-coefficients of p associated with the domain points in 4.1.6 is equivalent to setting the derivatives of p up to order $\eta - 4(m + \lfloor \frac{r}{3} \rfloor + 1)$ at v_T . Thus, by Lemma 2.5, the remaining undetermined B-coefficients of p are uniquely and stably determined.

Since some of the steps used to compute the B-coefficients corresponding to the domain points in the ball $D_\eta(v_T)$ are quite complicated, let us verify that these B-coefficients are not overdetermined. To do so we compare the dimension of the space of trivariate polynomials of degree η with the cardinalities of the degrees of freedom, i.e. the domain points in sets 4.1.1 - 4.1.6 which are contained in $D_\eta(v_T)$. We first consider the case $r = 2m$. In set 4.1.1 a total number of

$$\frac{250}{3}m^3 + 100m^2 + \frac{110}{3}m + 4$$

domain points is contained in $D_\eta(v_T)$. The cardinality of the subset of points in 4.1.2 contained in $D_\eta(v_T)$ is equal to

$$28m^3 + 42m^2 + 14m.$$

The set 4.1.3 consists of

$$\frac{176}{3}m^3 + 56m^2 - \frac{8}{3}m$$

domain points contained in $D_\eta(v_T)$. In set 4.1.4 a total number of

$$6 \left\lfloor \frac{r}{3} \right\rfloor \left(3m + 1 + \left\lfloor \frac{r}{3} \right\rfloor \right)$$

domain points is contained in $D_\eta(v_T)$. A total number of

$$\left\lfloor \frac{r}{3} \right\rfloor \left(74m^2 - 22m - 20m \left\lfloor \frac{r}{3} \right\rfloor - \frac{26}{3} - 16 \left\lfloor \frac{r}{3} \right\rfloor - \frac{22}{3} \left\lfloor \frac{r}{3} \right\rfloor^2 \right)$$

domain points from 4.1.5 lie in $D_\eta(v_T)$. The set 4.1.5 is completely contained in $D_\eta(v_T)$. Its cardinality is equal to

$$\begin{aligned} & \frac{343}{6}m^3 - \left(98 \left\lfloor \frac{r}{3} \right\rfloor + \frac{49}{2} \right) m^2 + \left(56 \left\lfloor \frac{r}{3} \right\rfloor^2 + 28 \left\lfloor \frac{r}{3} \right\rfloor + \frac{7}{3} \right) m \\ & - \frac{32}{3} \left\lfloor \frac{r}{3} \right\rfloor^3 - 8 \left\lfloor \frac{r}{3} \right\rfloor^2 - \frac{4}{3} \left\lfloor \frac{r}{3} \right\rfloor. \end{aligned}$$

By adding the above domain points together, one can see that the sum of the degrees of freedom is equal to $\dim \mathcal{P}_\eta^3 = \frac{1331}{6}m^3 + \frac{363}{2}m^2 + \frac{143}{3}m + 4$. It is clear that these degrees of freedom are linearly independent as they are associated with different domain

points. Thus all the B-coefficients over the domain points above determine the C^η supersmoothness at the split point v_T .

We have also obtained similar formulae for $r = 2m + 1$ and checked that no inconsistencies arise in the determination of the unknown B-coefficients corresponding to domain points in $D_\eta(v_T)$. Their details are omitted here.

To compute the dimension of $\mathcal{S}_d^{r,\rho,\mu,\eta}(T_A)$, we observe that

$$\begin{aligned} \#\mathcal{M}_T = & 4 \binom{\rho+3}{3} + 12 \binom{\mu+2}{3} \\ & + 4 \sum_{i=0}^r \left(\binom{d-i+2}{2} - 3 \binom{\rho-i+2}{2} - 3 \binom{\mu-i+1}{2} - 3i(\mu-i+1) \right) \\ & + 4 \sum_{i=1}^{\lfloor \frac{r}{3} \rfloor} \left(\binom{d-r-i+2}{2} - 3 \binom{\rho-r-i+2}{2} - 3 \binom{\mu-r+i+1}{2} \right. \\ & \left. - 3(r+i)(\mu-r+i+1) \right) + 6 \sum_{i=1}^{\lfloor \frac{r}{3} \rfloor} (\mu+2i) \\ & + \left(\eta - 4 \left(r + \lfloor \frac{r}{3} \rfloor + \eta - d \right) - 1 \right), \end{aligned}$$

which reduces to the number in (4.1). \square

Since all B-coefficients associated with the domain points within a distance of r from a face F can be uniquely determined using the B-coefficients corresponding to the domain points in 4.1.1 - 4.1.3, two neighboring tetrahedra T and \tilde{T} join with C^r continuity across the common face F . Thus, the macro-elements constructed above can be used to define a C^r macro-element space over a general tetrahedral partition. In the following, we present our construction of C^r macro-elements over an arbitrary tetrahedral partition. Let Δ be an arbitrary tetrahedral partition of a polyhedral domain Ω , and let \mathcal{V} , \mathcal{E} , and \mathcal{F} be its sets of vertices, edges, and faces, respectively. Let Δ_A be the refined partition obtained by applying the Alfeld split to each tetrahedron in Δ . Then, let \mathcal{V}_I be the set of split points in the interior of the tetrahedra, and \mathcal{F}_I the set of faces of the form $\langle u, v, w \rangle$, with $u, v \in \mathcal{V}$ and $w \in \mathcal{V}_I$, in Δ_A . Let

$$\begin{aligned} \mathcal{S}_d^{r,\rho,\mu,\eta}(\Delta_A) := & \{s \in C^r(\Omega) : s|_{\tilde{T}} \in \mathcal{P}_d^3 \text{ for all } \tilde{T} \in \Delta_A, \\ & s \in C^\mu(e) \text{ for all } e \in \mathcal{E}, \\ & s \in C^\rho(v) \text{ for all } v \in \mathcal{V}, \\ & s \in C^\eta(v_T) \text{ for all } v_T \in \mathcal{V}_I\}. \end{aligned}$$

Moreover, let $\#V$, $\#E$, and $\#F$ be the cardinalities of the sets \mathcal{V} , \mathcal{E} , and \mathcal{F} , respectively and let $\#N$ be the number of tetrahedra in Δ .

Theorem 4.2 *Let \mathcal{M}_Δ be the union of the following sets of domain points:*

$$4.2.1. \bigcup_{v \in \mathcal{V}} D_\rho^{T_v}(v), \text{ with } v \in T_v$$

$$4.2.2. \bigcup_{e := \langle u, v \rangle \in \mathcal{E}} E_\mu^{T_e}(e) \setminus (D_\rho(u) \cup D_\rho(v)), \text{ with } e \in T_e$$

$$4.2.3. \bigcup_{F \in \mathcal{F}} \bigcup_{j=0}^r F_j^{T_F}(F) \setminus \left(\bigcup_{v \in F} D_\rho(v) \cup \bigcup_{e \in F} E_\mu(e) \right), \text{ with } F \in T_F$$

$$4.2.4. \bigcup_{F \in \mathcal{F}_I} \{ \tilde{\zeta}_{\mu-2i+j, d-\rho-j, \mu+2i, 0}^{T_F} : i = 1, \dots, \lfloor \frac{r}{3} \rfloor, j = 1, \dots, \mu + 2i \}, \text{ with } F \in T_F$$

$$4.2.5. \bigcup_{T \in \Delta} \bigcup_{F \in T} \bigcup_{j=r+1}^{\lfloor \frac{r}{3} \rfloor} \left(F_j^T(F) \setminus \left(\bigcup_{v \in F} D_\rho(v) \cup \bigcup_{e \in F} E_{\mu+2j}(e) \right) \right)$$

$$4.2.6. \bigcup_{v \in \mathcal{V}_I} D_{\eta-4(m+\lfloor \frac{r}{3} \rfloor+1)}^{T_v}(v), \text{ with } v \in T_v$$

Then \mathcal{M}_Δ is a stable minimal determining set for $\mathcal{S}_d^{r, \rho, \mu, \eta}(\Delta_A)$ and

$$\dim \mathcal{S}_d^{r, \rho, \mu, \eta}(\Delta_A) = \begin{cases} (36m^3 + 36m^2 + 11m + 1)\#V + (9m^3 + 9m^2 + 2m)\#E \\ + \left(\frac{134m^3 + 57m^2 - 5m}{6} \right)\#F + \left(\frac{311m^3 - 99m^2 - 2m}{6} \right)\#N, \\ \text{for } r = 2m, \\ (36m^3 + 72m^2 + 47m + 10)\#V + (9m^3 + 18m^2 + 11m + 2)\#E \\ + \left(\frac{134m^3 + 228m^2 + 112m + 18}{6} \right)\#F + \left(\frac{311m^3 + 294m^2 + 85m + 6}{6} \right)\#N, \\ \text{for } r = 2m + 1. \end{cases} \quad (4.2)$$

Remark 4.3 When constructing trivariate finite elements (cf. [10] and [3]) normally the degrees of freedom as those in 4.1.1, 4.1.2, 4.1.3, and 4.1.6 are used. For our construction, we introduce two new kinds of degrees of freedom, 4.1.4 and 4.1.5, similar to those used in [12]. The degrees of freedom in 4.1.4 are used to determine the B-coefficients marked with \times , 4.1.5 for \bullet , as in Figure 3.

5. Examples

We will give some examples for the macro-elements constructed in the previous section for $r = 1, \dots, 4$, for one tetrahedron $T := \langle v_1, v_2, v_3, v_4 \rangle$. We also show that our macro-elements reduce to those from [9] for $r = 1$ and $r = 2$. More examples of C^r macro-elements (for $r = 5, 6$) can be found in [11]. where these examples for $r = 1, \dots, 6$ are described in detail.

5.1. C^1 macro-elements

The macro-elements are in the space $\mathcal{S}_d^{r,\rho,\mu,\eta}(T_A)$, which reduces to the superspline space $\mathcal{S}_5^{1,2,1,4}(T_A)$ by the third row of Table 3 and $m = 0$. Then, by (4.1), the dimension of the spline space is equal to 65. Moreover, the minimal determining set \mathcal{M}_T is the union of the following sets of domain points:

$$\text{C1.1 } D_2^{T_i}(v_i), i = 1, \dots, 4$$

$$\text{C1.2 } \bigcup_{e:=\langle u,v \rangle \in \mathcal{E}_T} E_1^{T_e}(e) \setminus (D_2(u) \cup D_2(v)), \text{ with } e \in T_e$$

$$\text{C1.3 } \bigcup_{j=0}^1 F_j^{T_i}(F_i) \setminus \left(\bigcup_{v \in F_i} D_\rho(v) \cup \bigcup_{e \in F_i} E_\mu(e) \right), i = 1, \dots, 4$$

$$\text{C1.4 } v_T$$

The superspline space constructed in this subsection is exactly the same as constructed in [9], chapter 18.3.

5.2. C^2 macro-elements

The macro-elements are in the space $\mathcal{S}_d^{r,\rho,\mu,\eta}(T_A)$ of supersplines, which reduces to the superspline space $\mathcal{S}_{13}^{2,6,3,12}(T_A)$ by the second row of Table 3 and $m = 1$. By (4.1), the dimension of the spline space is equal to 615 and the minimal determining set \mathcal{M}_T is the union of the following sets of domain points:

$$\text{C2.1 } D_6^{T_i}(v_i), i = 1, \dots, 4$$

$$\text{C2.2 } \bigcup_{e:=\langle u,v \rangle \in \mathcal{E}_T} E_3^{T_e}(e) \setminus (D_6(u) \cup D_6(v)), \text{ with } e \in T_e$$

$$\text{C2.3 } \bigcup_{j=0}^2 F_j^{T_i}(F_i) \setminus \left(\bigcup_{v \in F_i} D_6(v) \cup \bigcup_{e \in F_i} E_3(e) \right), i = 1, \dots, 4$$

$$\text{C2.4 } D_4^{T_1}(v_T)$$

The macro-element constructed in this subsection is exactly the same as the one constructed in [9], chapter 18.7. We use the same minimal determining set and the same superspline space. Moreover, our superspline space is also equal to the one in [3]. Since no minimal determining set was constructed in [3], we can not make the comparison.

5.3. C^3 macro-elements

The macro-elements are in the space $\mathcal{S}_d^{r,\rho,\mu,\eta}(T_A)$ of supersplines, which reduces to the superspline space $\mathcal{S}_{17}^{3,8,4,15}(T_A)$ by the third row of Table 3 with $m = 1$. The dimension of the spline space is, as in (4.1) with $m = 1$, equal to 1344 and the corresponding minimal determining set \mathcal{M}_T is the union of the following sets of domain points:

$$\text{C3.1 } D_8^{T_i}(v_i), i = 1, \dots, 4$$

$$\text{C3.2 } \bigcup_{e:=\langle u,v \rangle \in \mathcal{E}_T} E_4^{T_e}(e) \setminus (D_8(u) \cup D_8(v)), \text{ with } e \in T_e$$

$$\text{C3.3 } \bigcup_{j=0}^3 F_j^{T_i}(F_i) \setminus \left(\bigcup_{v \in F_i} D_8(v) \cup \bigcup_{e \in F_i} E_4(e) \right), i = 1, \dots, 4$$

$$\text{C3.4 } \bigcup_{F \in \mathcal{F}_{I,T}} \{\xi_{2+j,9-j,6,0}^{T_F} : j = 1, \dots, 6\}$$

$$\text{C3.5 } F_4^{T_i}(F_i) \setminus \left(\bigcup_{v \in F_i} D_8(v) \cup \bigcup_{e \in F_i} E_6(e) \right), i = 1, \dots, 4$$

$$\text{C3.6 } D_3^{T_1}(v_T)$$

5.4. C^4 macro-elements

We now describe the construction of C^4 macro-elements. When $r = 4$, by the second row of Table 3 and $m = 2$, the superspline space $\mathcal{S}_d^{r,\rho,\mu,\eta}(T_A)$ is reduced to the superspline space $\mathcal{S}_{25}^{4,12,6,23}(T_A)$. The dimension of the spline space is, as in (4.1) with $m = 2$, equal to 3700 and the corresponding minimal determining set \mathcal{M}_T is the union of the following sets of domain points:

$$\text{C4.1 } D_{12}^{T_i}(v_i), i = 1, \dots, 4$$

$$\text{C4.2 } \bigcup_{e:=\langle u,v \rangle \in \mathcal{E}_T} E_6^{T_e}(e) \setminus (D_{12}(u) \cup D_{12}(v)), \text{ with } e \in T_e$$

$$\text{C4.3 } \bigcup_{j=0}^4 F_j^{T_i}(F_i) \setminus \left(\bigcup_{v \in F_i} D_{12}(v) \cup \bigcup_{e \in F_i} E_6(e) \right), i = 1, \dots, 4$$

$$\text{C4.4 } \bigcup_{F \in \mathcal{F}_{I,T}} \{\xi_{4+j,13-j,8,0}^{T_F} : j = 1, \dots, 8\}$$

$$\text{C4.5 } F_5^{T_i}(F_i) \setminus \left(\bigcup_{v \in F_i} D_{12}(v) \cup \bigcup_{e \in F_i} E_8(e) \right), i = 1, \dots, 4$$

$$\text{C4.6 } D_7^{T_1}(v_T)$$

6. Nodal Degrees of Freedom

In this section we show how to construct a nodal minimal determining set for our macro-elements. The macro-elements are in the same superspline space as defined in section 4. First we need some additional notation.

Given a multi-index $\alpha := (\alpha_1, \alpha_2, \alpha_3)$, let D^α be the derivative $D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3}$. For each edge $e := \langle u, v \rangle$ of a tetrahedron $T \in \Delta$, let X_e be the plane perpendicular to e at the point u . We provide X_e with Cartesian coordinate axes

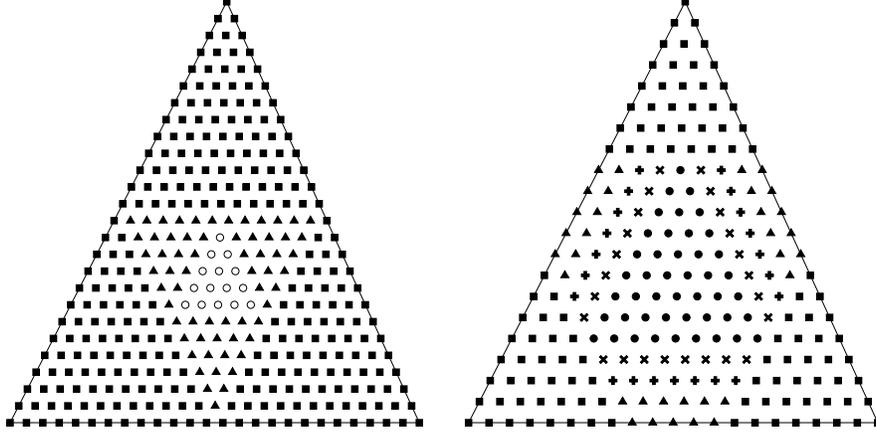


Figure 3: Faces of the shells $R_{25}(v_T)$ (left) and $R_{20}(v_T)$ (right). Domain points whose B-coefficients are determined from C4.1 are marked with \blacksquare , those from C4.2 with \blacktriangle , those from C4.3 with \circ , those from smoothness conditions using Lemma 2.1 with $\blacksquare+$, those from C4.4 and smoothness conditions using Lemma 2.1 with $\blacksquare\times$, and those from C4.5 with \bullet .

whose origin lies at the point u . Then, we define $D_{e^\perp}^\beta$ to be the corresponding directional derivative of order $|\beta| := \beta_1 + \beta_2$, in a direction lying in X_e . For an oriented triangular face $F := \langle u, v, w \rangle$ of T , we write D_F for a unit normal derivative associated with F . For each edge e of T , let $D_{e^{\hat{F}}}$ be the directional derivative associated with a unit vector perpendicular to e that lies in the interior face \hat{F} containing e . Moreover, for each edge e in a face F of T , let $D_{e^\perp F}$ be the directional derivative associated with a unit vector perpendicular to e that lies in F .

We also need some notations for certain points and sets of points.

For $i > 0$, we define

$$\zeta_{e,j}^i := \frac{(i-j+1)u + jv}{i+1}, \quad j = 1, \dots, i, \quad (6.1)$$

to be equally spaced points in the interior of e .

For $\alpha \geq 0$, we define

$$A_\alpha^F := \{\zeta_{ijk}^F \in \mathcal{D}_{d-\alpha, F} : i, j, k \geq \mu + 1 - \alpha + \lfloor \frac{\alpha}{2} \rfloor\} \quad (6.2)$$

to be a set of points in the interior of a triangle F .

If $\lfloor \frac{\alpha}{3} \rfloor > 0$, for $\beta \geq 1$, we also define the following set in the interior of a

triangle F :

$$B_\beta^F := \mathcal{D}_{d-r-\beta, F} \setminus \bigcup_{e \in F} E_{\mu-2\beta-1}^F(e). \quad (6.3)$$

For any point $u \in \mathbb{R}^3$, we write ϵ_u for the point evaluation functional defined by

$$\epsilon_u f := f(u). \quad (6.4)$$

Now we are ready to give a nodal minimal determining set for our macroelements. Therefore, let Δ be an arbitrary tetrahedral partition of a polyhedral domain Ω , and let \mathcal{V} , \mathcal{E} , and \mathcal{F} be its sets of vertices, edges, and faces, respectively. Let Δ_A be the refined partition obtained by applying the Alfeld split to each tetrahedron in Δ . Then, let \mathcal{V}_I be the set of split points in Δ_A .

Theorem 6.1 *Let \mathcal{N}_Δ be the union of the following nodal sets:*

$$6.1.1 \quad \bigcup_{v \in \mathcal{V}} \{\epsilon_v D^\alpha\}_{|\alpha| \leq \rho}$$

$$6.1.2 \quad \bigcup_{e \in \mathcal{E}} \bigcup_{i=1}^{\mu} \bigcup_{j=1}^i \{\epsilon_{\zeta_{e,j}^i} D_{e^\perp}^\beta\}_{|\beta|=i}$$

$$6.1.3 \quad \bigcup_{F \in \mathcal{F}} \{\epsilon_\zeta D_F^\alpha\}_{\zeta \in A_\alpha^F, \alpha=0, \dots, r}$$

$$6.1.4 \quad \bigcup_{F \in \mathcal{F}} \bigcup_{e \in F} \bigcup_{i=2}^r \bigcup_{j=1}^{\lfloor \frac{i}{2} \rfloor} \{\epsilon_{\zeta_{e,k}^{\mu+j}} D_{e^{\hat{F}}}^i D_{e^\perp F}^{\mu-i+j}\}_{k=1}^{\mu+j}$$

$$6.1.5 \quad \bigcup_{T \in \Delta} \bigcup_{e \in T} \bigcup_{i=1}^{\lfloor \frac{r}{3} \rfloor} \bigcup_{j=1}^{\mu+2i} \{\epsilon_{\zeta_{e,j}^i} D_{e^{\hat{F}}}^\beta\}_{|\beta|=\mu+2i}$$

$$6.1.6 \quad \bigcup_{T \in \Delta} \bigcup_{F \in T} \bigcup_{e \in F} \bigcup_{i=1}^{\lfloor \frac{r}{3} \rfloor} \bigcup_{j=1}^{r-3i-1} \{\epsilon_{\zeta_{e,k}^{\mu+2i+j}} D_{e^{\hat{F}}}^{r+i} D_{e^\perp F}^{\mu-r+i+j}\}_{k=1}^{\mu+2i+j}$$

$$6.1.7 \quad \bigcup_{T \in \Delta} \bigcup_{F \in T} \{\epsilon_\zeta D_F^{r+\beta}\}_{\zeta \in B_\beta^F, \beta = 1, \dots, \lfloor \frac{r}{3} \rfloor}$$

$$6.1.8 \quad \bigcup_{v_T \in \mathcal{V}_I} \{\epsilon_{v_T} D^\alpha\}_{|\alpha| \leq \eta - 4(m + \lfloor \frac{r}{3} \rfloor + 1)}$$

Then \mathcal{N}_Δ is a stable nodal minimal determining set for $S_d^{r,\rho,\mu,\eta}(\Delta_A)$.

Proof *It can easily be seen, that the cardinality of \mathcal{N}_Δ is equal to the dimension of the spline space $S_d^{r,\rho,\mu,\eta}(\Delta_A)$, as given in Theorem 4.2 in section 4. Therefore, it suffices to show that setting $\{\lambda s\}_{\lambda \in \mathcal{N}_\Delta}$ for a spline $s \in S_d^{r,\rho,\mu,\eta}(\Delta_A)$ determines all B-coefficients of s .*

For each vertex $v \in \mathcal{V}$ the B-coefficients of s associated with the domain points in the ball $D_\rho(v)$ can be stably computed from the values in 6.1.1.

For each edge $e \in \mathcal{E}$ the B-coefficients of s corresponding to the domain points in the tube $E_\mu(e)$ can be stably determined from the values in 6.1.2.

For each face $F \in \mathcal{F}$, the sets 6.1.3 and 6.1.4 contain values for one tetrahedron T_F containing F . For the face F of T_F , we have already determined the B-coefficients associated with the domain points within a distance of μ from the edges of F . Thus, the only undetermined B-coefficients of $s|_F$ are those associated with the domain points in A_0^F . Since 6.1.3 contains the values of s at the points in A_0^F , these B-coefficients can be computed by solving a linear system with matrix $M_0 := \left[B_v^{d,F}(\xi) \right]_{v,\xi \in A_0^F}$. The entries of the matrix depend only on barycentric coordinates, thus M_0 is independent of the size and shape of F . Moreover, by lemma 2.6 the matrix is nonsingular.

In the same way the remaining undetermined B-coefficients of s associated with domain points with a distance of one from F in T_F can be determined from the values of the first order derivatives in 6.1.3 corresponding to F . This involves solving a linear system with matrix $M_1 := \left[B_v^{d-1,F_1}(\xi) \right]_{v,\xi \in A_1^F}$, which is also independent of the size and shape of F and nonsingular by lemma 2.6.

Now, let F_i be the layer of domain points with a distance of i from a face F of T . Then, we have already determined the B-coefficients of s associated with the domain points within a distance of $\rho - i$ of the vertices of F_i and those associated with the domain points within a distance of $\mu - i$ from the edges of F_i . We use the values of the derivatives in 6.1.4 corresponding to F_i to stably determine the B-coefficients associated with the domain points with a distance of $\mu - i + j$ from the edges of F_i , for $j = 1, \dots, \lfloor \frac{i}{2} \rfloor$. Next, we use the values of the derivatives in 6.1.3 corresponding to F_i to compute the remaining B-coefficients of $s|_{F_i}$. Therefore, we have to solve a linear system with matrix $M_i := \left[B_v^{d-i,F_i}(\xi) \right]_{v,\xi \in A_i^F}$. By lemma 2.6, this matrix is nonsingular. Thus, we have determined all B-coefficients of s associated with the domain points within a distance of r from F in T_F . If F is an inner face of Δ , the B-coefficients of $s|_{\tilde{T}_F}$ associated with the domain points within a distance of r from F can be uniquely and stably determined by the C^r smoothness conditions at F , where \tilde{T}_F is the tetrahedron touching T_F at F .

At this point we have determined all B-coefficients of s , such that the smoothness conditions at the vertices, edges and faces of Δ are satisfied.

Now let T be a tetrahedron in Δ . Note that by the C^η smoothness at v_T , the B-coefficients of s associated with the domain points in $D_\eta(v_T)$ can be regarded as those of a polynomial p of degree η on the tetrahedron \tilde{T} bounded by the domain points $R_\eta(v_T)$. The tetrahedron \tilde{T} is also subdivided at v_T .

Considering p , we have already determined the B-coefficients associated with the domain points within a distance of m from the faces of \tilde{T} , those within a distance of $2m$ from the edges of \tilde{T} and those within a distance of $\rho - d + \eta$ from the vertices of \tilde{T} .

For $\lfloor \frac{r}{3} \rfloor > 0$, we also have to consider the B-coefficients associated with the domain points in the rings $R_{d-r-i}(v_T)$, $i = 1, \dots, \lfloor \frac{r}{3} \rfloor$. Thus, for $i = 1, \dots, \lfloor \frac{r}{3} \rfloor$, we repeat

the following:

We use Lemma 2.1 to stably determine the B-coefficients associated with the domain points within a distance of $2(m + i) - 1$ from the edges of \tilde{T} . Now, we can use the values of the derivatives in 6.1.5 associated with T to stably determine the B-coefficients of s associated with the domain points on the interior faces of T_A with a distance of $\mu + 2i$ from the corresponding edges of T . These domain points have a distance of $2(m + i)$ from the corresponding edge of \tilde{T} . Thus, using Lemma 2.1, we can determine the B-coefficients of p associated with the domain points within a distance of $2(m + i)$ from the edges of \tilde{T} .

Let \tilde{F}_i be the layer of domain points of p with a distance of $m + i$ from a face \tilde{F} of \tilde{T} . Using the values of the derivatives in 6.1.6 corresponding to \tilde{F}_i , the B-coefficients of p associated with the domain points with a distance of $\mu - r + i + j$, $j = 1, \dots, r - 3i - 1$, from the edges of \tilde{F}_i can be stably determined. Subsequently, we use the values of the derivatives in 6.1.7 corresponding to \tilde{F}_i to compute the remaining B-coefficients of $p|_{\tilde{F}_i}$. This involves solving a linear system with matrix $\tilde{M}_i := \left[B_v^{d-r-i, \tilde{F}_i}(\xi) \right]_{v, \xi \in B_{\tilde{F}_i}}$, which is nonsingular by lemma 2.6 and also independent of the size and shape of \tilde{F} .

Now, we have determined all B-coefficients of p within a distance of $m + \lfloor \frac{r}{3} \rfloor$ from the faces of \tilde{T} . Thus, using Lemma 2.5, the remaining undetermined B-coefficients of p can be stably determined from the values of the derivatives in 6.1.8 associated with T .

Thus, all B-coefficients of s can be uniquely and stably determined. \square

7. Remarks

Remark 7.1 For $r = 2$, our macro-elements differ from those constructed in [3], only in the way the B-coefficients associated with domain points in $R_{11}(v_T)$ are determined. We determine some of the B-coefficients on this shell by using the nodal data in 6.1.3 and those in 6.1.4. In [3], more derivatives at the faces of each tetrahedron are used and therefore the nodal data in 6.1.4 can be omitted. We use this more complicated nodal minimal determining set, since we extend our construction to arbitrary smoothness. Assuming that conjecture of Schumaker in [3] holds, it would be possible to omit 6.1.4 and use more derivatives at the faces of T also for $r > 2$.

Remark 7.2 In Theorem 6.1 we have shown, that for any sufficiently smooth function f there is a unique spline $s \in \mathcal{S}_d^{r, \rho, \eta, \eta}(\Delta_A)$ solving the Hermite interpolation problem

$$\lambda s = \lambda f, \quad \text{for all } \lambda \in \mathcal{N}_\Delta.$$

It can easily be seen, that the interpolation is local and stable and thus the Hermite interpolation scheme has optimal approximation order ([11]).

Remark 7.3 We would like to thank Peter Alfeld for writing a great JAVA program which can be used to explore trivariate splines spaces. We made extensive use of the program to test our results for $r = 1, \dots, 6$. The program can be found at <http://www.math.utah.edu/~alfeld/3DMDS/>.

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