A New Kind of Trivariate C[1] Macro-Element

Ming-Jun Lai [1)] and A. LeMéhauté [2)]

Abstract. We propose a construction of a trivariate C^1 macro-element over a special tetrahedral partition and compare our construction with known C^1 macro-elements which are summarized in this paper. Also, we propose an improvement of the Alfeld construction of a C^1 quintic macro-element such that the new scheme is able to reproduce all polynomials of total degree ≤ 5.

§1. Introduction

The study of trivariate spline functions was pioneered by Ženíček, LeMéhauté, Alfeld, Worsey, Farin among others. See references [Ženíček'73], [LeMehaute'84], [Alfeld'84], [Worsey and Farin'87], and [Worsey and Piper'88]. Most of these results involve the construction of trivariate C^1 macro-elements over a tetrahedral partition Δ or a refinement of Δ. They are generalizations of the well-known bivariate C^1 quintic Argyris element, C^1 Clough-Tocher element, or C^1 quadratic Powell-Sabin element. In this paper we shall present a new kind of trivariate C^1 macro-element which is generalized from the bivariate C^1 cubic FVS elements ([Fraejis en Veubeke'65] and [Sander'64]). Such an element has not been presented in the literature so far to the best of our knowledge. The new C^1 macro-element offers several advantages over the existing C^1 macro-elements cited above. (See Remarks 4.1–4.3 and 4.10.)

To describe the new macro-element, we begin with a special tetrahedral partition: Let $O = \langle v_1, v_2, \cdots, v_6\rangle$ be an octahedron such that the three diagonals of O intersect at a common point m_O inside O as shown in Fig. 1. In this case, v_1, v_2, v_3, v_4 are coplanar. So are v_2, v_4, v_5, v_6 and v_1, v_3, v_5, v_6. In this paper we shall restrict our attention to such tetrahedra which will be called *centrical octahedra.* We will show how to partition some common 3D solids into a collection of centrical octahedra. (See Examples 4.6–4.10.) By adding the three planes $\langle v_1, v_2, v_3, v_4\rangle$, $\langle v_1, v_3, v_5, v_6\rangle$, and $\langle v_2, v_4, v_5, v_6\rangle$, we obtain 8 tetrahedra in O. Let

[1)] Department of Mathematics, The University of Georgia, Athens, GA 30602, mjlai@math.uga.edu. Supported by the National Science Foundation under grant DMS-9870187.

[2)] CNRS UMR 6629, Department of Mathematics, University of Nantes, 44072 Nantes, France. alm@math.univ-nantes.fr

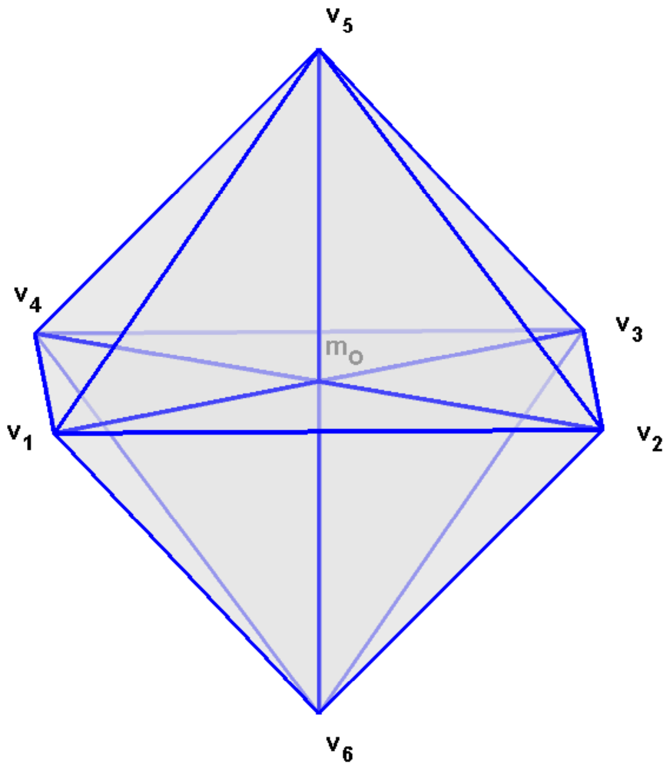

Fig. 1. An octahedron and its tetrahedral partition

$\oplus$ be the collection of these eight tetrahedra. We shall use piecewise trivariate polynomials of degree 5 over $\oplus$ to construct C^1 macro-elements.

Instead of using traditional nodal values to describe the construction of macro-elements, we shall use the B-form (cf. [de Boor'87]) to do so. Let us first briefly introduce the B-form for trivariate polynomials. Let $t = \langle v_1, v_2, v_3, v_4\rangle$ be a tetrahedron with nonzero volume. For any polynomial p of degree ≤ 5, we write

$$p = \sum_{i+j+k+\ell=5} c^t_{ijk\ell} B^t_{ijk\ell}$$

with $B^t_{ijk\ell}(x) = \dfrac{5!}{i!j!k!}\lambda_1^i\lambda_2^j\lambda_3^k\lambda_4^\ell$ the Bézier polynomials of degree 5, where λ_1, λ_2, λ_3, λ_4 are the barycentric coordinates of x with respect to t. This polynomial form is called the B-form. Note that the index i is associated with the first vertex v_1 in $\langle v_1, v_2, v_3, v_4\rangle$, index j is associated with the second vertex v_2, index k is associated with the third vertex v_3 and index ℓ is associated with last vertex v_4. The conditions which ensure that two polynomials defined on two neighboring tetrahedra join in a smooth fashion are called the smoothness conditions and may be given in terms of the coefficients of polynomials in B-form. For $t' = \langle v_2, v_3, v_4, v_5\rangle$ which share a common face $\langle v_2, v_3, v_4\rangle$ with t, let $q = \sum_{i+j+k+\ell=5} c^{t'}_{ijk\ell} B^{t'}_{ijk\ell}$. Then the C^1 smoothness conditions are as follows.

Lemma 1.1. *Let p and q be two quintic polynomials defined on t and t' which share a common face $\langle v_2, v_3, v_4\rangle$. Then p and q join in a C^1 fashion if and only if*

$$c^t_{0,j,k,\ell} = c^{t'}_{j,k,\ell,0}, \quad j+k+\ell = 5$$

and

$$c^t_{1,j,k,\ell} = \alpha c^{t'}_{j+1,k,\ell,0} + \beta c^{t'}_{j,k+1,\ell,0} + \gamma c^{t'}_{j,k,\ell+1,0} + \theta c^{t'}_{j,k,\ell,1}$$

for all $i+j+k = 4$, where $(\alpha, \beta, \gamma, \theta)$ is the vector of barycentric coordinates of v_5 with respect to $t = \langle v_1, v_2, v_3, v_4\rangle$. That is, $v_5 = \alpha v_1 + \beta v_2 + \gamma v_3 + \theta v_4$ and $\alpha + \beta + \gamma + \theta = 1$.

We refer to [de Boor'87] for a proof of this lemma and the C^k smoothness conditions for $k \geq 2$. In general, we shall use the B-form to express trivariate piecewise polynomial (spline) functions of any degree. Let Δ be a collection of tetrededra which is a triangulation in the sense that for any $t \in \Delta$ and $t' \in \Delta$, the intersection $t \cap t'$ is either empty or their common vertex or their common edge or their common face. Fix two integers $r \geq -1$ and $d > r$. Let

$$S^r_d(\Delta) = \{s \in C^r(\cup_{t\in\Delta} t) : s|_t \in \mathbb{P}_d \quad \forall t \in \Delta\}$$

be the trivariate spline space of degree d and smoothness r, where $\mathbb{P}_d$ stands for the space of all polynomials of degree $\leq d$. For each spline function $s \in S^{-1}_d(\Delta)$, we shall write

$$s|_t = \sum_{i+j+k+\ell=d} c^t_{ijk\ell} B^t_{ijk\ell}, \quad \forall t \in \Delta$$

with B-coefficients $c^t_{ijk\ell}$. For tetrahedron $t = \langle v_1, v_2, v_3, v_4\rangle$, let

$$\xi^t_{i,j,k,\ell} = \frac{iv_1 + jv_2 + kv_3 + \ell v_4}{d}$$

be the domain point of t associated with the index (i, j, k, ℓ) with $i+j+k+\ell = d$. For convenience, we shall associate $c^t_{ijk\ell}$ with $\xi^t_{ijk\ell}$. We denote by $\mathcal{I}_d(\Delta) := \{(\xi^t_{i,j,k,\ell} : i+j+k+\ell = d, t \in \Delta\}$ the collection of all domain points associated with $S^{-1}_d(\Delta)$. We shall use the concept of 3D minimal determining set which are a simple generalization of those used in the study of bivariate splines (cf. [Alfeld and Schumaker'87]). Let Γ be a proper subset of $\mathcal{I}_d(\Delta)$. Γ is a *determining set* for a spline subspace $\mathcal{S} \subset S^{-1}_d(\Delta)$ if any spline function $s \in \mathcal{S}$ whose B-coefficients associated with the domain points in Γ are zero is zero everywhere. Γ is a *minimal determining set* if Γ is a determining set and the cardinality of Γ is the smallest possible.

The paper is organized as follows: We first construct our C^1 quintic macro-elements over the special tetrahedral partition $\oplus$ in the next section. Then we shall review the existing constructions of trivariate C^1 macro-elements in §3. We use the B-form to explain these constructions and provide an improved version of the Alfeld C^1 quintic macro-element so that all quintic polynomials can be reproduced by using the modified macro-element. With the overview of the new and existing macro-elements, we are able to give several comparisons on the dimensions, approximation powers, and storages of these C^1 macro-elements in §4. Finally, we remark on how to partition some common 3D solids into a centrical octahedral partition.

§2. Construction of C^1 Quintic Macro-Element

Let O be a centrical octahedron and $\oplus$ the tetrahedral partition of O. The macro-element we are going to construct has the following smoothness properties: It is C^1 over the union of the eight tetrahedra of $\oplus$ and C^2 at the six vertices of O. Here, a function s is said to be C^2 at a vertex v if s is twice differentiable at v. We denote by $S_5^{1,2}(\oplus)$ the space of all spline functions in $S_5^{-1}(\oplus)$ which are C^1 across each interior triangular face of $\oplus$ and C^2 at vertices of O.

Next we need additional notation: letting $e = \langle v_1, v_2\rangle$ be an edge of t and $f = \langle v_1, v_2, v_3\rangle$ be a face of t, we denote

$$\begin{aligned}\mathcal{D}^t_m(v_1) &= \{\xi^t_{i,j,k,\ell} : i \geq d-m, i+j+k+\ell = d\},\\ \mathcal{E}^t_m(e) &= \{\xi^t_{i,j,k,\ell} : i+j \leq m, i+j+k+\ell = d\},\\ \mathcal{F}^t_m(f) &= \{\xi^t_{i,j,k,\ell} : \ell \leq m, i+j+k+\ell = d\}\end{aligned}$$

for integer $0 \leq m \leq d$. In this section, we fix $d = 5$. Similarly, we can define these sets for other vertices, other edges and other faces of tetrahedron t. Let m_O denote the intersection of the three diagonals of O. We now specify the following subsets to be formed into a minimal determining set Γ for $S_5^{1,2}(\oplus)$:

(1) For each vertex $v \in \oplus$ except for m_O, let $t_v \in \oplus$ be a tetrahedron having v as one of its vertices. Let $S_v := \mathcal{D}^{t_v}_2(v)$. We note that in terms of traditional nodal values, the determination of the B-coefficients which are associated with domain points in S_v of any spline function s is equivalent to the assignments of $\partial^\alpha s(v)$ for all $|\alpha| \leq 2$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3$ and

$$\partial^\alpha s(v) := \left(\frac{\partial}{\partial x}\right)^{\alpha_1} \left(\frac{\partial}{\partial y}\right)^{\alpha_2} \left(\frac{\partial}{\partial z}\right)^{\alpha_3} s(v).$$

(2) For each boundary edge $e \in \oplus$, let $t_e \in \oplus$ be a tetrahedron containing e. Writing $e = \langle u, v\rangle$, let

$$S_e := \mathcal{E}^{t_e}_1(e) \backslash \left(\mathcal{D}^{t_e}_2(u) \cup \mathcal{D}^{t_e}_2(v)\right).$$

We note that the determination of the B-coefficients associated with domain points in S_e of any spline function s may be replaced by the assignment of two normal derivatives of e which are perpendicular to each other at the midpoint of e if the traditional nodal values are used.

(3) For each boundary face $f \in \oplus$, let $t_f \in \oplus$ be a tetrahedron with f as one of its faces. Writing $f = \langle u_1, u_2, u_3 \rangle$ and $t_f = \langle u_1, u_2, u_3, u_4 \rangle$, let

$$S_f := \{\xi^{t_f}_{2,1,1,1}, \xi^{t_f}_{1,2,1,1}, \xi^{t_f}_{1,1,2,1}\}.$$

In terms of nodal values, the determination of the B-coefficients associated with the domain points in S_f of spline functions is the same as the assignment of the normal derivative of f at the three locations whose barycentric coordinates are $(2/5, 2/5, 1/5, 0), (1/5, 2/5, 2/5, 0), (2/5, 1/5, 2/5, 0)$ with respect to t_f. Here, for convenience, we have arranged that the last index ℓ of B-coefficients $c^t_{i,j,k,\ell}$ is associated with m_O.

(4) For m_O, let $t_{O,n}, n = 1, \cdots, 8$ be the 8 tetrahedra of $\oplus$. For convenience, we arrange that the last index ℓ of B-coefficients $c^{t_{O,n}}_{i,j,k,\ell}$ of any spline function is associated with m_O. Let $S_O := \{\xi^{t_{O,n}}_{1,1,1,2}, n = 1, \cdots, 8\}$. We note that the domain points can also be associated with nodal values, but not needed here. We now show that the set

$$\Gamma := \bigcup_{v \in O} S_v \cup \bigcup_{e \in O} S_e \cup \bigcup_{f \in O} S_f \cup S_O$$

is a minimal determining set for $S^{1,2}_5(\oplus)$. In fact, we have

Theorem 2.1. *Γ as defined above is a minimal determining set for $S^{1,2}_5(\oplus)$ and*

$$\dim(S^{1,2}_5(\oplus)) = 10 \times 6 + 2 \times 12 + 3 \times 8 + 8 = 116.$$

Proof: We use the following figures to show that Γ is a minimal determining set. Note that we use the domain points to show the associated B-coefficients of spline functions. Let $\{\xi^{t_{O,n}}_{i,j,k,\ell}, i+j+k+\ell = 5, n = 1, \cdots, 8\}$ be the domain points of over all tetrahedra in $\oplus$ with index ℓ being associated with m_O. We call the collection of the linear plane $L_{n,\ell}$ containing $\{\xi^{t_{O,n}}_{i,j,k,\ell}, i+j+k = 5-\ell\}$ for $n = 1, \cdots, 8$ *the $(\ell+1)$th layer around m_O*. For example, the first layer is the surface of the octahedron O. For another example, the second layer is the collection of the planes $L_{n,1}$ containing $\xi^{t_{O,n}}_{i,j,k,1}, i+j+k = 4$'s. In Fig. 2, the domain points on the first layer are shown. The domain points on the second layer are displayed in Fig. 3 and the domain points on the third layer are given in Fig. 4. Note that only the domain points on the top half of the octahedron are shown. Only the domain points that can be seen are displayed.

Assume that all B-coefficients associated with domain points in Γ are set. The B-coefficients marked with $\circ$'s in Figs 2, 3, 4 are uniquely determined either since their domain points are in S_v or by the C^2 smoothness conditions around the vertex v using the known B-coefficients whose associated domain points are in

(1). The B-coefficients marked with $\diamond$'s in Fig. 2 are uniquely determined either since their domain points are in S_e or by the C^1 smoothness condition around the edge e using the known coefficients determined from (1) and (2). Note that the smoothness conditions in Lemma 1.1 involve ratios of the volumes of certain tetrahedra with denominators being the volume of one of the given eight tetrahedra. The B-coefficients marked with $\times$'s in Fig. 3 are uniquely determined by the C^1 smoothness condition using the known coefficients marked with $\diamond$'s and $\circ$'s in Fig. 2. The B-coefficients marked with $\star$'s in Figure 3 are determined since their domain points are in (3). The coefficients marked with $\times$'s in Fig. 4 are determined by the C^1 smoothness condition and the B-coefficients marked with $\circ$, $\times$, and $\star$ in Fig. 3. Finally, the coefficients marked with $\triangle$'s are determined in (4). We now claim that the remaining B-coefficients are uniquely determined by the C^1 smoothness conditions.

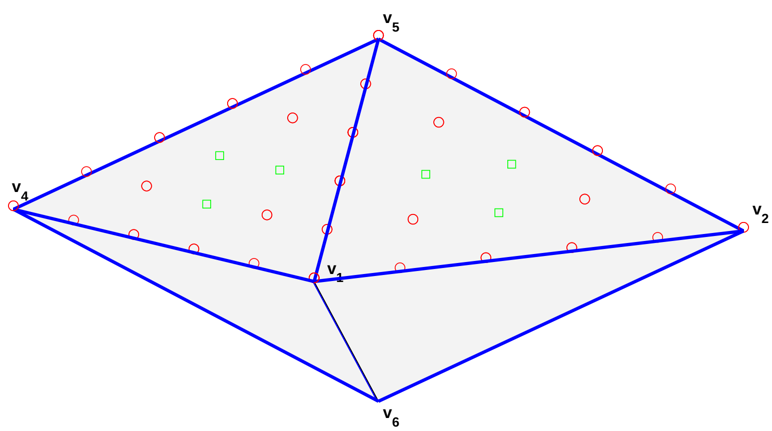

Fig. 2. Domain points (on the first layer) on the top half of an octahedron

Since the three diagonals of each octahedron intersect, we claim that the B-coefficients marked with $\triangle$'s determine the remaining B-coefficients as in the bivariate C^1 cubic FVS element. To prove the claim, we only need to show that the B-coefficient at m_O is uniquely determined by the eight coefficients whose associated domain points are in (4). For simplicity, we first assume that O is an octahedron such that $\langle v_1, v_2, v_3, v_4\rangle, \langle v_1, v_3, v_5, v_6\rangle, \langle v_2, v_4, v_5, v_6\rangle$ are square. Thus, the coefficients of any spline functions corresponding to the domain point set in (4) are at the corners of the cube inside the octahedron as shown in Fig. 5. We now show how to determine the 27 coefficients whose associated domain points located on and inside the cube. For simplicity, let s_1 be a spline in $S^0_5(\oplus)$ whose coefficients assocated with the domain points in Γ are all zero except for one of the eight domain points in (4) which is 1. Then all the coefficients of s_1 are zeros except for the coefficients on the cube which are shown in Fig. 5. That is, $c^{t_{O,1}}_{1,1,1,2} = 1$, $c^{t_{O,1}}_{0,1,1,3} = 1/2$, $c^{t_{O,1}}_{1,0,1,3} = 1/2$,

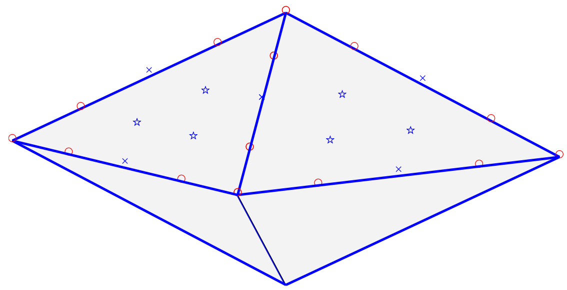

Fig. 3. Domain points (on the second layer) on the top half of an octahedron

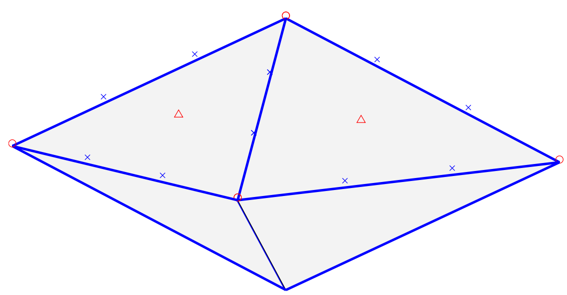

Fig. 4. Domain points (on the third layer) on the top half of an octahedron

$c^{t_O,1}_{1,1,0,3} = 1/2$, $c^{t_O,1}_{1,0,0,4} = 1/4$, $c^{t_O,1}_{0,1,0,4} = 1/4$, $c^{t_O,1}_{0,0,1,4} = 1/4$, and $c^{t_O,1}_{0,0,0,5} = 1/8$. It is easy to verify that s_1 is in $S^{1,2}_5(\oplus)$ by using the smoothness condition (cf. Lemma 1.1). For a general octahedron in $\oplus$, the cube will become a parallelepiped. The nonzero coefficients of s_1 will be $c^{t_O,1}_{1,1,1,2} = 1$, $c^{t_O,1}_{0,1,1,3} = 1/(1+\alpha)$, $c^{t_O,1}_{1,0,1,3} = 1/(1+\beta)$, $c^{t_O,1}_{1,1,0,3} = 1/(1+\gamma)$, $c^{t_O,1}_{1,0,0,4} = 1/((1+\beta)(1+\gamma))$, $c^{t_O,1}_{0,1,0,4} = 1/((1+\alpha)(1+\gamma))$, $c^{t_O,1}_{0,0,1,4} = 1/((1+\alpha)(1+\beta))$, and $c^{t_O,1}_{0,0,0,5} = 1/((1+\alpha)(1+\beta)(1+\gamma))$, where α, β, γ denote the ratios of the lengths of interior edges of O, i.e.

$$\alpha = \frac{|\langle v_3, m_O\rangle|}{|\langle v_1, m_O\rangle|}, \quad \beta = \frac{|\langle v_4, m_O\rangle|}{|\langle v_2, m_O\rangle|}, \quad \gamma = \frac{|\langle v_6, m_O\rangle|}{|\langle v_5, m_O\rangle|}.$$

Similarly, we can construct other spline functions s_i, $i = 2, \ldots, 8$ corresponding to nonzero coefficients associated with the domain points in (4). Any linear combination of these eight spline functions determines the coefficient at m_O. Therefore, we have the claim. It follows that the spline function determined by the values in Γ is globally C^1.

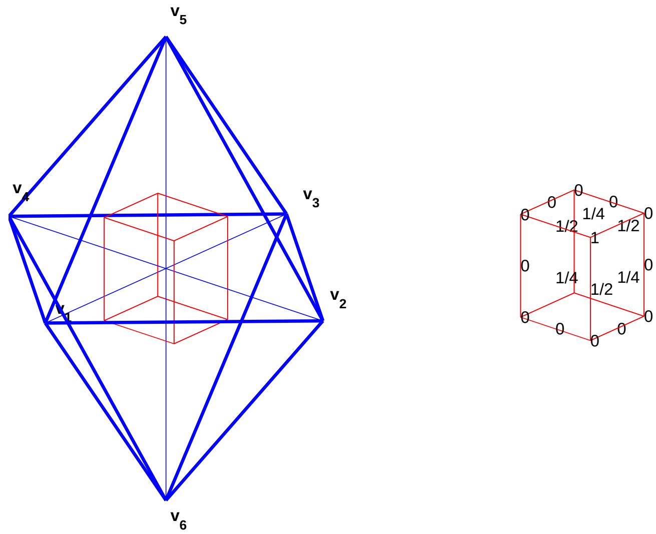

Fig. 5. Coefficients of spline function $s_1 \in S_5^{1,2}(\oplus)$

To construct a basis for $S_5^{1,2}(\oplus)$, we simply let $s_\gamma, \gamma \in \Gamma$ be the cardinal spline function such that the coefficients of s_γ associated with domain points in $\Gamma \backslash \gamma$ are zero while the coefficient of s_γ associated with γ is one. It is easy to see that such s_γ is well defined in $S_5^{1,2}(\oplus)$ by using the smoothness conditions. They are linearly independent. Thus, Γ is a minimal set, and the dimension of $S_5^{1,2}(\oplus)$ follows easily. This completes the proof. $\square$

Next we let $\triangle$ be a collection of centrical octahedra $O_i, i = 1, \ldots, N$. Suppose that $\triangle$ is regular in the sense that the intersection of any two octahedra O_i and O_j is either an empty set, or their common face, or their common edge, or their common vertex. Let $\oplus_i$ be the eight tetrahedra obtained from O_i as described in §1, and let

$$L(\triangle) = \cup_{i=1}^N \oplus_i$$

be the corresponding special tetrahedral partition. We will address how to use such centrical octahedra with half-octahedra to partition several common polygonal domains (cf. Examples 4.6–4.9). The above construction of C^1 quintic macro-element can be applied to such special tetrahedral partitions. To be precise, let $S_5^{1,2}(L(\triangle))$ be the space of all spline functions in $S_5^{-1}(L(\triangle))$ which is C^1 globally and C^2 at each vertex of $\cup_{i=1}^N O_i$. Let $\mathcal{I}_5(L(\triangle)) := \{(\xi^t_{i,j,k,\ell} : i + j + k + \ell = 5, t \in L(\triangle)\}$ be the domain set associated with $S_5^{-1}(L(\triangle))$.

Let us describe a minimal determining set $\Gamma \subset \mathcal{I}_5(L(\triangle))$ for $S_5^{1,2}(L(\triangle))$:

(1) For each vertex $v \in \triangle$, let $t_v \in L(\triangle)$ be a tetrahedron having v as one of its vertices. Let $S_v := \mathcal{D}_2^{t_v}(v)$.

(2) For each edge $e \in \triangle$, let $t_e \in L(\triangle)$ be a tetrahedron containing e. Writing $e = \langle u, v \rangle$, let

$$S_e := \mathcal{E}_1^{t_e}(e) \backslash \left(\mathcal{D}_2^{t_e}(u) \cup \mathcal{D}_2^{t_e}(v) \right).$$

(3) For each triangular face $f \in \triangle$, let $t_f \in L(\triangle)$ be a tetrahedron with f as one of its faces. Writing $f = \langle u_1, u_2, u_3 \rangle$ and $t_f = \langle u_1, u_2, u_3, u_4 \rangle$, let

$$S_f := \{\xi^{t_f}_{2,1,1,1}, \xi^{t_f}_{1,2,1,1}, \xi^{t_f}_{1,1,2,1}\}.$$

(4) For octahedron O_i of $\triangle$, let $t_{i,n}, n = 1, \cdots, 8$ be the 8 tetrahedra of O_i. For convenience, we arrange so that the last index ℓ of B-coefficients $c^{t_{i,n}}_{i,j,k,\ell}$ of any spline function is associated with m_{O_i}. Let $S_{O_i} := \{\xi^{t_{i,n}}_{1,1,1,2}, n = 1, \cdots, 8\}$.

Theorem 2.2. *Let*

$$\Gamma := \bigcup_{v \in \triangle} S_v \cup \bigcup_{e \in \triangle} S_e \cup \bigcup_{f \in \triangle} S_f \cup \bigcup_{i=1}^{N} S_{O_i}.$$

Then Γ is a minimal determining set for $S^{1,2}_5(L(\triangle))$, and

$$\dim S^{1,2}_5(L(\triangle)) = 10V + 2E + 3F + 8N,$$

where V, E, F denote the numbers of all vertices, edges, and faces of $\triangle = \{O_i, i = 1, \cdots, N\}$.

Proof: We first show that the Γ is a determining set. Let $s \in S^{1,2}_5(L(\triangle))$ be a spline function whose B-coefficients associated with domain points in Γ are zero. To see that s is identically zero, we look at s restricted to an octahedron O_i. We claim that $s|_{O_i} \equiv 0$. Indeed, for each vertex v of O_i, we know that the B-coefficients of s associated with the domain points in S_v are zero. These zero coefficients imply that the coefficients of s associated with domain points in $\mathcal{D}^t_2(v)$ are zero, either by the default, i.e., the domain points are already in Γ or by the C^2 smoothness conditions for those tetrahedra $t \in \oplus_i$ that have v as one of its vertices.

For each edge $e = \langle u, v \rangle \in O_i$, by the C^1 smoothness condition or the default, the zero coefficients of s asociated with the domain points in S_e imply that the coefficients of s associated with domain points in $\mathcal{E}^t_1(e) \backslash (\mathcal{D}^t_2(u) \cup \mathcal{D}^t_2(v))$ are zero, for those tetrahedra $t \in \oplus_i$ that share e as one of its edges.

For each face f of O_i, we write $f = \langle u, v, w \rangle$ and $t = \langle u, v, w, m_{O_i} \rangle$. The zero coefficients of s with domain points in S_f implies that the coefficients of s associated with domain points $\xi^t_{2,1,1,1}, \xi^t_{1,2,1,1}, \xi^t_{1,1,2,1}$ are zero by default or the C^1 smoothness condition.

Together with the eight zero coefficients of s with domain points in S_{O_i}, we use Theorem 2.1 to conclude that s is equal to zero. The above discussion shows that Γ is a determining set.

To see that Γ is a minimal determining set, we construct a basis for $S^{1,2}_5(L(\triangle))$ by letting s_γ be a spline function whose coefficients associated with domain points in Γ are zero except for γ and whose coefficient associated with γ is 1. We then use the C^1 and C^2 smoothness conditions to determine the remaining coefficients.

It is easy to show that such s_γ is well-defined and belongs to $S_5^{1,2}(L(\triangle))$. Then the linear independence of $s_\gamma, \gamma \in \Gamma$ implies that Γ is a minimal determining set. Hence, the dimension of $S_5^{1,2}(L(\triangle))$ follows immediately. $\square$

It is easy to see that s_γ is locally supported. That is, the support of s_γ is the union of all octahedra which share a common vertex v if $\gamma \in S_v$, or the union of all octahedra which share a common edge e if $\gamma \in S_e$, or the union of all octahedra which share a common face f if $\gamma \in S_f$, or the octahedron O_i if $\gamma \in S_{O_i}$.

Note that one can construct locally supported splines $s_{v,\beta} \in S_5^{1,2}(L(\Delta))$ which satisfies interpolation properties:

$$\partial^\alpha s_{v,\beta}(u) = \begin{cases} 1, & \text{if } u = v \text{ and } \alpha = \beta \\ 0, & \text{otherwise} \end{cases}$$

for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| \le 2$, where $\beta = (\beta_1, \beta_2, \beta_3)$ with $|\beta| \le 2$ and v is a vertex of $\triangle$. For simplicity, let V also stand for the collection of all vertices of $\triangle$. It follows that for any twice differentiable function g, we can construct its C^1 spline interpolant as

$$s_g = \sum_{v \in V} \sum_{|\alpha| \le 2} \partial^\alpha g(v) s_{v,\alpha}.$$

We remark that we are not able to construct a C^1 macro-element over $L(\triangle)$ using polynomials of degree strictly less than 5.

§3. Other Constructions of Trivariate C^1 Macro-elements

In this section, we collect all the constructions (to our knowledge) of trivariate C^1 macro-elements over tetrahedral partitions or some refinements of tetrahedral partitions. We shall use the concept of minimal determining sets to explain these constructions. The purpose of our discussion is to enable us to compare our macro-elements with the existing ones.

3.1. C^1 Splines of Degree 9 over Tetrahedral Partitions

Let Δ be a tetrahedral partition of a polygonal domain in $\mathbb{R}^3$. The first construction of C^1 macro-elements over tetrahedral partitions was obtained by Ženiček in [Ženiček'73]. See also [LeMéhauté'84]. In our notation, these macro-elements are in $S_9^{1,2,4}(\Delta)$ which is the space of all spline functions in $S_9^{-1}(\Delta)$ that are C^1 over the union of all tetrahedra in Δ, C^2 around each edge of Δ, and C^4 at each vertex of Δ. Here, a function s is said to be C^2 around an edge if s is twice differentiable at each point of e and C^4 at a vertex v if s is four times differentiable at v. Let us briefly explain the construction in terms of minimal determining sets.

Let $\mathcal{I}_9(\Delta) := \{\xi^t_{ijk\ell}, i+j+k+\ell = 9, t \in \Delta\}$ be the collection of the domain points associated with $S_9^{-1}(\Delta)$. In this subsection, we fix $d = 9$ in the definition of $\mathcal{D}^t_m(v), \mathcal{E}^t_m(e)$, and $\mathcal{F}^t_m(f)$ as defined in §2. We shall specify the following domain point sets to form a minimal determining set for $S_9^{1,2,4}(\Delta)$ (cf. Fig. 6).

(1) For each vertex $v \in \Delta$, choose a tetrahedron t_v in Δ such that t_v contains v. Let

$$S_v := \mathcal{D}_4^{t_v}(v).$$

We note that the determination of the coefficients of a spline s associated with domain points in S_v is equivalent to the assignment of all derivatives of order 4 of s at v. The domain points in $S_{v_1}, S_{v_2}, S_{v_3}, S_{v_4}$ are marked with $\circ$'s in Fig. 6.

(2) For each edge $e \in \Delta$, choose a tetrahedron t_e in Δ such that t_e contains e. Writing $e = \langle u, v \rangle$, we let

$$S_e := \mathcal{E}_2^{t_e}(e) \backslash \left(\mathcal{D}_4^{t_e}(u) \cup \mathcal{D}_4^{t_e}(v) \right).$$

The domain points in S_e for all edges of a tetrahedron are marked with $\diamond$'s in Fig. 6.

(3) For each face $f \in \Delta$, choose a tetrahedron t_f in Δ containing f. Writing $f = \langle u, v, w \rangle$ and $t_f = \langle u, v, w, x \rangle$, we let

$$\begin{aligned} S_f :=& \mathcal{F}_1^{t_f}(f) \\ & \backslash \Big(\mathcal{D}_4^{t_f}(u) \cup \mathcal{D}_4^{t_f}(v) \cup \mathcal{D}_4^{t_f}(w) \cup \mathcal{E}_2^{t_f}(\langle u, v \rangle) \cup \mathcal{E}_2^{t_f}(\langle v, w \rangle) \cup \mathcal{E}_2^{t_f}(\langle u, w \rangle) \Big). \end{aligned}$$

The domain points in S_f for all faces are marked with $\star$'s in Fig. 6.

(4) For each tetrahedron t, let S_t be the remaining coefficients on t, i.e.,

$$S_t = \{(t, i, j, k, \ell), i \geq 2, j \geq 2, k \geq 2, \ell \geq 2\}.$$

The domain points in S_t are marked with $\triangle$'s in Fig. 6.

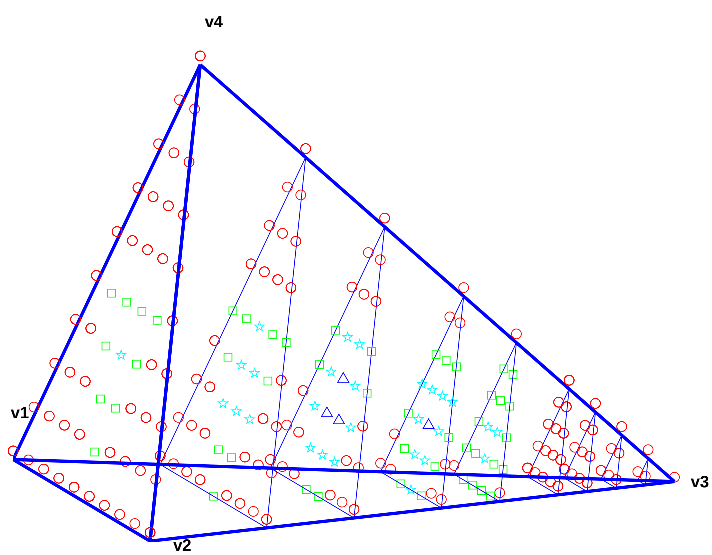

Fig. 6. Indication of domain point subsets on a tetrahedron

Let

$$\Gamma := \bigcup_{v\in\Delta} S_v \cup \bigcup_{e\in\Delta} S_e \cup \bigcup_{f\in\Delta} S_f \cup \bigcup_{t\in\Delta} S_t.$$

Then it can be shown as in the previous section that Γ forms a minimal determining set. Using Γ, we can construct a basis $s_\gamma, \gamma \in \Gamma$ for $S_9^{1,2,4}(\Delta)$ as before. Thus, we have

Theorem 3.1. *Let Δ be a tetrahedral partition of a polygonal domain Ω in $\mathbb{R}^3$. Then there exists a locally supported basis for $S_9^{1,2,4}(\Delta)$, and*

$$\dim S_9^{1,2,4}(\Delta) = 35V + 8E + 7F + 4T,$$

where V, E, F, T denote the number of vertices, edges, faces, and tetrahedra of Δ.

It is well-known that one can use $S_9^{1,2,4}(\Delta)$ to construct C^1 spline interpolants. We omit the details here.

3.2. The C^1 Quintic Macro-elements over Clough-Tocher Tetrahedral Partition

Let Δ be a tetrahedral partition of a polygonal domain Ω in $\mathbb{R}^3$. In [Alfeld'84], Alfeld generalized the well-known bivariate Clough-Tocher split of triangles to the trivariate setting. He split each tetrahedron t into four subtetrahedra at the center m_t of t using the triangular planes each of which consists an edge of t and m_t. For simplicity, we denote the refinement of a tetrahedral partition Δ by $A(\Delta)$. In his paper [Alfeld'84], Alfeld constructed a kind of C^1 quintic spline interpolant over $A(\Delta)$. It is easy to see that his construction directly yields C^1 quintic macro-elements. One of the properties of his interpolant is the reproduction of all trivariate cubic polynomials. After modifying his construction of C^1 quintic interpolatory scheme, we find a new version of C^1 quintic interpolants which are able to reproduce all quintic polynomials. In this subsection, we discuss the improved version of the Alfeld C^1 quintic interpolants. Let

$$S_5^{1,2,3}(A(\Delta)) = \{s \in S_5^{-1}(A(\Delta)) : s \in C^1(\Omega), s \in C^2(v), v \in \Delta, s \in C^3(m_t), t \in \Delta\}$$

be the space of all spline functions which are C^1 globally while C^2 at each vertex of Δ and C^3 at the center m_t of each tetrahedron $t \in \Delta$. Let

$$\mathcal{I}_5(A(\Delta)) = \{\xi^t_{ijk\ell}, i + j + k + \ell = 5, t \in A(\Delta)\}$$

be the collection of the domain points of $S_5^{-1}(A(\Delta))$.

To find a minimal determining set for the spline space $S_5^{1,2,3}(\Delta))$, we choose the following domain points. (Note that we fix $d = 5$ when using the notation of $\mathcal{D}^t_m(v), \mathcal{E}^t_m(e)$, and $\mathcal{F}^t_m(f)$ in this subsection.)

(1) For each vertex v of Δ, choose a tetrahedron t_v in $A(\Delta)$ containing v. Let

$$S_v := \mathcal{D}^{t_v}_2(v).$$

(2) For each edge e of Δ, choose a tetrahedron t_e in $A(\Delta)$ having an edge e. Writing $e = \langle u, v \rangle$. Let

$$S_e := \mathcal{E}_1^{t_e}(e) \backslash \left(\mathcal{D}_2^{t_e}(u) \cup \mathcal{D}_2^{t_e}(v) \right).$$

(3) For each face f of Δ, choose a tetrahedron t_f in $A(\Delta)$ such that t_f contains f. Writing $f = \langle u, v, w \rangle$ and $t_f = \langle f, x \rangle$, let

$$S_f := \{\xi^{t_f}_{2,1,1,1}, \xi^{t_f}_{1,2,1,1}, \xi^{t_f}_{1,1,2,1}\}.$$

(4) For each tetrahedron $t = \langle u, v, w, x \rangle \in \Delta$, let m_t be the center of t and $t_1 = \langle u, v, w, m_t \rangle, t_2 = \langle v, w, x, m_t \rangle, t_3 = \langle w, u, x, m_t \rangle$, and $t_4 = \langle u, v, x, m_t \rangle$ be four tetrahedra in $A(\Delta)$ contained in t. Let

$$S_t := \{\xi^{t_1}_{1,1,1,2}, \xi^{t_2}_{1,1,1,2}, \xi^{t_3}_{1,1,1,2}, \xi^{t_4}_{1,1,1,2}\}.$$

Theorem 3.2. *Let $A(\Delta)$ be the Alfeld refinement of tetrahedral partition Δ. Then*

$$\Gamma := \bigcup_{v \in \Delta} S_v \cup \bigcup_{e \in \Delta} S_e \cup \bigcup_{f \in \Delta} S_f \cup \bigcup_{t \in \Delta} S_t$$

is a minimal determining set for $S_5^{1,2,3}(A(\Delta))$, and

$$\dim S_5^{1,2,3}(A(\Delta)) = 10V + 2E + 3F + 4T,$$

where V, E, F, T denote the number of all vertices, edges, faces, and tetrahedra of Δ. Also, there exists a locally supported basis for $S_5^{1,2,3}(A(\Delta))$.

Proof: We use Figs. 7–9 to help explain that Γ is a minimal determining set. In these figures, we use the domain points to indicate the coefficients of spline functions. Only domain points on one tetrahedron t are shown. Let $\{\xi^{t_n}_{i,j,k,\ell}, i + j + k + \ell = 5, n = 1, \cdots, 4\}$ be the domain points on four subtetrahedra in t with index ℓ being associated with m_t. We call the collection of the linear plane $L_{n,\ell}$ containing $\{\xi^{t_n}_{i,j,k,\ell}, i + j + k = 5 - \ell\}$ for $n = 1, \cdots, 4$ the $(\ell + 1)$th layer around m_t. For example, the first layer is the surface of tetrahedron t. For simplicity, only the coefficients which can be seen are displayed. In Fig. 7, the coefficients $c^{t_\ell}_{ijk0}, i + j + k = 5, n = 1, 2, 3, 4$ on the first layer are shown. Fig. 8 shows the coefficients $c^{t_\ell}_{i,j,k,1}, i + j + k = 4, \ell = 1, 2, 3, 4$ on the second layer. and in Fig. 9 the coefficients $c^{t_\ell}_{i,j,k,2}, i + j + k = 3, \ell = 1, 2, 3, 4$ are displayed.

Assume that the coefficients of a spline function whose domain points are in Γ are determined. The coefficients marked with $\circ$'s are determined either in (1) or using the C^2 smoothness condition around the vertices based on the coefficients already determined in (1). The coefficients marked with $\diamond$'s are determined either in (2) or using the C^1 condition around edges based on the coefficients already determined in (2). The coefficients marked with $\times$'s in Fig. 8 are obtained by the C^1 smoothness condition across the interior faces based on the coefficients marked with $\circ$ and $\diamond$'s in Fig. 7. The coefficients marked with $\star$'s are determined either in (3) or using the C^1 smoothness condition across a face based on the coefficients already determined in (3). The coefficients marked with $\times$'s in Fig. 9 are determined using the C^1 smoothness condition across interior faces. Finally, the coefficients marked with $\triangle$'s are determined in (4). The coefficients in Fig. 9 completely determine a cubic polynomial. Consequently, we obtain the coefficients $c^{t_n}_{ijk\ell}, i = j + k + \ell = 5, \ell \geq 3, n = 1, 2, 3, 4$. Hence, the spline function determined above is C^3 at m_t. We have shown that the set Γ is a determining set. That is, if all the coefficients in Γ are zero, the above arguments show that the remaining coefficients of a spline have to be zero. Also, the arguments show that we can construct a set of cardinal functions $\{s_\gamma, \gamma \in \Gamma\}$ which are linear independent. Thus Γ is a minimal determining set. That is, the collection of all cardinal basis functions form a basis for $S^{1,2,3}_5(A(\Delta))$. Also, it is easy to see that all basis functions are locally supported.

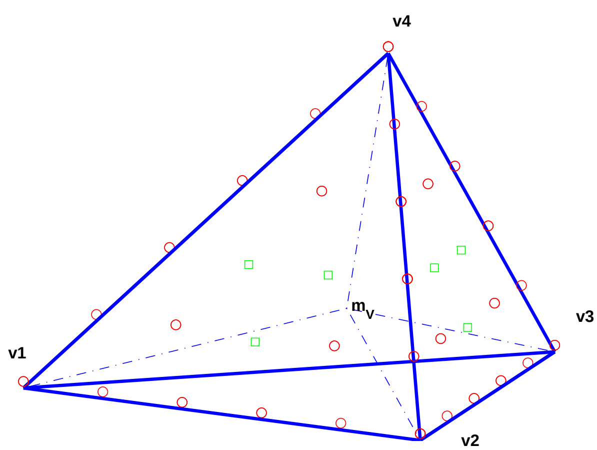

Fig. 7. Domain point (on the first layer) on the Alfeld split of a tetrahedron

The computation of the dimension of the spline space $S^{1,2,3}_5(A(\Delta))$ is straightforward. This completes the proof. $\square$

Next we construct an interpolation scheme which is able to reproduce all polynomials of degree ≤ 5. We begin with the following

Lemma 3.3. *Let $T = \langle v_1, v_2, v_3, v_4 \rangle$ be a tetrahedron and v_5 be another point which does not lie on any of four planes each of which is spanned by one of faces of T. Given $f_{i,\alpha}, |\alpha| \leq 1, i = 1, 2, 3, 4, 5$, there exists a unique cubic polynomial p satisfying the following interpolation conditions:*

$$D^\alpha p(v_i) = f_{i,\alpha}, i = 1, 2, \cdots, 5, \quad |\alpha| \leq 1.$$

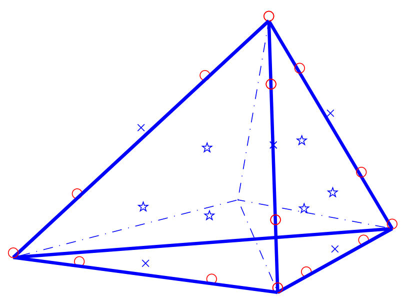

Fig. 8. Domain point (on the second layer) on the Alfeld split of a tetrahedron

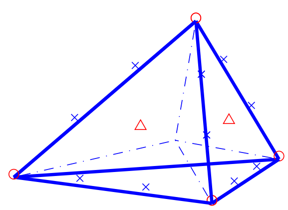

Fig. 9. Domain points (on the third layer) on the Alfeld split of a tetrahedron

This is a well-known result. We refer to [LeMéhauté'84] for a detailed proof. We are now ready to describe an interpolation scheme.

For each edge e, let m_e be the midpoint of e and let $e_{\perp,1}$ and $e_{\perp,2}$ be two directions which are perpendicular to e and are linearly independent to each other. For each face $f = \langle v_1, v_2, v_3 \rangle$, let f_1, f_2, f_3 be the three domain points $\{(iv_1 + jv_2 + kv_3)/5, (i,j,k) = (2,2,1), (1,2,2), (2,1,2)\}$ on f. Let n_f be a normal unit vector of f. For each tetrahedron t, let m_t be the center point of t. Our interpolation scheme is as follows: For a function $g \in C^2(\Omega)$, let $S_g \in S_5^{1,2,3}(A(\Delta))$ satisfy

(1) For each vertex v of Δ,

$$D^\alpha S_g(v) = D^\alpha g(v), \quad \forall |\alpha| \le 2;$$

(2) For each edge e of Δ,

$$D_{e_{\perp,i}} S_g(m_e) = D_{e_{\perp,i}} g(m_e), \quad i = 1, 2;$$

Here, $D_{e_{\perp,i}}$ denotes the derivative along direction $e_{\perp,i}$, $i = 1, 2$;

(3) For each face f of Δ,

$$D_{n_f} S_g(f_j) = D_{n_f} g(f_j), \quad j = 1, 2, 3;$$

Here, D_{n_f} denotes the derivative along direction n_f.

(4) For each tetrahedron t,

$$D^\alpha S_g(m_t) = D^\alpha g(m_t), \quad \forall |\alpha| \le 1.$$

We use the interpolation conditions (1)–(3), C^2 smoothness conditions at the vertices of Δ, C^1 smoothness conditions around the edges and across the faces of Δ to determine the coefficients of S_g. Indeed, the interpolation conditions (1) and C^2 smoothness conditions at the vertices of Δ determine the coefficients of S_g whose domain points are in $\mathcal{D}_2^t(v)$ for each tetrahedron $t \in A(\Delta)$ which shares vertex v for all vertices of Δ. We use the interpolation conditions (2) and C^1 smoothness conditions to determine the coefficients whose domain points in $\mathcal{E}_1^t(e)$ for each tetrahedron $t \in A(\Delta)$ which shares edge e for all edges of Δ. We then use the interpolation conditions (3) and C^1 smoothness conditions to determine the coefficients whose domain points in $\mathcal{F}_1^t(f)$ for each tetrahedron $t \in A(\Delta)$ which shares face f for all faces of Δ. To determine the remaining coefficients of S_g, we consider S_g restricted on tetrahedron $t \in \Delta$. There are four interior edges inside t connecting to the center m_t of t. By the C^1 smoothness conditions around each of the four interior edges, we obtain the coefficients whose associated domain points are in the collection of $\{\xi^{t_\ell}_{i,j,k,2}, i+j+k=3\} \backslash \xi^{t_\ell}_{1,1,1,2}, \ell = 1,2,3,4$ as well as $\{\xi^{t_\ell}_{2,0,0,3}, \xi^{t_\ell}_{0,2,0,3}, \xi^{t_\ell}_{0,0,2,3}, \ell = 1,2,3,4\}$, where $t_\ell, \ell = 1,2,3,4$ denote the four subtetrahedra of t. Note that four of them in the first of the above two groups have already been determined by the interpolation conditions in (1). The coefficients just determined in the previous sentence can be converted to the function and first order derivative values at four vertices of t. Together with the interpolation conditions in (4), we can apply Lemma 3.3 to get a unique cubic polynomial p_f satisfying the interpolation conditions at the five vertices. We then find the coefficients of p_f over four subtetrahedra t_ℓ's and use these coefficients for the remaining coefficients of S_g. It follows that S_g is C^3 at m_t. Thus we know $S_g \in S_5^{1,2,3}(A(\Delta))$. In particular, $S_g = g$ for each polynomial g of degree ≤ 5. This show that the interpolation scheme improves the one in [Alfeld'84] which can reproduce only polynomials of degree ≤ 3.

3.3. C^1 Cubic Splines over Another Clough-Tocher Refinement of Tetrahedral Partitions

In [Worsey and Farin'87], Worsey and Farin split each tetrahedron of Δ even further than Alfeld did. That is, they first split tetrahedron t at its center m_t of the inscribed sphere of t into four subtetrahedra by triangular planes each of which consists of one edge of t and m_t. Then for each interior face shared by two tetrahedra t and t', they connect m_t and $m_{t'}$ by a line which intersects the common face f at n_f and split the subtetrahedron of t containing f into three subsubtetrahedra using the triangular planes each of which consists of n_f, m_t and one vertex of f. Similarly, they split the subtetrahedron of t' containing f into three subtetrahedra in the same way. Since m_t and $m_{t'}$ are the centers of the inscribed spheres, it is always true that n_f is strictly inside the face f. For a boundary face f, they split the subtetrahedra of t containing f into three subsubtetrahedra at the center n_f of f using triangular planes each of which consists of n_f, m_f, and one vertex of f. This is another 3D generalization of the well-known Clough-Tocher refinement of triangulation. Let us use $WF(\Delta)$ to denote such a refinement.

In [Worsey and Farin'87], locally supported spline functions in $S^1_3(WF(\Delta))$ were constructed. In our notation, we specify the following two domain point sets to be formed into a minimal determining set: Note that we fix $d = 3$ when we use the notation $\mathcal{D}^t_m(v)$ and $\mathcal{E}^t_m(e)$ in this subsection.

(1) For each vertex v of Δ, choose a tetrahedron t_v of $WF(\Delta)$ containing v. Let

$$S_v := \mathcal{D}^{t_v}_1(v).$$

(2) For each edge e of Δ, choose a tetrahedron t_e of $WF(\Delta)$ containing e. Writing $e = \langle u, v\rangle$, let

$$S_e := \mathcal{E}^{t_e}_1(e)\backslash \left(\mathcal{D}^{t_e}_1(u) \cup \mathcal{D}^{t_e}_1(v)\right).$$

Then letting $\Gamma = \cup_{t\in\Delta} S_v \bigcup \cup_{e\in\Delta} S_e$, we can prove that Γ is a minimal determining set for $S^1_3(WF(\Delta))$. We leave the details to the interested reader. Thus, we may summarize the results above in

Theorem 3.4. *Let Δ be a tetrahedral partition and let $WF(\Delta)$ be the Worsey-Farin refinement of Δ. Then the dimension of $S^1_3(WF(\Delta))$ is*

$$\dim S^1_3(WF(\Delta)) = 4V + 2E.$$

There exists a locally supported basis for $S^1_3(WF(\Delta))$. Furthermore, each spline function in $S^1_3(WF(\Delta))$ is in C^2 at m_t for all $t \in \Delta$.

3.4. C^1 Quadratic Splines on a Powell-Sabin Refinement

In [Worsey and Piper'88], Worsey and Piper refined each tetrahedron in Δ even further than [Worsey and Farin'87] to construct C^1 spline functions using piecewise quadratic polynomials. The tetrahedral partition has to satisfy a stringent condition. For a general tetrahedral partition, their construction will not be in C^1 globally.

§4. Remarks and Examples

We now compare the constructions of all C^1 macro-elements. The only assumption we make is that a polygonal domain Ω admits a partition using our centrical octahedra. Let us say $\triangle = \cup_{i=1}^n O_i$. To partition $\triangle$ into tetrahedra, we subdivide each octahedron O_i into four tetrahedra by triangular planes each of which consists of one diagonal d_i of O_i and one of the other four vertices of O_i, i.e., the vertices of O_i which are not on d_i. Thus, $O_i = t^1_{O_i} \cup t_{O_i,2} \cup t_{O_i,3} \cup t_{O_i,4}$, where $t_{O,i,j}, j = 1,2,3,4$ denote the four tetrahedra. Let $T(\Delta)$ denote the induced tetrahedral partition which consists of $4N$ tetrahedra. Recall that $L(\triangle)$ is the special tetrahedral refinement of the octahedral partition $\triangle$. Let V, E, F, N denote the numbers of vertices, edges, faces, and octahedra of $\triangle$ and let V_T, E_T, F_T, N_T denote the numbers of vertices, edges, faces, and tetrahedra in $T(\triangle)$. Note that $V_T = V, E_T = E + N, F_T = F + 4N, N_T = 4N$.

Remark 4.1. *Under the assumption above, we have the following dimensions of various C^1 spline spaces:*

$$\begin{aligned}
\dim S_9^{1,2,4}(T(\triangle)) &= 35V_T + 8V_T + 7F_T + 4N_T = 35V + 8E + 7F + 52N,\\
\dim S_5^{1,2,3}(A(T(\triangle))) &= 10V_T + 2E_T + 3F_T + 4N_T = 10V + 2E + 3F + 30N,\\
\dim S_3^1(WF(T(\triangle))) &= 4V_T + 2E_T = 4V + 2E + 2N,\\
\dim S_5^{1,2}(L(\triangle)) &= 10V + 2E + 3F + 8N. \quad \square
\end{aligned}$$

The space $S_3^1(WF(T(\triangle))$ has the smallest dimension and also has the lowest approximation power (cf. Remark 4.2). The space with the second smallest dimension is $S_5^{1,2}(L(\triangle))$. $\square$

Remark 4.2. *Concerning the approximation order in the maximum norm, we may use the well-known Bramble-Hilbert lemma to conclude the following:*

$$\begin{aligned}
dist(f, S_9^{1,2,4}(T(\triangle))) &\le C|\triangle|^{10},\\
dist(f, S_5^{1,2,3}(A(T(\triangle)))) &\le C|\triangle|^{6},\\
dist(f, S_3^1(WF(T(\triangle)))) &\le C|\triangle|^{4},\\
dist(f, S_5^{1,2}(L(\triangle))) &\le C|\triangle|^{6}.
\end{aligned}$$

Here, the approximation constants are dependent on the geometric shape of the tetrahedral partition. Excluding the geometric factor, $S_9^{1,2,4}(T(\Delta))$ has the highest approximation power, but has the largest dimension (cf. Remark 4.1). Our spline space $S_5^{1,2}(L(\Delta)))$ has the second highest approximation power and the second smallest in dimension. $\square$

Remark 4.3. *We now compare the number of coefficients of any spline on one octahedron using these four spline spaces. Let Δ be an octahedron and $\oplus$ be a tetrahedral partition of Δ by splitting O into 8 subtetrahedra. Let $T(\Delta)$ be a* *tetrahedral partition of Δ by splitting into 4 subtetrahedra. We have the following cardinality formulae:*

$$S_9^0(T(\Delta)) = 630$$
$$S_5^0(A(T(\Delta))) = 382$$
$$S_3^0(WF(T(\Delta)) = 276$$
$$S_5^0(L(\Delta)) = 231$$

For the purpose of evaluation, the smaller the cardinality the better. Thus, our spline space $S_5^{1,2}(L(\Delta))$ will be most efficient for evaluation. □

Remark 4.4. *Awanou and Lai have implemented the improved Alfeld interpolation scheme in [Awanou and Lai'02] in MATLAB. The computational experiments show that the interpolation scheme does reproduce all quintic polynomials and has excellent approximation properties.* □

Remark 4.5. *Lai and Wenston have implemented the Farin and Worsey C^1 cubic spline space in MATLAB (cf. [Lai and Wenston'01]). They applied the spline space for numerical solution of 3D biharmonic equations.* □

We next discuss how to partition some bounded domain Ω in $\mathbb{R}^3$ into centrical octahedral partitions. In general, we need to combine several halves of octahedra with the centrical octahedra to complete the task. Once we have a centrical octahedral partition Δ of Ω, we have a special tetrahedral refinement $L(\Delta)$ as discussed before. Let us use the following examples to illustrate how to partition some common domains.

Example 4.6. *Let $\Omega = [0,1]^3$ be a unit cube. Let $v = (0.5, 0.5, 0.5)^T$. For each edge e of Ω, let f_e be the triangular plane spanned by e and v. We use all f_e to partition Ω. Thus, we obtain six half–octahedra which partition Ω. Subdividing each half–octahedron into four tetrahedra by two triangular faces each of which consists of one diagonal of the square face and v, we obtain a tetrahedral partition of Ω. It is clear that such a tetrahedral partition admits our construction of C^1 quintic spline functions. Indeed, for each half-octahedron O, let $t^{O,i}, i = 1,2,3,4$ be the four tetrahedra partitioning O. Writing $t^{O,i} = \langle v, v_i, v_{i+1}, m_O\rangle$ with $\langle v_1, v_2, v_3, v_4\rangle$ being the square face of the half-octahedron O, we use a method similar to that in §2 to choose domain points. That is, we choose S_v for each vertex v of O. (In this case, we have only 5 vertices instead of 6.) We choose S_e for each boundary edge e of O. (We will have 8 edges instead of 12 before.) We choose S_f for each triangular face f of O. (There are 4 faces now.) Also, we choose $S_O = \{\xi^{t_{O,i}}_{1,1,1,2}, i = 1,2,3,4\}$. In addition, we choose 12 more domain points $\xi^{t_{O,i}}_{0,1,1,3}, \xi^{t_{O,i}}_{0,3,1,1}, \xi^{t_{O,i}}_{0,1,3,1}, i = 1,2,3,4$ to form a minimal determining set for $S_5^{1,2}(\cup_{i=1}^4 t^{O,i})$. The proof is the same as before and we omit the details.* □

Example 4.7. *Let $\Omega = [0,1]^3$. In general, we may subdivide Ω into many small parallelepipeds by using planes parallel to the three coordinate planes, e.g., x=0.1,*

..., x=0.9, y=0.1, ..., 0.9, z=0.1, ..., 0.9. We obtain 1000 parallelepipeds. We then partition each parallelepiped as in Example 4.6 into six half–octahedra. Now every pair of neighboring half–octahedra which share a common square face form an octahedron. Then subdividing each octahedron into 8 tetrahedra as in §1 and each half octahedron on the boundary of Ω *as in Example 4.6, we obtain a special tetrahedral partition which admits our construction of* C^1 *quintic macro-elements. It is clear that we can partition* Ω *using non-equally-spaced parallel planes. Also,* Ω *may be a rectangular parallelepiped.* □

Example 4.8. *Let* Ω *be a prism. To partition* Ω *using octahedra, we add another prism to* Ω *such that the union of two prisms forms a rectangular parallelepiped* Ω_1*. Then we partition* Ω_1 *using the method in Example 4.7. Note that we use equally-spaced parallel planes. This results in a tetrahedral partition* $L(\Delta)$ *of* Ω_1 *which admits our construction of* C^1 *quintic spline space. In particular, the tetrahedral partition* Ω_1 *induces a tetrahedral partition over the original* Ω*. The restriction of* $S_5^{1,2}(L(\Omega_1))$ *on* Ω *is the desirable* C^1 *quintic spline space over the prism.* □

Example 4.9. *Let* Ω *be a tetrahedron. To partition* Ω*, we add another two tetrahedra such that three tetrahedra form a prism* Ω_1*. Then we can use the method in Example 4.8 to obtain* $L(\Delta)$ *of* Ω_1*. Note that* $L(\Delta)$ *induces a tetrahedral partition over the original* Ω*.* □

Remark 4.10. *For a general polygonal domain, we should use both tetrahedra and octahedra to partition it and then combine the improved Alfeld interpolation scheme and our* C^1 *quintic macro-elements to build up a spline space* $S_5^{1,2}$*. From Remarks 4.1–4.3, such a mixed spline space will have the same approximation power as the spline space based on the Alfeld* C^1 *quintic interpolation scheme, but will have a smaller dimension and few coefficients. Thus, the mixed spline space will be more efficient.* □

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