Construction of $C^1$ Polygonal Splines over Quadrilateral Partitions: Part I

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Abstract

We are interested in constructing smooth bivariate spline functions over a polygonal partition, and we begin by considering a quadrilateral partition. Vertex splines, introduced in [8] are splines supported over a collection of triangles sharing a vertex. In this paper, we extend the concept of vertex splines to the partition of polygons and describe how to construct $C^1$ vertex splines over a collection of parallelograms first. Although when $\Omega$ is a rectangle, we can use the tensor product of B-splines to construct smooth vertex splines. The significance of our research is to explain how to construct $C^1$ vertex splines over a collection of parallelograms which are not necessarily parallel among themselves. Together with special vertex splines, e.g. edge and face splines, approximation properties of these vertex splines will be shown. Numerical interpolation results and an application to smooth surface reconstruction will be provided. In the end, we describe how to extend the construction of vertex splines of more general $C^r$ smoothness for $r \geq 1$. The results in this paper is the first installment of constructing vertex splines over quadrilateral partition.

1 Introduction

Recently, there have been efforts to use finite element-like functions over polygonal partitions to numerically solve partial differential equations (see [12], and the references therein). Some efforts were made based on virtual finite elements over polygons (see [2], [3], [4]). Several researchers used discontinuous Galerkin methods and a weak Galerkin method over polygons for numerical solutions of PDEs (see [28], [22]). Other attempts have been made to use continuous generalized barycentric coordinates (GBCs) defined on polygons for numerical solution of partial differential equations (see [25], [21], [12], [18]). In the interest of numerically solving PDEs of higher order, we should consider construction of smooth elements over polygonal or polyhedral partitions. Such a construction has not been available in the literature to the best of the authors’ knowledge.

Another motivation for smooth polygonal splines can be found in geometric design, where tensor-product B-spline surfaces have been widely standardized to represent functions and surfaces in research and industries such as aircraft and car body design. However, they are not flexible enough for some geometric modeling, such as a suitcase corner, because the B-spline surface is formed by the union of many collections of exactly four quadrilateral B-spline patches. A standard task in geometric design is to construct a $C^1$ surface to blend tensor-product B-spline patches over quadrilateral meshes which meet at several vertices of a valence other than 4, called extra-ordinarily points (EPs). While this task can be solved by recursive subdivision (e.g. [7]), a blending surface with a finite small number

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of patches is often preferable. Bi-quintic spline surfaces have been constructed by manually adjusting coefficients of 6 types (see [13]) or by solving a minimization for several different types of functionals (see [16] and [15]). It is desirable to have a concrete construction instead of solution of minimization over each extraordinary point.

Approximation theory on multivariate splines has been studied for many years. In particular, the theory of spline functions over triangulations has been fully studied (see [19]), and many applications, including numerical solution of PDE and scattered data fitting, have been thoroughly explored (see [1]). Recently, a construction of locally supported spline functions over polygonal partitions was carried out in [12]. Generalized barycentric coordinates (GBCs) defined on polygons (see [11]) can be pieced together to form continuous vertex spline functions which are supported over the collection of all polygons sharing a common vertex. The locally supported spline functions are continuous, but this construction cannot ensure even $C^1$ smoothness (see [12] for details). It is natural to extend the construction and explore how to construct smoother locally-supported spline functions. Recall in [8], the concept of vertex splines (smooth spline functions supported over a cell of triangles sharing a vertex) was first introduced. Some $C^1$ quintic vertex splines were constructed (see [17] for a detailed construction). Although vertex splines over a collection of parallelograms were considered in [17], no concrete construction was carried out. In fact, constructing smooth vertex splines over polygonal partition has not been well-studied since then.

While our over-arching goal is to describe a method to build such splines over more arbitrary partitions of quadrilaterals (and even polygons of more than 4 sides), the purpose of this paper is to describe a construction of $C^1$ splines over a specialized quadrilateral partition, particularly a collection of parallelograms as in Figure 1, and to report some interesting properties of these splines.

Our construction is based on Wachspress GBCs in our construction first. It happens that, in this special setting of parallelograms, the spline functions which we will construct are in fact bi-quintic polynomials, so we could use the theory of bi-quintic B-splines to explain our construction. However, over more general quadrilaterals, the GBCs will actually be rational functions, and this will be necessary to successfully construct the splines in terms of GBC’s. For this reason, we will study the GBC based construction even in the setting of parallelograms, in order to provide meaningful connections to our next paper, part II on the construction of smooth polygonal splines over quadrilaterals.

Our construction is based on Wachspress coordinates of degree 5. They can be used to form vertex splines satisfying various interpolatory conditions at a vertex, and can be used in a natural and convenient way for reproduction of polynomials. Let $\mathcal{P}$ be a collection of parallelograms whose nonempty intersection of two parallelograms is either a common vertex or an common edge of the two parallelograms. At each vertex $v \in \mathcal{P}$ we shall construct $C^1$ vertex splines $\psi_v, \psi_{x,v}, \psi_{y,v}, \psi_{x^2,v}, \psi_{xy,v}$,
and $\psi_{y^2,v}$ such that they are supported over $\Omega_v$, the union of all parallelograms in $P$ which share the common vertex $v$, and satisfy some interpolatory conditions. For example, $\psi_v$ is a $C^1$ function over $\Omega$, supported only in $\Omega_v$, and satisfies

$$\psi_v(w) = \delta_{v,w}, \nabla \psi_v(w) = 0, \nabla^2 \psi_v(w) = 0$$

(1)

for all vertices $w$ of $P$. Similarly, $\psi_{x,v}$ is a $C^1$ function over $\Omega$ which is supported over $\Omega_v$ satisfying

$$\psi_{x,v}(w) = 0, \nabla \psi_{x,v}(w) = (\delta_{v,w}, 0), \nabla^2 \psi_{x,v}(w) = 0$$

(2)

for all vertices $w$ of $P$. The remaining functions are defined similarly. See their figures in the next sections. We shall also explain our construction using bi-quintic B-spline surfaces later.

Let $S(P) = \{\sum_{v \in P} \sum_{\alpha,\beta \leq 2} c_{\alpha,\beta,v} \psi_{x^\alpha y^\beta,v}, c_{\alpha,\beta,v} \in \mathbb{R}\}$ be the $C^1$ vertex spline space over the partition $P$. It is clear that $P$ can be uniformly refined; hence, we let $P_k$ be the uniform refinement of $P_{k-1}$ starting with a partition $P_1$ of parallelograms of $\Omega$. We shall study the approximation properties of $S(P_k)$ by showing that $f - Q_k(f) \to 0$ for some $Q_k(f) \in S(P_k)$ as $k \to \infty$. Since $S(P)$ is not able to reproduce all quintic polynomials, we shall add more special vertex splines which are supported over two parallelograms sharing a common edge or supported over a single parallelogram. They will be called edge splines and face splines. We shall also study the approximation properties of these splines together with $S(P)$.

The paper is organized as follows. We first present the construction of various vertex, edge, and face splines which are locally supported in $P$. Then we explain two constructions of quasi-interpolatory operators, one based on the vertex splines, and another based on all the splines we constructed. Results of numerical approximation using these quasi-interpolatory splines will be demonstrated. Finally, we extend the construction to $C^r$ vertex splines when the degree $d \geq 3r + 2$.

## 2 Construction of Various Vertex Splines

Given a partition $P$ of parallelograms over a polygonal domain $\Omega$, let $V$ be the set of all vertices in $P$. For a vertex $v \in V$, denote by $\Omega_v$ the union of the parallelograms in $P$ which contain $v$. Our construction uses Wachspress generalized barycentric coordinates (see [29]), so for convenience, let us briefly introduce the associated notation.

Let $P_n = \langle v_1, \ldots, v_n \rangle$ be a convex polygon. We use the definition given in [11]: Any functions $\phi_i$, $i = 1, \ldots, n$, will be called generalized barycentric coordinates (GBCs) of $P_n$ if, for all $x \in P_n$, $\phi_i(x) \geq 0$ and

$$\sum_{i=1}^{n} \phi_i(x) = 1, \quad \text{and} \quad \sum_{i=1}^{n} \phi_i(x)v_i = x. \quad (3)$$

When $n = 3$, $P_n$ is a triangle, and the coordinates $\phi_1$, $\phi_2$, $\phi_3$ can be uniquely determined by (3), and are the usual barycentric coordinates. For $n > 3$, the coordinates $\phi_i$ are not uniquely determined by (3) alone, but they share a basic property that they are piecewise linear on the boundary of $P_d$:

$$\phi_i(v_j) = \delta_{ij}, \quad \text{and} \quad \phi_i((1 - \mu)v_j + \mu v_{j+1}) = (1 - \mu)\phi_i(v_j) + \mu \phi_i(v_{j+1}) \text{ for } \mu \in [0, 1]. \quad (4)$$

Wachspress (rational) coordinates are the most commonly used GBCs. For any $x \in P_n$, let $A_i(x)$ be the signed area of the triangle $\langle x, v_i, v_{i+1} \rangle$, and $C_i = A_i(v_{i-1})$. Then define the functions

$$w_i = C_i \prod_{j=1}^{n-2} A_{i+j}, \quad i = 1, \ldots, n \text{ and } W = \sum_{i=1}^{n} w_i.$$
where the functions $A_i$ are indexed cyclically (i.e. $A_{n+1} = A_1$). Then the functions $\phi_i = w_i/W$, $i = 1, \ldots, n$ are the Wachspress GBCs, which are rational functions. See [11] for several other representations of these coordinates.

First, however, we will note an interesting property of Wachspress coordinates on parallelograms. While Wachspress coordinates are generally defined as rational functions, we can actually say the following:

**Lemma 1** Wachspress functions on parallelograms are quadratic polynomials. In fact, it is a tensor production of two linear polynomials.

**Proof.** Let $P = \langle v_1, v_2, v_3, v_4 \rangle$ be a parallelogram. Since $P$ is a parallelogram, then each subtriangle of its vertices has the same area; that is, there is a constant $C$ such that $C_i = C$ for all $i = 1, 2, 3, 4$. Then we can simplify the expression of $\phi_i$ by

$$
\phi_i = \frac{A_{i+1}A_{i+2}}{\sum_{j=1}^{4} A_{j+1}A_{j+2}}
$$

Now, the functions $A_j$ are linear polynomials, and notice that, where $e_j$ is the edge of $P$ joining the vertices $v_j$ and $v_{j+1}$, $A_j|_{e_j} = 0$. Moreover, since $P$ is a parallelogram, it is easy to see that the functions $A_j$ are constant along the edge opposite $e_j$, $e_{j+2}$, and in fact $A_j|_{e_{j+2}} = C$. This implies, then, that $A_j = C - A_{j+2}$. Then we can simplify the sum in the denominator of $\phi_i$ by

$$
\sum_{j=1}^{4} A_{j+1}A_{j+2} = C^2,
$$

so we have

$$
\phi_i = \frac{A_{i+1}A_{i+2}}{C^2},
$$

which is a quadratic polynomial. \qed

By Lemma 1, whatever function $\psi_v$ we build, it will actually be a piecewise polynomial. It is natural to wonder at this point why one would not simply use the well-established methods of tensor-product B-splines to describe the results in this paper. The answer lies in our future goals: we will use similar methods to the ones detailed in this paper to do the construction over a more general quadrilaterals as it is known that tensor-product splines are not defined on arbitrary quadrilaterals. In fact, in the more general case, we will not use polynomials at all, but instead we will use the true rational function structure of the Wachspress coordinates.

We now describe various polygonal splines in the following subsections. We construct three types of splines: one is supported over the union $\Omega_v$ of all parallelograms which share a common vertex $v$, one is supported over two parallelograms which share a common edge, and one is completely supported over a single parallelogram. We shall use Mathematica to do the construction and once we obtain the formula, we will implement them in MATLAB for easy numerical computation.

### 2.1 Value Interpolation Functions $\psi_v$

Our first goal will be to construct a function $\psi_v$ with the following properties:

1. $\psi_v(w) = \delta_{v,w}$ for $w \in V$, 

for some constants $J$ properties listed above regardless of the geometry of edges, but do affect the gradient, are given a coefficient of $K$ some $v$.

With this in mind, we can create a list of linearly independent degree-3 monomials of Wachspress polynomials of $Wachspress$ coordinates on each parallelogram $P$ in $Ω_v$, and we will show that defining $ψ_v$ piecewise as the appropriate polynomial over each parallelogram will satisfy all 5 properties listed above regardless of the geometry of $P$ and $Ω$.

Now, choose a parallelogram $P_1$ in $Ω_v$, and let $P_1$ have vertices $v_1, v_2, v_3, v_4$. Then $v = v_i$ for some $i = 1, 2, 3, 4$. We’ll first construct a function $ψ_i$ on $P_1$, and we’ll do so in a way that an analogous construction on another parallelogram $P_2 ∈ Ω_v$ will join $C^1$-smoothly at $v$ and over a shared edge.

Considering property 2 from above, we know that we will desire that $ψ_i|_{e_{i+1}} = ψ_i|_{e_{i+2}} = 0$, in order to have continuity of $ψ_v$ over all of $Ω$, particularly at the boundary of $Ω_v$. Moreover, we will also need $∇ψ_i|_{e_{i+1}} = ∇ψ_i|_{e_{i+2}} = 0$ in order to satisfy property 3 at the boundary of $Ω_v$. Since $φ_i|_{e_{i+1}} = φ_i|_{e_{i+2}} = 0$, then we will require that $φ_i^2$ divides $ψ_i$, so we’ll have $ψ_i = φ_i^2Q_i$ for some $Q_i$.

It would be convenient to know the degree of $Q_i$. Without putting any more restrictions on the domain, we will in fact need for $Q_i$ to be of degree at least 6, which means that as a polynomial of Wachspress coordinates it should be of degree at least 3.

Since $\sum_{j=1}^{4} φ_j = 1$, then we can express any degree $d$ polynomial of Wachspress coordinates by a linear combination of monomials of the form $φ_{i+k}φ_{i+2}φ_{i+3}$ where $j + k + l + m = d$ for non-negative integer powers $j, k, l, m$. However, not all of these terms are linearly independent; in fact, it is easy to see that, in the case of parallelograms,

$$φ_iφ_{i+2} = \frac{1}{C^2} ∏_{j=1}^{4} A_j = φ_{i-1}φ_{i+1}.$$ 

With this in mind, we can create a list of linearly independent degree-3 monomials of Wachspress coordinates, and we arrange them strategically to form the following template for $ψ_i$:

$$ψ_i = φ_i^2(J_0φ_i^3 + φ_i^2(J_1φ_{i+1} + J_2φ_{i-1}) + φ_i(J_3φ_{i+1}^2 + J_4φ_{i-1}^2) + J_5φ_{i+1}^3 + J_6φ_{i-1}^3
+ φ_i(K_0φ_i^2 + φ_i(K_1φ_{i+1} + K_2φ_{i-1}) + K_3φ_{i+1}^2 + K_4φ_{i-1}^2)
+ φ_i^2(S_0φ_i + S_1φ_{i+1} + S_2φ_{i-1} + S_3φ_{i+2}))$$

for some constants $J_m, K_d, S_p$; $m = 0, ..., 6, n = 0, ..., 4, p = 0, 1, 2, 3$.

The arrangement of (5) is done very intentionally. All terms which have value on the edges $e_i$ and $e_{i-1}$, and therefore affect the value and gradient of $ψ_v$ on these edges, are given a coefficient $J_m$ for some $m$. We often call these the edge terms. The terms which do not affect the value of $ψ_v$ on these edges, but do affect the gradient, are given a coefficient of $K_n$ for some $n$. We call these smoothness terms. Finally, the remaining terms, which affect neither the value nor gradient on the edges, are given a coefficient of $S_p$ for some $p$, and are called combinatorial terms.

The edge term coefficients are the easiest to determine. Property 1 above easily gives us that $J_0 = 1$, since the only term in $ψ_i$ which is valued at any vertex is $J_0φ_i^3$, which is valued $J_0$ at $v_i$. More
edge term coefficients can be found by considering the gradient at the vertex $v_i$. Since $\phi_j(v_i) = \delta_{i,j}$, then we can quickly take the gradient of $\psi_i$ at $v_i$ to retrieve

$$
\nabla \psi_i|_{v_i} = 5\nabla \phi_i|_{v_i} + J_1 \nabla \phi_{i+1}|_{v_i} + J_2 \nabla \phi_{i-1}|_{v_i} + K_0 \nabla \phi_{i+2}|_{v_i}.
$$

(6)

Since $\phi_{i+2}|_{e_i} = \phi_{i+2}|_{e_{i-1}} = 0$, then $\nabla \phi_{i+2}|_{v_i} = 0$. The other gradients, however, will be nonzero at $v_i$. Where $n_j$ is the outward unit normal to the edge-directional vector $e_j = v_{j+1} - v_j$, we have

$$
\nabla \phi_i|_{v_i} = \left(\frac{1}{C^2}(\nabla A_{i+1}A_{i+2} + A_{i+1}\nabla A_{i+2})\right)
\Rightarrow \nabla \phi_i|_{v_i} = -\left(\frac{1}{2C^2}(C|e_{i+1}|n_{i+1} + C|e_{i+2}|n_{i+2})\right) = \frac{1}{2C}(|e_i|n_i + |e_{i-1}|n_{i-1}).
$$

Similarly we can determine that

$$
\nabla \phi_{i+1} = \left(\frac{1}{C^2}(\nabla A_{i+2}A_{i-1} + A_{i+2}\nabla A_{i-1})\right)
\text{ that is, } \nabla \phi_{i+1}|_{v_i} = -\left(\frac{1}{2C}|e_{i-1}|n_{i-1}\right)
$$

and

$$
\nabla \phi_{i-1} = \left(\frac{1}{C^2}(\nabla A_{i}A_{i+1} + A_{i}\nabla A_{i+1})\right)
\text{ that is, } \nabla \phi_{i-1}|_{v_i} = -\left(\frac{1}{2C}|e_i|n_{i}\right).
$$

Then (6) yields

$$
\nabla \psi_i|_{v_i} = (5 - J_1)|e_{i-1}|n_{i-1} + (5 - J_2)|e_i|n_{i}.
$$

Consider that the function we build in another parallelogram in $\Omega_v$ should share the same gradient at $v$. However, clearly the gradient we have computed here depends not only on the values of $J_1$ and $J_2$, but on the length and direction of the edges of $P_1$. We wish to avoid any further geometric restrictions, and so the only reasonable ways to make the gradients match would either be to choose values of $J_1$ and $J_2$ which depend on the surrounding parallelograms rather than just on $P_1$, or to simply let the gradient at $v$ be 0. We will take the latter route, and add a new property to our list:

6. $\nabla \psi_i|_{v} = 0$.

Hence we choose $J_1 = J_2 = 5$.

Now the remaining coefficients can be determined by considering property 4 above. Within $P_1$, this property can be translated to mean that $\sum_{i=1}^{4} \psi_i = 1$. Let us focus only on edge $e_i$ for now. Since $\psi_{i+2}|_{e_i} = \psi_{i-1}|_{e_i} = 0$, we only need that $\psi_i|_{e_i} + \psi_{i+1}|_{e_i} = 1$. Considering the edge terms from each of these, along with the fact that $\phi_{i-1}|_{e_i} = \phi_{i+2}|_{e_i} = 0$, we can write

$$
\psi_i|_{e_{i}} + \psi_{i+1}|_{e_i} = \phi_i^5 + 5\phi_i^4\phi_{i+1} + (J_{3,i} + J_{5,i+1})\phi_i^3\phi_{i+1}^2 + (J_{5,i} + J_{3,i+1})\phi_i^2\phi_{i+1}^3 + 5\phi_i\phi_{i+1}^4 + \phi_{i+1}^5.
$$

Comparing this to the constant 1 is not difficult when you consider that $\phi_i|_{e_{i}} + \phi_{i+1}|_{e_i} = 1$, so in particular $(\phi_i + \phi_{i+1})^5|_{e_i} = 1$. Then we can see that

$$
(1 - (\psi_i + \psi_{i+1}))|_{e_i} = (10 - (J_{3,i} + J_{5,i+1}))\phi_i^3\phi_{i+1}^2 + (10 - (J_{5,i} + J_{3,i+1}))\phi_i^2\phi_{i+1}^3.
$$

Of course, this alone is not enough to determine the values of $J_3$ and $J_5$ uniquely, but thinking ahead, there is another useful requirement we can impose. Even if we were to have some completed functions $\psi_i$ for each vertex, then every function in the linear span of the $\psi_i$ will have gradient 0 at every vertex, and so if we wished to contain even linear polynomials in this span, we will need functions which can adjust the gradient at the vertices. Not only is this possible, but we can also even adjust the Hessian at the vertices nicely, as we will show later. With this in mind, it will also be helpful to add a seventh property to our list:
7. $\nabla^2 \psi_v |_{v \in V} = 0$.

Now, let’s consider $\frac{\partial^2 \psi_i}{\partial e_i^2} |_{v_{i+1}}$. Combining the facts that $\phi_{i-1} |_{e_i} = \phi_{i+2} |_{e_i} = 0$, $\phi_i |_{v_{i+1}} = 0$, and $\phi_i$ and $\phi_{i+1}$ are linear on $e_i$, then this computation is fairly easy:

$$\left. \frac{\partial^2 \psi_i}{\partial e_i^2} \right|_{v_{i+1}} = 2J_5 \left( \frac{\partial \phi_i}{\partial e_i} \right)^2 |_{v_{i+1}} = \frac{2J_5}{|e_i|^2},$$

which is 0 if and only if $J_5$ is 0. This would imply, then, that $J_3 = 10$, and a similar computation will show that this will force $\frac{\partial^2 \psi_i}{\partial e_i^2} |_{v_i} = 0$. A nearly identical analysis on $e_{i-1}$ yields that $J_4 = 10$ and $J_6 = 0$ will force $\frac{\partial^2 \psi_i}{\partial e_i^2} |_{v_{i-1}} = 0$, and we finally have all the edge term coefficients.

Computing the smoothness term coefficients is a bit more complicated. $K_0$ will be the easiest, and can be determined using property 7. Consider $\frac{\partial^2 \psi_i}{\partial e_i \partial e_{i-1}} |_{v_i}$:

$$\left. \frac{\partial^2 \psi_i}{\partial e_i \partial e_{i-1}} \right|_{v_i} = 20 \frac{\partial \phi_i}{\partial e_i} \frac{\partial \phi_i}{\partial e_{i-1}} + 5 \frac{\partial^2 \phi_i}{\partial e_i \partial e_{i-1}} + 20 \frac{\partial \phi_i}{\partial e_i} \frac{\partial \phi_{i-1}}{\partial e_{i-1}} + 20 \frac{\partial \phi_i}{\partial e_i} \frac{\partial \phi_{i+1}}{\partial e_{i-1}}$$

$$+ 5 \left( \frac{\partial^2 \phi_{i+1}}{\partial e_i \partial e_{i-1}} + \frac{\partial^2 \phi_{i-1}}{\partial e_i \partial e_{i-1}} \right) + K_0 \frac{\partial^2 \phi_{i+2}}{\partial e_i \partial e_{i-1}}$$

$$= -20 - 5 + 20 + 20 + 10 - K_0 = 25 - K_0 |e_i| |e_{i-1}|$$

Since we desire that $\nabla^2 \psi_i |_{v_i} = 0$, then we must set $K_0 = 25$. One can check that the mixed edge-direction derivatives are conveniently already 0 at the other vertices, so we have satisfied property 7 at all vertices now.

The remaining smoothness term coefficients will be determined by property 3. Since Wachspress coordinates are smooth on the interior of the polygon over which they are defined, we need only worry about $C^1$ smoothness at the shared edges and vertices between parallelograms. Since we have fully controlled the gradient at all the vertices, we need only worry about the shared edges between adjacent parallelograms.

Choose a parallelogram $P_2 \in \Omega_v$ which is adjacent to $P_1$. Without loss of generality, assume that $v = v_{i,P_1} = v_{i,P_2}$ and $v_{i+1,P_1} = v_{i-1,P_2}$, so that $e_i,P_1 = e_{i-1,P_2}$ and $e_i,P_2 = -e_{i-1,P_2}$. See Figure 2.

Now consider $v_i,P_1$ and $v_i,P_2$ on the shared edge. Since Wachspress coordinates are linear polynomials on edges, then both $\psi_{i,P_1}$ and $\psi_{i,P_2}$ are degree-5 polynomials on the shared edge, and are in fact the same polynomial. To enforce $C^1$-smoothness, then, we can take the derivative of both functions on this edge in the outward normal direction, and require that $\frac{\partial \psi_{i,P_1}}{\partial n_{i,P_1}} |_{e_i} + \frac{\partial \psi_{i,P_2}}{\partial n_{i-1,P_2}} |_{e_{i-1,P_2}} = 0$. First we compute the outward normal derivatives of the Wachspress coordinates. Where $\theta_j$ is the interior angle of the parallelogram at vertex $v_j$, we retrieve the following:

$$\frac{\partial \phi_i}{\partial n_i} |_{e_i} = \frac{1}{2C} (|e_i| \phi_i - |e_{i-1}| \cos(\theta_i)),\]
$$\frac{\partial \phi_{i+1}}{\partial n_i} |_{e_i} = \frac{1}{2C} (|e_i| \phi_{i+1} + |e_{i-1}| \cos(\theta_i)),\]
$$\frac{\partial \phi_{i-1}}{\partial n_i} |_{e_i} = -\frac{|e_i|}{2C} \phi_i,\]
$$\frac{\partial \phi_{i+2}}{\partial n_i} |_{e_i} = -\frac{|e_i|}{2C} \phi_{i+1}.$$

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Now we compute the directional derivatives of $\psi_i$:

$$
\frac{\partial \psi_i}{\partial n_i} |_{e_i} = \frac{\partial \phi_i}{\partial n_i} (30 \phi_i^2 \phi_{i+1}^2) + \frac{\partial \phi_{i-1}}{\partial n_i} (-20 \phi_i^3 \phi_{i+1}) + \frac{\partial \phi_{i+2}}{\partial n_i} (20 \phi_i^4 + (K_1 - 20) \phi_i^3 \phi_{i+1} + K_3 \phi_i^2 \phi_{i+1}^2)
$$

$$
= \phi_i^2 \phi_{i+1}^2 \left( (50 - K_1) \frac{|e_i|}{2C} - 30 \frac{|e_{i-1}| \cos(\theta_i)}{2C} \right) + \phi_i^2 \phi_{i+1}^2 \left( -K_3 \frac{|e_i|}{2C} - 30 \frac{|e_{i-1}| \cos(\theta_i)}{2C} \right);
$$

$$
\frac{\partial \psi_i}{\partial n_{i-1}} |_{e_{i-1}} = \frac{\partial \phi_i}{\partial n_{i-1}} (30 \phi_i^2 \phi_{i-1}^2) + \frac{\partial \phi_{i+1}}{\partial n_{i-1}} (-20 \phi_i^3 \phi_{i-1}) + \frac{\partial \phi_{i+2}}{\partial n_{i-1}} (20 \phi_i^4 + (K_2 - 20) \phi_i^3 \phi_{i-1} + K_4 \phi_i^2 \phi_{i-1}^2)
$$

$$
= \phi_i^2 \phi_{i-1}^2 \left( (50 - K_2) \frac{|e_{i-1}|}{2C} - 30 \frac{|e_i| \cos(\theta_i)}{2C} \right) + \phi_i^2 \phi_{i-1}^2 \left( -K_4 \frac{|e_{i-1}|}{2C} - 30 \frac{|e_i| \cos(\theta_i)}{2C} \right).
$$

Then, since $\phi_i |_{e_i} = \phi_i |_{e_{i-1}}$, and $\phi_{i+1} |_{e_i} = \phi_{i-1} |_{e_{i-1}}$, we have

$$
\frac{\partial \psi_i |_{e_i} \phi_{i+1} |_{e_i}}{\partial n_i} = \frac{\partial \psi_i |_{e_{i-1}} \phi_{i+1} |_{e_{i-1}}}{\partial n_{i-1}} = 0.
$$

Of course, we have more than enough freedom to make this sum evaluate to 0, but we still desire that the coefficients of a function associated with one parallelogram be independent of the geometry of other parallelograms. Then we ought to set

$$
K_1 = 50 - 30 \frac{|e_{i-1}|}{|e_i|} \cos(\theta_i), \quad K_2 = 50 - 30 \frac{|e_i|}{|e_{i-1}|} \cos(\theta_i),
$$

$$
K_3 = -30 \frac{|e_{i-1}|}{|e_i|} \cos(\theta_i), \quad K_4 = -30 \frac{|e_i|}{|e_{i-1}|} \cos(\theta_i).
$$

Figure 2: Two adjacent parallelograms
Now we are almost finished. As things stand, we have (after a little simplification)

\[
\psi_i = \phi_i^2 \left( \phi_i^3 + 5\phi_i^2 (\phi_{i+1} + \phi_{i-1}) + 10\phi_i (\phi_{i+1}^2 + \phi_{i-1}^2) \right) \\
+ \phi_{i+2} \left( 25\phi_i^2 + \phi_i \left( \left( 50 - 30 \frac{|e_{i-1}|}{|e_i|} \cos(\theta_i) \right) \phi_{i+1} + \left( 50 - 30 \frac{|e_i|}{|e_{i-1}|} \cos(\theta_i) \right) \phi_{i-1} \right) \\
- 30 \cos(\theta_i) \left( \frac{|e_{i-1}|}{|e_i|} \phi_{i+1}^2 + \frac{|e_i|}{|e_{i-1}|} \phi_{i-1}^2 \right) \right) \\
+ \phi_{i+2}^2 (S_0\phi_i + S_1\phi_{i+1} + S_2\phi_{i-1} + S_3\phi_{i+2}).
\]

Now we consider the combinatorial coefficients, which can be determined by property 4. Since

\[
\sum_{i=1}^{n} \phi_i = 1,
\]

then we find that

\[
\left( \sum_{i=1}^{n} \phi_i \right)^5 = 1,
\]

which helps us compare. If we compute \( 1 - \sum_{i=1}^{n} \psi_i \), and denote

\[
B = \phi_i\phi_{i+2} = \phi_{i+1}\phi_{i-1},
\]

then we find that

\[
1 - \sum_{i=1}^{n} \psi_i = -B^2 \sum_{i=1}^{n} (S_{0,i} + S_{1,i-1} + S_{2,i+1} + S_{3,i+2} - 100).
\]

While we have very much freedom, we choose to force \( S_1 = S_2 = S_3 = 0 \), and we will stick to this choice for the functions we'll build later in the paper as well. Hence we have \( S_0 = 100 \). We have now completed our construction with final result

\[
\psi_i = \phi_i^2 \left( \phi_i^3 + 5\phi_i^2 (\phi_{i+1} + \phi_{i-1}) + 10\phi_i (\phi_{i+1}^2 + \phi_{i-1}^2) \right) \\
+ \phi_{i+2} \left( 25\phi_i^2 + \phi_i \left( \left( 50 - 30 \frac{|e_{i-1}|}{|e_i|} \cos(\theta_i) \right) \phi_{i+1} + \left( 50 - 30 \frac{|e_i|}{|e_{i-1}|} \cos(\theta_i) \right) \phi_{i-1} \right) \\
- 30 \cos(\theta_i) \left( \frac{|e_{i-1}|}{|e_i|} \phi_{i+1}^2 + \frac{|e_i|}{|e_{i-1}|} \phi_{i-1}^2 \right) \right) \\
+ 100\phi_{i+2}^2 (S_0\phi_i + S_1\phi_{i+1} + S_2\phi_{i-1} + S_3\phi_{i+2}).
\]

The discussion of this subsection serves as a proof of the following theorem:

**Theorem 1** Let \( P \) be a parallelogram partition of \( \Omega \), with set of vertices \( V \). Then for a vertex \( v \in V \), the function

\[
\psi_v(x) := \begin{cases} 
\psi_{v,P}(x) & x \in P, P \in \Omega_v \\
0 & x \notin \Omega_v
\end{cases}
\]

is a piecewise polynomial with the following properties:

1. \( \psi_v(w) = \delta_{v,w} \) for \( w \in V \);
2. \( \nabla \psi_v|_w = 0 \) for \( w \in V \);
3. \( \nabla^2 \psi_v|_w = 0 \) for \( w \in V \);
4. \( \text{supp}(\psi_v) \subseteq \Omega_v \);
5. \( \psi_v \in C^1(\Omega_v) \);
6. \( \sum_{v \in V} \psi_v = 1 \).

Using the cell \( \Omega_v \) of parallelograms shown in Figure 3 with \( v \) being the interior vertex of the union of all parallelograms in \( \Omega_v \), the function \( \psi_v \) is plotted in Figure 4.
2.2 Gradient Interpolation Functions $\psi_{x,v}$ and $\psi_{y,v}$

As mentioned in the previous section, it is hard to construct decent interpolations with only the functions $\psi_v$, since the gradient at every vertex will always be 0. For example, the graph in Figure 5 is an approximation of the linear polynomial $x$ given by

$$I_{x,0} = \sum_{v \in V} v_x \psi_v,$$

where $v = (v_x, v_y)$.

Despite interpolating the values of $x$ at each vertex, it is clear that the graph in Figure 5 is not a very good approximation of a linear function overall. Thus, if we wish to interpolate linear polynomials,
we should create some functions that can adjust the gradient at each vertex while not disturbing the value. We shall construct functions $\psi_{x,v}$ and $\psi_{y,v}$ which have the properties

1. $\psi_{x,v}|_{w \in V} = \psi_{y,v}|_{w \in V} = 0$
2. $\nabla \psi_{x,v}|_{w \in V} = \langle \delta_{v,w}, 0 \rangle$ and $\nabla \psi_{y,v}|_{w \in V} = \langle 0, \delta_{v,w} \rangle$
3. $\nabla^2 \psi_{x,v}|_{w \in V} = \nabla^2 \psi_{y,v}|_{w \in V} = 0$
4. $\text{supp}(\psi_{x,v}), \text{supp}(\psi_{y,v}) \subseteq \Omega_v$
5. $\psi_{x,v}, \psi_{y,v} \in C^1(\Omega)$
6. $\sum v_x \psi_v + \psi_{x,v} = x$ and $\sum v_y \psi_v + \psi_{y,v} = y$.

We will show how to construct $\psi_{x,v}; \psi_{y,v}$ has an analogous construction to the previous subsection.

Again we first restrict to a parallelogram $P_1$ in $\Omega_v$ with $v = v_i$ in $P_1$, and we will construct a function $\psi_{x,i}$ in $P_1$. To preserve locality, we can again use the template given in (5), and solve for the coefficients. We shall start with the edge term coefficients as before.

Clearly $J_0 = 0$ by property 1 above, since $\psi_{x,i}|_v = J_0$. Taking derivatives in the edge directions at $v_i$ is simple:

$$\frac{\partial \psi_{x,i}}{\partial e_i}|_{v_i} = \frac{J_1}{|e_i|}, \quad \text{and} \quad \frac{\partial \psi_{x,i}}{\partial e_{i-1}}|_{v_i} = -\frac{J_2}{|e_{i-1}|}.$$  

By property 2, we know that we want, for example, $\frac{\partial \psi_{x,i}}{\partial e_i}|_{v_i} = \frac{e_{i,x}}{|e_i|}$, where $e_i = (e_{i,x}, e_{i,y}) = (v_{i+1,x} - v_{i,x}, v_{i+1,y} - v_{i,y})$. Thus we set $J_1 = e_{i,x}$. Similarly, we set $J_2 = -e_{i-1,x}$.

Consider $\frac{\partial^2 \psi_{x,i}}{\partial e_i^2}|_{v_{i+1}} = \frac{2J_5}{|e_i|^2}$. By property 3, we wish for this to be zero, so we set $J_5 = 0$. Similarly, considering $\frac{\partial^2 \psi_{x,i}}{\partial e_{i-1}^2}|_{v_{i-1}}$ leads us to set $J_6 = 0$.

Consider now

$$\frac{\partial^2 \psi_{x,i}}{\partial e_i^2}|_{v_i} = \frac{2J_3 - 8e_{i,x}}{|e_i|^2}, \quad \text{and} \quad \frac{\partial^2 \psi_{x,i}}{\partial e_{i-1}^2}|_{v_i} = \frac{2J_4 + 8e_{i-1,x}}{|e_{i-1}|^2},$$

so by property 3 we should set $J_3 = 4e_{i,x}$ and $J_4 = -4e_{i-1,x}$. Now we have solved for all the edge term coefficients.

Consider now the mixed edge-direction derivative

$$\frac{\partial^2 \psi_{x,i}}{\partial e_i \partial e_{i-1}}|_{v_i} = \frac{1}{|e_i||e_{i-1}|}(4e_{i,x} + e_{i-1,x} - 4e_{i-1,x} - e_{i-1,x} - K_0).$$

By property 3, we should set $K_0 = 5(e_{i,x} - e_{i-1,x})$.

As before, finding the rest of the smoothness term coefficients will require us to consider outward
normal derivatives on the edges $e_i$ and $e_{i-1}$. We first compute $\frac{\partial \psi_{x,i}}{\partial n_i} |_{e_i}$:

\[
\frac{\partial \psi_{x,i}}{\partial n_i} |_{e_i} = \frac{\partial \phi_{i+1}}{\partial n_i} (e_{i,x} \phi_i^4 + 4e_{i,x} \phi_i^3 \phi_{i+1} - 12e_{i,x} \phi_i^2 \phi_{i+1}^2) \\
+ \frac{\partial \phi_{i-1}}{\partial n_i} (-e_{i-1,x} \phi_i^4 - 4e_{i,x} \phi_i^3 \phi_{i+1} - 12e_{i,x} \phi_i^2 \phi_{i+1}^2) \\
+ \frac{\partial \phi_{i+2}}{\partial n_i} (5(e_{i,x} - e_{i-1,x}) \phi_i^4 + (K_1 - 4e_{i,x}) \phi_i^3 \phi_{i+1} + (K_3 - 12e_{i,x}) \phi_i^2 \phi_{i+1}^2)
\]

\[
= \phi_i^5 \left( \frac{|e_i|}{2C} e_{i-1,x} + \frac{|e_{i-1}| \cos(\theta_i)}{2C} e_{i,x} \right) + 5 \phi_i^4 \phi_{i+1} \left( \frac{|e_i|}{2C} e_{i-1,x} - \frac{|e_{i-1}| \cos(\theta_i)}{2C} e_{i,x} \right) \\
+ \phi_i^3 \phi_{i+1}^2 \left( \frac{|e_i|}{2C} (20e_{i,x} - K_1) - \frac{|e_{i-1}| \cos(\theta_i)}{2C} 8e_{i,x} \right) - \phi_i^2 \phi_{i+1}^3 \left( K_3 \frac{|e_i|}{2C} + \frac{|e_{i-1}| \cos(\theta_i)}{2C} 12e_{i,x} \right).
\]

Using the standard geometric identities

\[
2C = |e_i||e_{i-1}| \sin(\theta_i) \text{ and } \frac{e_{i-1,x}}{|e_{i-1}|} = \frac{e_{i,y} \sin(\theta_i) - e_{i,x} \cos(\theta_i)}{|e_i|}, \tag{7}
\]

this can be simplified to

\[
\frac{\partial \psi_{x,i}}{\partial n_i} |_{e_i} = \phi_i^5 \left( \frac{e_{i,y}}{|e_i|} \right) + 5 \phi_i^4 \phi_{i+1} \left( \frac{e_{i,y}}{|e_i|} \right) \\
+ \phi_i^3 \phi_{i+1}^2 \left( \frac{50e_{i,x} - K_1}{2C} - \frac{|e_{i-1}| \cos(\theta_i)}{2C} 8e_{i,x} \right) - \phi_i^2 \phi_{i+1}^3 \left( \frac{12e_{i,x}}{2C} \right).
\]

Similarly, we can compute

\[
\frac{\partial \psi_{x,i}}{\partial n_{i-1}} |_{e_{i-1}} = \phi_i^5 \left( \frac{e_{i-1,y}}{|e_{i-1}|} \right) + 5 \phi_i^4 \phi_{i-1} \left( \frac{e_{i-1,y}}{|e_{i-1}|} \right) \\
+ \phi_i^3 \phi_{i-1}^2 \left( \frac{20e_{i-1,x} + K_2}{2C} - \frac{|e_{i-1}| \cos(\theta_i)}{2C} 8e_{i-1,x} \right) - \phi_i^2 \phi_{i-1}^3 \left( \frac{12e_{i-1,x}}{2C} \right).
\]

Then if we choose a parallelogram $P_2 \in \Omega_v$ which shares an edge with $P_1$, and we assume without
loss of generality that they share the edge \( e_{i,P_1} = e_{i-1,P_2} \), then we compute the sum

\[
\frac{\partial \psi_{x,i,P_1}}{\partial n_{i,P_1}} \bigg|_{e_{i,P_1}} + \frac{\partial \psi_{x,i,P_2}}{\partial n_{i-1,P_2}} \bigg|_{e_{i-1,P_2}} = \phi_i^{\delta P_1} \left( e_{i,y,P_1} + e_{i-1,y,P_2} \right) + \phi_i^{\delta P_1} \phi_{i+1,P_1} \left( e_{i,y,P_1} + e_{i-1,y,P_2} \right) + \phi_i^{\delta P_1} \phi_{i+1,P_1} \left( \frac{K_{i,P_1}}{2C_{P_1}} \right) - 8e_{i,x,P_1} \frac{e_{i-1,P_1}}{2C_{P_1}} (\cos(\theta_i) - 2C_{P_1})
\]

Since \( e_{i,P_1} = -e_{i-1,P_2} \), then the first two terms are 0. For now, let us choose the following:

\[
K_1 = \left( 20 - 8 \frac{|e_{i-1}|}{|e_i|} \cos(\theta_i) \right) e_{i,x} + \tilde{K}_1,
K_2 = \left( 20 - 8 \frac{|e_i|}{|e_{i-1}|} \cos(\theta_i) \right) e_{i-1,x} + \tilde{K}_2,
K_3 = -12e_{i,x} \frac{|e_{i-1}|}{|e_i|} \cos(\theta_i) + \tilde{K}_3,
K_4 = 12e_{i-1,x} \frac{|e_{i}|}{|e_{i-1}|} \cos(\theta_i) + \tilde{K}_4,
\]

where \( \tilde{K}_j \) is another constant for \( j = 1, 2, 3, 4 \). Then we’ll have

\[
\frac{\partial \psi_{x,i,P_1}}{\partial n_{i,P_1}} \bigg|_{e_{i,P_1}} + \frac{\partial \psi_{x,i,P_2}}{\partial n_{i-1,P_2}} \bigg|_{e_{i-1,P_2}} = -\phi_i^{\delta P_1} \phi_{i+1,P_1} \left( \frac{K_{i,P_1}}{2C_{P_1}} \right) - 8e_{i,x,P_1} \frac{e_{i-1,P_1}}{2C_{P_1}} (\cos(\theta_i) - 2C_{P_1})
\]

While it may seem that we should simply set \( \tilde{K}_j = 0 \) for all \( j = 1, 2, 3, 4 \), we will see that this is not the right choice. The remaining coefficients will affect property 6.

Since \( \sum_{j=1}^{4} v_{j,x} \phi_j = x \), we can express \( x = \left( \sum_{j=1}^{4} v_{j,x} \phi_j \right)^4 \) to find that

\[
x - \left( \sum_{j=1}^{4} v_{j,x} \psi_j + \psi_{x,j} \right) = -B \left( \sum_{j=1}^{4} \phi_j \left( \tilde{K}_{1,j} + \tilde{K}_{4,j+1} + 10e_{j-1,x} + 10 \frac{|e_{j-1}|}{|e_j|} \cos(\theta_j) e_{j,x} \right) \phi_{i+1}
\]

\[
+ \left( \tilde{K}_{2,j} + \tilde{K}_{3,j-1} + 10e_{j,x} - 10 \frac{|e_j|}{|e_{j-1}|} \cos(\theta_j) e_{j-1,x} \right) \phi_{i-1}
\]

\[
- B^2 \left( \sum_{j=1}^{4} \phi_j \left( S_{0,j} + S_{1,j-1} + S_{2,j+1} + S_{3,j+2} - 40e_{j,x} + 40e_{j-1,x} \right) \right).
\]

We maintain the requirement that \( S_1 = S_2 = S_3 = 0 \), so we’ll set \( S_0 = 40(e_{i,x} - e_{i-1,x}) \). However, it is difficult to determine the appropriate way to set the values of the other coefficients. To make some future computations easier, we will set

\[
\tilde{K}_1 = -\frac{17}{6} \left( e_{i-1,x} + \frac{|e_{i-1}|}{|e_i|} \cos(\theta_i) e_{i,x} \right), \quad \tilde{K}_2 = \frac{17}{6} \left( e_{i,x} + \frac{|e_i|}{|e_{i-1}|} \cos(\theta_i) e_{i-1,x} \right),
\]

\[
\tilde{K}_3 = -\frac{43}{6} \left( e_{i-1,x} + \frac{|e_{i-1}|}{|e_i|} \cos(\theta_i) e_{i,x} \right), \quad \tilde{K}_4 = \frac{43}{6} \left( e_{i,x} + \frac{|e_i|}{|e_{i-1}|} \cos(\theta_i) e_{i-1,x} \right).
\]
Using the same geometric identities (7) will show that the smoothness relationship is satisfied, and we retrieve the final result

\[
\psi_{x,i} = \phi_i^2 \left( \phi_i^2 (e_{i,x} \phi_{i+1} - e_{i-1,x} \phi_{i-1}) + 4 \phi_i (e_{i,x} \phi_{i+1}^2 - e_{i-1,x} \phi_{i-1}^2) \right) \\
+ \phi_{i+2} \left( 5(e_{i,x} - e_{i-1,x}) \phi_i^2 + \phi_i \left( \left( \frac{20}{6} \frac{|e_{i-1}|}{|e_i|} |e_i| \cos(\theta_i) \right) e_{i,x} - \frac{17}{6} e_{i-1,x} \right) \phi_{i+1} \\
- \left( \frac{20}{6} \frac{|e_{i-1}|}{|e_i|} |e_i| \cos(\theta_i) \right) e_{i-1,x} - \frac{17}{6} e_{i-1,x} \right) \phi_{i-1} \right) \\
- \left( \frac{115}{6} \frac{|e_{i-1}|}{|e_i|} e_{i,x} + \frac{43}{6} e_{i-1,x} \right) \phi_{i+1}^2 + \left( \frac{115}{6} \frac{|e_{i-1}|}{|e_i|} e_{i-1,x} + \frac{43}{6} e_{i,x} \right) \phi_{i-1}^2 \right) \\
+ 40(e_{i,x} - e_{i-1,x}) \phi_i \phi_{i+2}^2 \right).
\]

As desired, we can recover \( x \) by \( \sum_{v \in V} v_x \psi_v + \psi_{x,v} \), shown below:

Figure 6: The plot of \( \psi_{x,v} \) over the partition shown in Figure 3

A similar analysis can determine \( \psi_{y,i} \), which happens to be identical except it uses the \( y \) components.

As desired, we can recover \( x \) by \( \sum_{v \in V} v_x \psi_v + \psi_{x,v} \), shown below:

Figure 7: The plot of \( \sum_{v \in V} v_x \psi_v + \psi_{x,v} \)
of \( \mathbf{e}_i \) and \( \mathbf{e}_{i-1} \) instead of the \( x \) components:

\[
\psi_{y,i} = \phi_i^2 \left( \phi_i^2 (e_{i,y} \phi_{i+1} - e_{i-1,y} \phi_{i-1}) + 4\phi_i (e_{i,y} \phi_{i+1}^2 - e_{i-1,y} \phi_{i-1}^2) \right)
+ \phi_{i+2} \left( 5(e_{i,y} - e_{i-1,y}) \phi_i^2 + \phi_i \left( \left( 20 - \frac{65}{6} |\mathbf{e}_{i-1}| \right) |\mathbf{e}_i| \cos(\theta_i) \right) e_{i,y} - \frac{17}{6} e_{i-1,y} \right) \phi_{i+1}
- \phi_{i-1} \left( \left( 20 - \frac{65}{6} |\mathbf{e}_i| \cos(\theta_i) \right) e_{i-1,y} - \frac{17}{6} e_{i,y} \right) \phi_{i-1}
- \left( \frac{115}{6} \frac{|\mathbf{e}_{i-1}|}{|\mathbf{e}_i|} e_{i,y} + \frac{43}{6} e_{i-1,y} \right) \phi_{i+1}^2 + \left( \frac{115}{6} \frac{|\mathbf{e}_i|}{|\mathbf{e}_{i-1}|} e_{i-1,y} + \frac{43}{6} e_{i,y} \right) \phi_{i-1}^2
+ 40(e_{i,y} - e_{i-1,y}) \phi_i \phi_{i+2}^2 \right).
\]

Figure 8: The plot of \( \psi_{y,v} \) over the partition shown in Figure 3

As expected, we can recover \( y \) by \( \sum_{v \in V} v_y \psi_v + \psi_{y,v} \):

Figure 9: The plot of \( \sum_{v \in V} v_y \psi_v + \psi_{y,v} \)

### 2.3 Hessian Interpolation Functions \( \psi_{x^2,v}, \psi_{y^2,v}, \text{ and } \psi_{xy,v} \)

Finally we extend our span to include \( x^2, y^2, \) and \( xy \). Since our functions thus far have 0 Hessian, we will need to build functions \( \psi_{x^2,v}, \psi_{y^2,v}, \) and \( \psi_{xy,v} \) that satisfy the following properties:

1. \( \psi_{x^2,v}|_{w \in V} = \psi_{y^2,v}|_{w \in V} = \psi_{xy,v}|_{w \in V} = 0 \)
2. \( \nabla \psi_{x^2,v}|_{w \in V} = \nabla \psi_{y^2,v}|_{w \in V} = \nabla \psi_{xy,v}|_{w \in V} = 0 \)
The previous sections. We simply list the results below:

3. $\nabla^2 \psi_{x^2,v}|_{w \in V} = \begin{bmatrix} \delta_{v,w} & 0 \\ 0 & 0 \end{bmatrix}$, $\nabla^2 \psi_{y^2,v}|_{w \in V} = \begin{bmatrix} 0 & 0 \\ 0 & \delta_{v,w} \end{bmatrix}$, and $\nabla^2 \psi_{xy,v}|_{w \in V} = \begin{bmatrix} 0 & \delta_{v,w} \\ \delta_{v,w} & 0 \end{bmatrix}$

4. $\text{supp}(\psi_{x^2,v})$, $\text{supp}(\psi_{y^2,v})$, $\text{supp}(\psi_{xy,v}) \subseteq \Omega_v$

5. $\psi_{x^2,v}, \psi_{y^2,v}, \psi_{xy,v} \in C^1(\Omega)$

6. $\sum_{v \in V} v_x^2 \psi_v + 2v_x \psi_{x,v} + 2\psi_{x^2,v} = x^2$, $\sum_{v \in V} v_y^2 \psi_v + 2v_y \psi_{y,v} + 2\psi_{y^2,v} = y^2$, and $\sum_{v \in V} v_x v_y \psi_v + v_y \psi_{x,v} + v_x \psi_{y,v} + \psi_{xy,v} = xy$.

For brevity, we’ll suppress the computations here, as they are of a similar flavor to those done in the previous sections. We simply list the results below:

$$
\psi_{x^2,i} = \phi_i^2 \left( \phi_i \left( \frac{1}{2} e_{i,x}^2 \phi_{i+1}^2 + \frac{1}{2} e_{i-1,x}^2 \phi_{i-1}^2 \right) ight) + \phi_{i+2} \left( -e_{i,x} e_{i-1,x} \phi_i^2 + \phi_i \left( \left( \frac{5}{2} + \frac{5}{3} \frac{|e_{i-1}|}{|e_i|} \cos(\theta_i) \right) e_{i,x}^2 + \frac{13}{6} e_{i,x} e_{i-1,x} \right) \phi_{i+1} 
+ \left( \frac{5}{2} + \frac{5}{3} \frac{|e_i|}{|e_{i-1}|} \cos(\theta_i) \right) e_{i-1,x}^2 + \frac{13}{6} e_{i,x} e_{i-1,x} \phi_{i-1} \right) 
- \left( \frac{5}{2} \frac{|e_{i-1}|}{|e_i|} \cos(\theta_i) e_{i,x}^2 + e_{i,x} e_{i-1,x} \right) \phi_{i+1}^2 
- \left( \frac{5}{2} \frac{|e_i|}{|e_{i-1}|} \cos(\theta_i) + e_{i,x} e_{i-1,x} \right) \phi_{i-1}^2 
+ (5(e_{i,x}^2 + e_{i-1,x}^2) - 16 e_{i,x} e_{i-1,x}) \phi_{i+2}^2 \phi_i \right)
$$

$$
\psi_{y^2,i} = \phi_i^2 \left( \phi_i \left( \frac{1}{2} e_{i,y}^2 \phi_{i+1}^2 + \frac{1}{2} e_{i-1,y}^2 \phi_{i-1}^2 \right) ight) + \phi_{i+2} \left( -e_{i,y} e_{i-1,y} \phi_i^2 + \phi_i \left( \left( \frac{5}{2} + \frac{5}{3} \frac{|e_{i-1}|}{|e_i|} \cos(\theta_i) \right) e_{i,y}^2 + \frac{13}{6} e_{i,y} e_{i-1,y} \right) \phi_{i+1} 
+ \left( \frac{5}{2} + \frac{5}{3} \frac{|e_i|}{|e_{i-1}|} \cos(\theta_i) \right) e_{i-1,y}^2 + \frac{13}{6} e_{i,y} e_{i-1,y} \phi_{i-1} \right) 
- \left( \frac{5}{2} \frac{|e_{i-1}|}{|e_i|} \cos(\theta_i) e_{i,y}^2 + e_{i,y} e_{i-1,y} \right) \phi_{i+1}^2 
- \left( \frac{5}{2} \frac{|e_i|}{|e_{i-1}|} \cos(\theta_i) + e_{i,y} e_{i-1,y} \right) \phi_{i-1}^2 
+ (5(e_{i,y}^2 + e_{i-1,y}^2) - 16 e_{i,y} e_{i-1,y}) \phi_{i+2}^2 \phi_i \right)
$$
Figure 11: The plot of \( \sum_{v \in V} v_x^2 \psi_v + 2v_x \psi_x,v + 2\psi_x^2,v \)

Figure 12: The plot of \( \psi_{y^2,v} \) over the partition shown in Figure 3

Figure 13: The plot of \( \sum_{v \in V} v_y^2 \psi_v + 2v_y \psi_y,v + 2\psi_y^2,v \)
\[ \psi_{xy,i} = \phi_i^2 \left( \psi_i(e_i x e_i y e_{i+1} + e_{i-1} x e_{i-1} y e_{i-1}) \right) + \right. \\
\left. \phi_{i+2} \left( - (e_i x e_{i-1}, y + e_i y e_{i-1}, x) \phi_i^2 \right) \right. \\
\left. + \phi_i \left( \left( \left( 5 + \frac{10}{3} \frac{|e_{i-1}|}{|e_i|} \cos(\theta_i) \right) e_i x e_i y + \frac{13}{6} (e_i x e_{i-1}, y + e_i y e_{i-1}, x) \phi_{i+1} \right) \right) \\
\left. + \left( \left( 5 + \frac{10}{3} \frac{|e_i|}{|e_{i-1}|} \cos(\theta_i) \right) e_{i-1} x e_{i-1} y + \frac{13}{6} (e_{i-1} x e_{i-1}, y + e_i y e_{i-1}, x) \phi_{i-1} \right) \right) \\
\left. - \left( 5e_i x e_i y \frac{|e_{i-1}|}{|e_i|} \cos(\theta_i) + (e_i x e_{i-1}, y + e_i y e_{i-1}, x) \right) \phi_{i+1}^2 \right) \\
\left. - \left( 5e_{i-1} x e_{i-1}, y \frac{|e_i|}{|e_{i-1}|} \cos(\theta_i) + (e_{i-1} x e_{i-1}, y + e_i y e_{i-1}, x) \right) \phi_{i-1}^2 \right) \\
\left. + (10(e_i x e_i y + e_{i-1} x e_{i-1}, y) - 16(e_i x e_{i-1}, y + e_i y e_{i-1}, x)) \phi_{i+2} \right) \phi_i \right) \\
\]

2.4 Edge and Face Splines

In addition to the six vertex splines \( \psi_v, \psi_{x,v}, \psi_{y,v}, \psi_{x^2,v}, \psi_{xy,v}, \psi_{y^2,v}, \) we need more in order to reproduce all degree 5 polynomials. We can manipulate the remaining degrees of freedom using edge splines (supported over two parallelograms sharing a common edge) and face splines (supported over each parallelogram). The face splines can be defined as follows:
For a given parallelogram $P \in \mathcal{P}$ with Wachspress coordinates $\phi_j, j = 1, 2, 3, 4$, let

$$\psi_{P,i} = B^2 \phi_i, i = 1, 2, 3, 4,$$

where $B = \phi_1 \phi_3 = \phi_2 \phi_4$ as before. The derivatives of $\psi_{P,i}$ up to the second order are zero at any vertex of $P$, and the outward normal derivatives of $\psi_{P,i}$ are zero on every edge of $P$. The graphs of some of these functions are shown in Fig. 16.

Next we consider edge splines over $P$. We can define edge splines $\psi_{e,i,1}^P$ and $\psi_{e,i,2}^P$ as follows:

$$\psi_{e,i,1}^P = \phi_1^3 \phi_{i+1} \phi_{i+2} = B \phi_1^2 \phi_{i+1}, \quad \text{and} \quad \psi_{e,i,2}^P = \phi_1^2 \phi_{i+1} \phi_{i+2} = B \phi_i \phi_{i+1}^2.$$ (9)

The graph of such a function is shown in Figure 17.

Let us explore how to use these new kinds of splines for quasi-interpolation.

The edge splines can be used to control the outward normal derivatives at two points on edge $e_i$. Note that the values, gradients, and Hessians of all 8 edge splines of $P$ are zero at every vertex of $P$. Moreover, their values are zero on every edge of $P$, and their gradients are zero on every edge except one: in particular,

$$\nabla \psi_{e,i,1}|_{e_j} = \nabla \psi_{e,i,2}|_{e_j} = 0 \text{ when } i \neq j.$$

Thus we need only consider the gradient of each function on one edge:

$$\frac{\partial \psi_{e,i,1}^P}{\partial n_i}|_{e_i} = -\frac{|e_i|}{2C} \phi_i^3 \phi_{i+1}, \quad \text{and} \quad \frac{\partial \psi_{e,i,2}^P}{\partial n_i}|_{e_i} = -\frac{|e_i|}{2C} \phi_i^2 \phi_{i+1}^3.$$
which are maximized at the points \( v_{i,1} = \frac{3}{5} v_i + \frac{2}{5} v_{i+1} \) and \( v_{i,2} = \frac{2}{5} v_i + \frac{3}{5} v_{i+1} \), respectively, so we will choose these to be the two points at which we control the normal derivatives.

Given a function \( f \) that we wish to interpolate, name our \( C^1 \) interpolant in \( P \) using only our previous vertex splines over \( P \) by

\[
S^P(f) = \sum_{i=1}^{4} f(v_i) \psi_i + f_x(v_i) \psi_{x,i} + f_y(v_i) \psi_{y,i} + f_{xx}(v_i) \psi_{x^2,i} + f_{xy}(v_i) \psi_{xy,i} + f_{yy}(v_i) \psi_{y^2,i}.
\]

We define a new interpolant

\[
S^P_e(f) = S^P(f) + \sum_{i=1}^{4} \psi^P_{e_i}(f),
\]

where \( \psi^P_{e_i}(f) = \kappa^P_{i,1}(f) \psi^P_{e_{i,1}} + \kappa^P_{i,2}(f) \psi^P_{e_{i,2}} \) for some constants \( \kappa^P_{i,1}(f) \) and \( \kappa^P_{i,2}(f) \) depending on \( f \).

We can compute the outward normal derivative of \( I_v \) at \( v_{i,1} \) and \( v_{i,2} \), and we’ll choose \( \kappa_{i,1} \) and \( \kappa_{i,2} \) so that \( \partial S^P_e |_{v_{i,j}} = \partial f |_{v_{i,j}} \) for \( i = 1, 2, 3, 4 \) and \( j = 1, 2 \).

After a long computation, we find that the appropriate constants are

\[
\kappa^P_{i,1}(f) = \frac{55}{180} (\nabla f(v_i) - 3 \nabla f(v_{i,1}) + 2 \nabla f(v_{i,2}), e_{i,-1} + \frac{|e_{i,-1}| | e_i |}{| e_i |} \cos(\theta_i) e_i),
\]

\[
\kappa^P_{i,2}(f) = \frac{55}{180} (\nabla f(v_{i+1}) + 2 \nabla f(v_{i,1}) - 3 \nabla f(v_{i,2}), e_{i,-1} + \frac{|e_{i,-1}| | e_i |}{| e_i |} \cos(\theta_i) e_i),
\]

where \((\cdot, \cdot)\) is the usual vector dot product.

Thinking globally, then, for each interior edge \( e \) of the partition \( \mathcal{P} \), let \( P \) and \( Q \) be the parallelograms incident to \( e \), with \( e = e_i \) in \( P \) and \( e = e_j \) in \( Q \). Define the function

\[
\psi_e(f) := \begin{cases} 
\psi^P_{e_i}(f), & x \in P; \\
\psi^Q_{e_j}(f), & x \in Q; \\
0, & \text{otherwise}.
\end{cases}
\]

Whenever \( e \) is a boundary edge, we use a similar definition, simply omitting the middle line so that \( \psi_e(f) \) is only supported within the unique boundary parallelogram containing \( e \).

We will show that, for an interior edge \( e \) which is \( e_i \) in \( P \) and \( e_j \) in \( Q \),

\[
\frac{\partial \psi^P_{e_i}(f)}{\partial n_{i,P}} + \frac{\partial \psi^Q_{e_j}(f)}{\partial n_{j,Q}} = 0,
\]

so \( \psi_e(f) \in C^1(\Omega) \) for any function \( f \in C^1(\Omega) \) and edge \( e \) of \( \mathcal{P} \). Then, where \( E \) is the collection of all edges in \( \mathcal{P} \), the two global interpolants

\[
S_I(f) = \sum_{v \in V} f(v) \psi_v + f_x(v) \psi_{x,v} + f_y(v) \psi_{y,v} + f_{xx}(v) \psi_{x^2,v} + f_{xy}(v) \psi_{xy,v} + f_{yy}(v) \psi_{y^2,v}
\]

and \( S_e(f) = S_I(f) + \sum_{e \in E} \psi_e(f) \) are both in \( C^1(\Omega) \).
In particular, suppose two parallelograms \( P \) and \( Q \) in \( \mathcal{P} \) share an edge \( e \), which is \( e_i \) in \( P \) and \( e_j \) in \( Q \). Then, since \( n_{i,P} = -n_{j,Q} \), we have
\[
\frac{\partial}{\partial n_{i,P}} S^P_e(f)|_{e_{i,P}} + \frac{\partial}{\partial n_{j,Q}} S^Q_e(f)|_{e_{j,Q}} = \frac{\partial}{\partial n_{i,P}} (S^P_e(f) - S^Q_e(f))
\]
\[
+ \kappa^P_{i,1}(f)\psi^P_{e_{i,1}} - \kappa^Q_{i,2}(f)\psi^Q_{e_{i,2}} + \kappa^P_{i,2}(f)\psi^P_{e_{i,2}} - \kappa^Q_{i,1}(f)\psi^Q_{e_{i,1}} |_{e_{i,P}}.
\]
We know that the vertex spline interpolants join smoothly on the shared edge, so we only need to show that
\[
\frac{\partial}{\partial n_{i,P}} \left( \kappa^P_{i,1}(f)\psi^P_{e_{i,1}} - \kappa^Q_{i,2}(f)\psi^Q_{e_{i,2}} + \kappa^P_{i,2}(f)\psi^P_{e_{i,2}} - \kappa^Q_{i,1}(f)\psi^Q_{e_{i,1}} \right) |_{e_{i,P}} = 0.
\]
Since \( v^P_{i,1} = v^Q_{j,1} \) and \( v^P_{i,2} = v^Q_{j,1} \), the left-hand side is equal to
\[
- \phi^3_{i,P}\phi^2_{i+1,P} \left( \frac{|e_{j,Q}|}{2C^Q} \kappa^Q_{j,2}(f) + \frac{|e_{i,P}|}{2C^P} \kappa^P_{i,1}(f) \right) - \phi^2_{i,P}\phi^3_{i+1,P} \left( \frac{|e_{j,Q}|}{2C^Q} \kappa^Q_{j,1}(f) + \frac{|e_{i,P}|}{2C^P} \kappa^P_{i,2}(f) \right)
\]
\[
= - \frac{5^5}{180} \phi^3_{i,P}\phi^2_{i+1,P} \left( \langle \nabla f(v^P_{i,1}) - 3\nabla f(v^P_{i,1}), 2\nabla f(v^P_{i,2}) \rangle, \frac{|e_{j,Q}|}{2C^Q} \frac{|e_{i,P}|}{2C^P} \right)
\]
\[
+ \langle \nabla f(v^P_{i,1}) - 3\nabla f(v^P_{i,1}), 2\nabla f(v^P_{i,2}) \rangle, \frac{|e_{j,Q}|}{2C^Q} \frac{|e_{i,P}|}{2C^P} \right),
\]
\[
- \frac{5^5}{180} \phi^2_{i,P}\phi^3_{i+1,P} \left( \langle \nabla f(v^P_{i,1}) + 2\nabla f(v^P_{i,1}), 3\nabla f(v^P_{i,2}) \rangle, \frac{|e_{j,Q}|}{2C^Q} \frac{|e_{i,P}|}{2C^P} \right)
\]
\[
+ \langle \nabla f(v^P_{i,1}) + 2\nabla f(v^P_{i,1}), 3\nabla f(v^P_{i,2}) \rangle, \frac{|e_{j,Q}|}{2C^Q} \frac{|e_{i,P}|}{2C^P} \right).
\]
Therefore \( S^P_e \) and \( S^Q_e \) join smoothly. This discussion serves as a proof of the following theorem:

**Theorem 2** For an edge \( e \) in \( \mathcal{P} \) and sufficiently differentiable function \( f \), define \( \psi_e(f) \) as in (10) above. Then the function
\[
S_e(f)(x) = S_I(f)(x) + \sum_{e \in E} \psi_e(f)(x)
\]
is in \( C^1(\Omega) \), and satisfies the five following properties:

1. \( S_e(f)(v) = f(v) \)
2. \( \nabla S_e(f)(v) = \nabla f(v) \)
3. \( \nabla^2 S_e(f)(v) = \nabla^2 f(v) \)
4. \( \frac{\partial}{\partial n_e} S_e(f)(e_1) = \frac{\partial}{\partial n_e} f(e_1) \)
5. \( \frac{\partial}{\partial n_e} S_e(f)(e_2) = \frac{\partial}{\partial n_e} f(e_2) \)

for all vertices \( v \) and edges \( e \) in \( \mathcal{P} \), where \( n_e \) is the normal direction to \( e \), and \( e_1 \) and \( e_2 \) are the points \( \frac{2}{3} \) and \( \frac{1}{3} \) of the way along the edge \( e \).

Since the face splines of \( P \) have no value or gradient on the edges of \( P \), we do not need to be concerned with \( C^1 \) smoothness when analyzing them. We need only find how to use them for quasi-interpolation.
We will use these to interpolate values at some points on the interior of each parallelogram. For a given parallelogram $P$, the function $\psi_{P,v}$ is maximized at the point $p_i = (9v_i + 6v_{i+1} + 4v_{i+2} + 6v_{i-1})/25$ for $i = 1, 2, 3, 4$; these will be the points at which we interpolate values. For a given function $f$, define the new interpolant over $P$ by

$$S_5^P(f) = S_e(f) + \sum_{i=1}^{4} \sigma_i^P(f) \psi_{P,i},$$

so we need to determine the coefficients $\sigma_i^P(f)$ such that $S_5^P(f)|_{p_i} = f|_{p_i}$ for $i = 1, 2, 3, 4$. We simply evaluate $S_e(f)$ at each point $p_i$ and solve the linear system given by

$$\sum_{i=1}^{4} \sigma_i^P(f) \psi_{P,i}(p_j) = f(p_j) - S_e(f)(p_j), j = 1, 2, 3, 4. \quad (12)$$

We now summarize the discussion above to have the following theorem:

**Theorem 3** Given a sufficiently differentiable function $f$, compute $S_e(f)$ and solve the linear system given in (12) over each parallelogram $P \in \mathcal{P}$ for the coefficients $\sigma_i^P(f), i = 1, 2, 3, 4$. Then the function

$$S_5(f) := S_I(f) + \sum_{P \in \mathcal{P}} \sum_{i=1}^{4} \sigma_i^P(f) \psi_{P,i} + \sum_{e \in E} \psi_e(f)(x), \quad (13)$$

satisfies all 5 properties listed in Theorem 2 along with the property that

$$S_5(f)(p_i^P) = f(p_i^P), \quad i = 1, 2, 3, 4$$

for every $P \in \mathcal{P}$.

Now we can easily show that the following theorem holds:

**Theorem 4** Let $\Psi_v$ be the collection of all the vertex splines $\psi_u, \psi_{x,v}, \psi_{y,v}, \psi_{x^2,v}, \psi_{xy,v}, \psi_{y^2,v}$ for each vertex $v$ of $\mathcal{P}$. Let $\Psi_e$ be the collection of all the edge splines $\psi_{e,1}$ and $\psi_{e,2}$ for edges $e$ of $\mathcal{P}$. Let $\Psi_p$ be the collection of all the parallelogram splines $\psi_{P,1}, \psi_{P,2}, \psi_{P,3}, \psi_{P,4}$ for parallelograms $P$ of $\mathcal{P}$.

Then $\text{span}(\Psi_v) \oplus \text{span}(\Psi_e) \oplus \text{span}(\Psi_p) \supseteq \Pi_5$, where $\Pi_5$ is the space of all bivariate polynomials of degree 5 or less. In particular, $S_5(p) = p$ for any polynomial $p \in \Pi_5$.

This can be easily proven by showing that $S_5(p) = p$ for the usual monomial basis of $\Pi_5$ using one’s favorite computer algebra program, e.g. Mathematica.

### 3 Approximation Properties and Numerical Results

Let $\mathcal{P}$ be a collection of parallelograms and $\mathcal{P}_k$ is the $k$th uniform refinement of $\mathcal{P}_{k-1}$ with $\mathcal{P}_0 = \mathcal{P}$. Define $S(\mathcal{P})$ be the span of all vertex splines we constructed in the previous subsections.

That is, where $V$ is the collection of all vertices in $\mathcal{P}$ and $E$ is the collection of all edges of $\mathcal{P}$, define

$$S_5(\mathcal{P}) = \text{span} \left( S_I(\mathcal{P}) \cup \{\psi_{e,1}, \psi_{e,2}, e \in E\} \cup \{\psi_{P,i}, i = 1, 2, 3, 4, P \in \mathcal{P}\} \right), \quad (14)$$

where $S_I(\mathcal{P})$ is an interpolatory spline space defined by

$$S_I(\mathcal{P}) = \text{span}\{\psi_v, \psi_{x,v}, \psi_{y,v}, \psi_{x^2,v}, \psi_{xy,v}, \psi_{y^2,v}, \forall v \in V\}. \quad (15)$$

Similarly we have interpolatory spaces $S_I(\mathcal{P}_k)$ and interpolatory splines $S_k(u)$ from $S_I(\mathcal{P}_k)$ for all $k \geq 1$.

We would like to show the following:
Theorem 5 Let $\Omega$ be the union of all parallelograms in $\mathcal{P}$. For any $f \in C^3(\Omega)$, the quasi-interpolant $S_k(f) \in \mathcal{S}_I(\mathcal{P}_k)$ satisfies
\[
\|f - S_k(f)\|_{\infty,\Omega} \leq C|f|_{3,\infty,\Omega}2^{-3k},
\]
where $C$ is a positive constant independent of $f$.

Proof. We can use the same technique in [19] to establish the proof. The detail is left to the interested reader. \hfill \Box

For the approximation property in $L^2$ norm, we can follow the discussions in [18] to establish the following:

Theorem 6 Suppose that $\mathcal{P}$ is fixed and let $\mathcal{P}_k, k \geq 1$ be the uniform refinements of $\mathcal{P}$. Then for any $u \in H^3(\Omega)$, there exists a polygonal spline $Q(u) \in \mathcal{S}_I(\mathcal{P}_k)$ such that
\[
\|u - Q(u)\|_{2,\Omega} \leq C|u|_{3,2,\Omega}2^{-3k}
\]
and
\[
|u - Q(u)|_{1,2,\Omega} \leq C|u|_{3,2,\Omega}2^{-2k}
\]
for constant $C$ which is independent of $u$, but may be dependent on the boundary of $\Omega$ if $\Omega$ is nonconvex.

Proof. One can follow the ideas and steps in the proof of Theorem 6 in [18]. We leave the proof to the interested reader. \hfill \Box

If one uses the whole space $\mathcal{S}_5(\mathcal{P})$, the approximation power is, of course, better than using $\mathcal{S}_I(\mathcal{P})$ only. We can do so using the constructed quasi-interpolatory spline given in Theorem 3.

That is, given a set of the data on an unknown function $u$, e.g. data of locations and function values, we approximate $u$ by $S_5(u)$ described below. In this situation, we need to estimate these coefficients in (13) as well as function values $u(v_i)$ and first order and second order derivatives at $v_i$ in order to use the quasi-interpolatory operator $S_5$. In this setting, we can use a two-stage method (cf. [26]). For the data set $\mathcal{D} = \{(x_i, y_i, u(x_i, y_i)), i = 1, \ldots, N\}$, we assume that $(x_i, y_i) \in \Omega$ for a bounded domain $\Omega$ for all $i = 1, \ldots, N$. We can choose a partition $\mathcal{P}$ consisting of parallelograms to contain the region $\Omega$ of interest so that all data locations are within these parallelograms. Note that $\Omega$ may be just the union of all parallelograms. For each vertex $v$ of $\mathcal{P}$, we use the given data values nearby $v$ to estimate $u(v), \nabla u(v), \nabla^2 u(v)$ by using the best quadratic polynomial fitting to the data values by using a least square method. Similarly, we can approximate $u(v_{P,j}), i = 1, \ldots, 4$ by using the best quadratic polynomial fit over the data values over each parallelogram $P$, or over all points within all the parallelograms sharing $e$. In this way, we can use the formula in (13). Similar to the proof of Theorem 6 in [18], we can establish

Theorem 7 Suppose that $\mathcal{P}$ is fixed and let $\mathcal{P}_k, k \geq 1$ be the uniform refinements of $\mathcal{P}$. Then for any $u \in H^3(\Omega)$, there exists a polygonal spline $Q(u) \in \mathcal{S}_5(\mathcal{P}_k)$ such that
\[
\|u - Q(u)\|_{2,\Omega} \leq C|u|_{6,2,\Omega}2^{-6k}
\]
and
\[
|u - Q(u)|_{1,2,\Omega} \leq C|u|_{6,2,\Omega}2^{-5k}
\]
for constant $C$ which is independent of $u$, but may be dependent on the boundary of $\Omega$ if $\Omega$ is nonconvex.
In the end of this section, we shall use the quasi-interpolatory splines constructed in this paper to interpolate various functions, and present some numerical results. We’ll use the parallelogram tiling shown in Figure 3 along with its uniform refinements to show the order of convergence, which is consistent with the result in Theorem 5.

**Example 1** In this example, we report the number of quadrilaterals in the partition for each refinement, along with the mesh size $h$, which we have defined as the largest diameter of any parallelogram in the partition. We report the root mean square error $\|u - S(u)\|_{RMS}$ computed over $500 \times 500$ points on interior of the partition, along with the convergence rate in terms of $h$; since we expect $L^2$ convergence of $O(h^3)$, we should expect a rate equal to 3. We first use three trigonometric functions to test the convergence of our quasi-interpolants. $u_1 = \sin(x)\sin(y)$, $u_2 = \sin(\pi x)\sin(\pi y)$, and $u_3 = \sin(2\pi x)\sin(2\pi y)$. Let $S(u)$ be the quasi-interpolatory vertex spline defined in (6) for function $u$, while $S_5(u)$ is the quasi-interpolatory spline defined in (13).

<table>
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<tr>
<th>Quads</th>
<th>$h$</th>
<th>$|u_1 - S(u_1)|_{RMS}$</th>
<th>rate</th>
<th>$|u_2 - S(u_2)|_{RMS}$</th>
<th>rate</th>
<th>$|u_3 - S(u_3)|_{RMS}$</th>
<th>rate</th>
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<td>2.14e-03</td>
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<td>3.04</td>
<td>1.12e-03</td>
<td>3.03</td>
</tr>
</tbody>
</table>

Table 1: The convergence over two refinements for $u_1$, $u_2$, $u_3$

Next we repeat the same experiments for functions which are more difficult to approximate. Consider test functions $u_4 = \sin(\pi(x^2 + y^2))$, $u_5 = (10 + x + y)^{-1}$, and $u_6 = (1 + x^2 + y^2)^{-1}$.

<table>
<thead>
<tr>
<th>Quads</th>
<th>$h$</th>
<th>$|u_4 - S(u_4)|_{RMS}$</th>
<th>rate</th>
<th>$|u_5 - S(u_5)|_{RMS}$</th>
<th>rate</th>
<th>$|u_6 - S(u_6)|_{RMS}$</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2.24e+0</td>
<td>1.02e-04</td>
<td>0.00</td>
<td>1.30e-03</td>
<td>6.44</td>
<td>1.13e-01</td>
<td>2.73</td>
</tr>
<tr>
<td>20</td>
<td>1.12e+0</td>
<td>1.91e-06</td>
<td>5.73</td>
<td>3.35e-05</td>
<td>5.30</td>
<td>1.31e-03</td>
<td>6.43</td>
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<tr>
<td>80</td>
<td>5.59e-01</td>
<td>3.09e-08</td>
<td>5.95</td>
<td>5.37e-07</td>
<td>5.95</td>
<td>3.34e-05</td>
<td>5.30</td>
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<tr>
<td>320</td>
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<td>4.89e-10</td>
<td>5.98</td>
<td>6.17e-09</td>
<td>5.98</td>
<td>1.06e-05</td>
<td>5.38</td>
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</tbody>
</table>

Table 2: The convergence over two refinements for more difficult functions $u_4$, $u_5$ and $u_6
4 A tensor-product spline approach

We present another explanation of the construction of our $C^1$ splines in this special case over parallelograms. In particular, we offer a description of the smoothness conditions across interior edges. One major difficulty for constructing locally supported spline functions is that the smoothness conditions across interior edges surrounded by an interior vertex are connected together. Fortunately, when using bi-quintic splines the smoothness conditions can be decoupled and hence $C^1$-smooth vertex splines can be constructed.

More generally, when the degree $d \geq 3r + 2$ for any $r \geq 1$, $C^r$ smooth vertex splines can be constructed. This is similar to the setting of bivariate splines over triangulations (see [5] or [19]). Let us use the Bernstein-Bézier coefficients of tensor-product polynomials to explain this point.

Let us recall some notation. For a parallelogram $T_2 = \langle y_1, y_2, y_3, y_4 \rangle$, where $y_1, y_2, y_3, y_4 \in \mathbb{R}^2$ are its four vertices, we assume that $\langle y_1, y_2 \rangle \parallel \langle y_3, y_4 \rangle$ and $\langle y_1, y_3 \rangle \parallel \langle y_2, y_4 \rangle$ without loss of generality. For each $x \in \langle y_1, y_2, y_3, y_4 \rangle$, it is clear that $x$ is uniquely expressed as

$$ x = y_1 + \mu_1(y_2 - y_1) + \mu_2(y_3 - y_1), $$

where $\mu_1, \mu_2$ are two nonnegative numbers. Define $\mu = (\mu_1, \mu_2)$, called the barycentric coordinate of $x$ with respect to $T_2$. For any $\sigma = (\sigma_1, \sigma_2) \in \mathbb{Z}_+^2$ and $\alpha = (\alpha_1, \alpha_2) \leq (\sigma_1, \sigma_2)$, denote

$$ \tilde{\Phi}_\alpha^\sigma(\mu) = \binom{\sigma_1}{\alpha_1} \binom{\sigma_2}{\alpha_2} (\mu_1)^{\alpha_1} (1-\mu_1)^{\sigma_1-\alpha_1} (\mu_2)^{\alpha_2} (1-\mu_2)^{\sigma_2-\alpha_2}. $$

Then $\tilde{\Phi}_\alpha^\sigma$ is a polynomial of $x$. We denote by $\Pi_\sigma(T_2)$ the space of all polynomials in the form

$$ P_\sigma(x) = \sum_{\alpha \leq \sigma} a_\alpha \tilde{\Phi}_\alpha^\sigma(\mu). $$

Here, $P_\sigma$ is called the Bernstein representation or B-form of a polynomial of coordinate degree $\sigma$ with respect to $T_2$.

If $\sigma = (d, d)$, we simply write $\Pi_d(T_2)$ to be the space of tensor product polynomials $P_\sigma$ of coordinate degree $\sigma$. In addition, the point set

$$ \left\{ (y_1 + \frac{\alpha_1}{\sigma_1}(y_2 - y_1) + \frac{\alpha_2}{\sigma_2}(y_3 - y_1), a_\alpha) : (\alpha_1, \alpha_2) \leq (\sigma_1, \sigma_2) \right\} $$

in $\mathbb{R}^3$ is called the B-net of $P_\sigma$ on $T_2$, and $a_\alpha, \alpha \leq \sigma$ are called the B-coefficients of $P_\sigma$, which may be simply shown as in Figure 18, where $\sigma = (5, 5)$.

Next we explain the smoothness conditions across the common edge of two parallelograms. Suppose that $P_d$ and $Q_d$ are defined on two adjacent parallelograms $T_1 = \langle x_1, x_2, x_3, x_4 \rangle$ and $T_2 = \langle x_3, x_4, x_5, x_6 \rangle$ which share a common edge $\langle x_3, x_4 \rangle$. More precisely, let

$$ P_d(x) = \sum_{\beta \leq (d, d)} a_\beta \tilde{\Phi}_\beta^d(\lambda), \quad x = x_3 + \lambda_1(x_1 - x_3) + \lambda_2(x_4 - x_3), $$

and

$$ Q_d(x) = \sum_{\beta \leq (d, d)} b_\beta \tilde{\Phi}_\beta^n(\mu), \quad x = x_3 + \mu_1(x_5 - x_3) + \mu_2(x_4 - x_3). $$

Write $x_5 - x_3 = \lambda_1^0(x_1 - x_3) + \lambda_2^0(x_4 - x_3)$. Then, we have $D_{x_5-x_3} = \lambda_1^0 D_{x_1-x_3} + \lambda_2^0 D_{x_4-x_3}$. Let $F$ be a function defined as follows:

$$ F(x) = \begin{cases} P_d(x) & \text{if } x \in T_1 \\ Q_d(x) & \text{if } x \in T_2. \end{cases} $$

It is straightforward to show the following:
Figure 18: The $B$-coefficients of $P_5$

**Lemma 2** $F \in C^r(T_1 \cup T_2)$ if and only if

$$(D_{x_5-x_3})^k Q_d \bigg|_{T_1 \cap T_2} = (\lambda_1^0 D_{x_1-x_3} + \lambda_2^0 D_{x_1-x_3})^k P_d \bigg|_{T_1 \cap T_2}$$

for $0 \leq k \leq r$.

Next we define the so-called degree raising operators $\bar{R}_2^k$ by

$$\bar{R}_2^k a_{i,j} = \sum_{\nu=0}^{j} \binom{j}{\nu} \binom{d-j}{d-k-\nu} a_{i\nu}, \quad k \geq 0.$$  

Then the smoothness conditions between $P_d$ and $Q_d$ easily follow, and are summarized in Lemma 3:

**Lemma 3** $F \in C^r(T_1 \cup T_2)$ if and only if

$$\Delta_j^k b_{(0,l)} = \sum_{i+j=k} \binom{k}{i} (\lambda_1^0)^i (\lambda_2^0)^j \frac{d! (d-k)!}{(d-i)! (d-j)!} \Delta_i^j \bar{R}_2^k a_{0,l},$$

for $0 \leq k \leq r$, and $0 \leq l \leq d$.

**Proof.** On the left-hand side, we have

$$(D_{x_5-x_3})^k Q_d \bigg|_{T_1 \cap T_2} = \frac{d!}{(d-k)!} \sum_{l=0}^{d} \Delta_j^k b_{(0,l)} \tilde{\Phi}_{(0,l)}^{(0,d)}(0, \mu_2)$$

On the right-hand side,

$$\sum_{i+j=k} \binom{k}{i} (\lambda_1^0 D_{x_1-x_3})^i (\lambda_2^0 D_{x_1-x_3})^j P_d \bigg|_{T_1 \cap T_2}$$

$$= \sum_{i+j=k} \binom{k}{i} (\lambda_1^0 \Delta_1)^i (\lambda_2^0 \Delta_2)^j a_{(0,m)} \tilde{\Phi}_{(0,m)}^{(0,d-j)}(0, \lambda_2)$$

$$= \sum_{i+j=k} \binom{k}{i} \frac{d!}{(d-j)!} \frac{d!}{(d-i)!} (\lambda_1^0 \Delta_1)^i (\lambda_2^0 \Delta_2)^j \bar{R}_2^k a_{(0,l)} \tilde{\Phi}_{(0,l)}^{(0,n)}(0, \lambda_2).$$

26
where we have used the degree raising technique. Comparing the coefficients on both sides, we obtain the smoothness conditions.

Computation of the degree raising operator $\overline{R}_2^j$ in (21) can be carried out as follows:

$$
(\lambda_2 + (1 - \lambda_2))^j \sum_{m=0}^{d-j} a_{(0,m)} \tilde{\Phi}_{(0,m)}^{(0,d-j)}
= \sum_{m=0}^{d-j} a_{(0,m)} \sum_{i=0}^{j} \binom{j}{i} \lambda_2^{i} (1 - \lambda_2)^{j-i} \left( \binom{d-j}{m} \right) \lambda_2^{m} (1 - \lambda_2)^{d-j-m}
= \sum_{m=0}^{d-j} \sum_{i=0}^{j} \binom{j}{i} a_{(0,m)} \left( \binom{d-j}{m} \right) \lambda_2^{m+i} (1 - \lambda_2)^{d-m-i}
= \sum_{k=0}^{d} \sum_{m+i=k}^{d-j} \binom{j}{i} \binom{d-j}{m} \binom{d}{k} a_{(0,m)} \tilde{\Phi}_{(0,k)}^{(0,d)}(0,\lambda_2)
= \sum_{k=0}^{d} \sum_{m+i=k}^{d-j} \binom{j}{i} \binom{d-j}{m} \binom{d}{k} a_{(0,m)} \tilde{\Phi}_{(0,k)}^{(0,d)}(0,\lambda_2).
$$

For $C^1$ smoothness condition, with $\alpha = \lambda_1^0$ and $\beta = \lambda_0^0$, we have

$$
b_{1,j} - b_{0,j} = \alpha (a_{1,j} - a_{0,j}) + \beta [(a_{0,j} - a_{0,j-1})j + (a_{0,j+1} - a_{0,j}) (d-j)]/d
$$

for all $j = 0, \cdots, d$. In particular, for $d = 5$, we have

$$
\begin{align*}
b_{1,0} - b_{0,0} &= \alpha (a_{1,0} - a_{0,0}) + \beta (a_{0,1} - a_{0,0}) \\
b_{1,j} - b_{0,j} &= \alpha (a_{1,j} - a_{0,j}) + \beta [(a_{0,j} - a_{0,j-1})j + (a_{0,j+1} - a_{0,j}) (5-j)]/5, 1 \leq j \leq 4, \\
b_{1,5} - b_{0,5} &= \alpha (a_{1,5} - a_{0,5}) + \beta (a_{0,5} - a_{0,4})
\end{align*}
$$

The geometric meaning of the smoothness conditions in (23) for $j = 0$ and $j = 5$ is similar to that of the smoothness conditions for $C^1$ quintic splines over a triangulation. However, the supports of smoothness conditions (23) for coefficients located inside the parallelogram, i.e. $j = 1, 2, 3, 4$ are different from the triangular spline case.

We are ready to explain that when the degree $d = 5$, the smoothness conditions across of interior edges can be decoupled. For convenience, we simply use one parallelogram to do so. We shall use $C^2$ interpolatory conditions at each of the four vertices to determine the 6 coefficients closed to the vertex (see the B-coefficients within the red triangle for four triangles in Figure 19). Next the two coefficients at the two blue circles near each edge can be determined by using the $C^1$ smoothness conditions as shown in Figure 19. Finally, the four coefficients in the cyan boxes in the center of the parallelogram are free which can be chosen so that a locally supported spline function such as $\psi_v$ to satisfy some properties or the span of all vertex splines reproduce bi-quintic polynomials. The detail of construction of vertex splines in terms of B-coefficients is left to the interested reader.

Note that in Figure 19, we have three different types of coefficients instead of 6 types in [13]. Certainly, when an edge of the parallelogram in Figure 19 is a boundary edge, the two B-coefficients in the blue circle nearest the edge are free to choose. The best way to choose them is in such a way that bi-quintic polynomials can be reproduced.
Figure 19: The three types of B-coefficients of $P_3$: the B-coefficients in the red triangles are determined by using $C^2$ interpolation conditions at the nearest vertices, the B-coefficients in the blue circles are determined by using $C^1$ smoothness conditions across the nearest edges, and the B-coefficients in cyan boxes are free parameters or by the conditions to preserving bi-quintic polynomials.

Furthermore, following the proof in [5] for the triangulation setting, when certain B-coefficients of $F$ are given, we apply the smoothness conditions (21) to make $F$ smooth across edge $[x_3, x_4]$. We have following two lemmas whose proofs can be found in [17].

**Lemma 4 (See [17])** For $1 \leq n < d$. Assume that $x_4 \notin [x_2, x_6]$. Suppose that the following B-coefficients of $P_n$ and $Q_d$ are given: $a_{(i,j)}, b_{(i,j)}, i + j \leq n - 1$ as well as $a_{(k,n-k)}, b_{(k,n-k)}, k = 0, \ldots, n - 2l - 2$. Further, suppose that $a_{(i,j)}, b_{(i,j)}, i + j \leq n - 1$ satisfy the smoothness conditions up to order $n - 1$ and $a_{(k,n-k)}, b_{(k,n-k)}$ with some other $a_{(ij)}$’s, $b_{(ij)}$’s satisfy the smoothness conditions up to order $n - 2l - 2$. Then for any given $a_{(n-k,k)}, b_{(n-k,k)}, 0 \leq k \leq l$, there exists a unique set of $a_{(n-l-k,k+1)}, b_{(n-l-k,k+1)}, 1 \leq k \leq l + 1$, such that $a_{(ij)}, b_{(ij)}, i + j \leq n$ satisfy the smoothness conditions up to order $n$.

**Lemma 5 (See [17])** Fix $1 \leq n < d$. Assume that $x_4 \in [x_2, x_6]$. Suppose that the B-coefficients $\{a_{(ij)} : i + j \leq n - 1\}$ and $\{b_{(ij)} : i + j \leq n - 1\}$ satisfy the smoothness conditions (21) up to order $n - 1$. Furthermore, suppose that $\{a_{(k,n-k)}, b_{(k,n-k)} : 0 \leq k \leq l\}$ are given and satisfy the smoothness conditions (21) up to order $l$, where $l < n$. Then for any $\{a_{(k,n-k)} : l + 1 \leq k \leq n\}$, there exists a unique set of coefficients $\{b_{(k,n-k)} : l + 1 \leq k \leq n\}$ such that $\{a_{(ij)} : i + j \leq n\}$ and $\{b_{(ij)} : i + j \leq n\}$ satisfy the smoothness conditions (21).

Finally, based on Lemmas 4 and 5, we use the same method in [5] or [19] to establish the main result in this section. The detailed proof is left to the interested reader.

**Theorem 8** Let $\mathcal{P}$ be a collection of parallelograms. Let $S_6^r(\mathcal{P})$ be the space of $C^r$ smooth spline functions of coordinate degree $(n, n)$ over the partition $\mathcal{P}$. For $n \geq 3r + 2$, there exists locally supported vertex spline functions which span the space $S_6^r(\mathcal{P})$.

5 An Application

In this section, we present a few examples to use our vertex splines for constructing surfaces.

**Example 2** Our first example is to construct a torus. As a torus $T$ is a parametric surface defined over $[0, 2\pi] \times [0, 2\pi]$ which can be partitioned into a simple collection of parallelograms as shown in
Figure 20. The surface of our vertex spline $L$ interpolating $T$ at vertices of the partition up to the 2nd order is also shown in Figure 20.

Similarly, we can generate tori of different shapes as shown in Figure 21.

6 Possible Extensions

We shall list a few possible extensions and open research problems.

- 1. Although our collection of parallelograms are fairly general, they are not general enough to partition any polygonal domain, e.g. a triangular domain. A polygon $\Omega$ can only be partitioned using parallelograms if the boundary of $\Omega$ satisfies some properties (see [14]). We plan to add triangles to an initial partition of parallelograms of a polygonal domain and then use $C^1$ quintic splines over triangles. In this way, we will be able to construct smooth surfaces over such a mixed partition.
• 2. One can extend the ideas in [1] to deal with the smooth tensor-product splines over any collection of arbitrary parallelograms with smoothness $r \geq 1$ and degree $d \geq 3r + 2$ as studied in Section 4. That is, one can represent tensor-product splines of coordinate degree $(d, d)$ and add all B-coefficients together as a vector. In addition, one can implement the smoothness conditions described in Lemma 3. Then one can use a minimization approach to find the coefficient vector satisfying the smoothness conditions. We leave the study and the implementation to the interested reader.

• 3. Our next paper will show how to enforce $C^1$ smoothness over a general quadrilateral partition, which will require Wachspress degree 7 rather than 5. We have already successfully constructed the analogous function $\psi_v$ in the case of degree 6 over such a general partition, but we have discovered that one cannot adequately decouple smoothness conditions in a way which allows for a reasonable span - in particular, one cannot even produce analogous functions $\psi_{x,v}$ and $\psi_{y,v}$ using only degree 6. Please look forward to this following paper.

• 4. A more interesting thing to do would be to extend our construction into a 3D domain over some class of hexahedral partitions. We leave it to the interested reader.

References


