On the Approximation Power of Splines on Triangulated Quadrangulations

Ming-Jun Lai [1] and Larry L. Schumaker [2]

Abstract. We study the approximation properties of the bivariate spline spaces $\mathcal{S}^r_{3r}(\diamondplus)$ of smoothness r and degree $3r$ defined on triangulations $\diamondplus$ which are obtained from arbitrary nondegenerate convex quadrangulations by adding the diagonals of each quadrilateral.

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§1. Introduction

Suppose $\Diamond$ is a nondegenerate convex *quadrangulation* of a polygonal domain Ω in $\mathbb{R}^2$ (see Sect. 6 for details on quadrangulations and techniques for constructing them). Let $\diamondplus$ be the triangulation obtained by inserting the diagonals of each quadrilateral of $\Diamond$. The purpose of this paper is to study the spline spaces

$$\mathcal{S}^r_d(\diamondplus) := \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_d,\ \forall T \in \diamondplus\},$$

for $d = 3r$, where $\mathcal{P}_d$ denotes the space of polynomials of total degree at most d.

Bivariate spline spaces defined over general triangulations have been heavily studied in the literature, and have proven to be useful in a variety of applications including data fitting and the numerical solution of boundary-value problems. The special case of triangulated quadrangulations has received less attention, but there have been several recent papers, see Remark 7.1. There are several reasons why the spaces $\mathcal{S}^r_{3r}(\diamondplus)$ are of particular interest:

1) As we shall show, the spaces $\mathcal{S}^r_{3r}(\diamondplus)$ possess full approximation power. This is in contrast to spaces of splines $\mathcal{S}^r_d(\triangle)$ based on general triangulations $\triangle$,

[1] Department of Mathematics, The University of Georgia, Athens, GA 30602, mjlai@math.uga.edu. Supported by the National Science Foundation under grant DMS-9303121.

[2] Department of Mathematics, Vanderbilt University, Nashville, TN 37240, s@mars.cas.vanderbilt.edu. Supported by the National Science Foundation under grant DMS-9500643 and by NATO under grant CRG-951291.

where full approximation power can only be guaranteed for $d \geq 3r+2$, see [3], and is known to fail for certain triangulations when $d < 3r+2$, see [4].

2) For $r = 1, 2$, the spaces $\mathcal{S}^r_{3r}(\diamondplus)$ have smaller dimensions than the triangle-based spline spaces which are typically used in practice for these choices of smoothness, see Remark 7.2. This implies reduced complexity in applications.

3) The spaces $\mathcal{S}^r_{3r}(\diamondplus)$ are nested with respect to dyadic refinement, see Sect. 6. This allows the construction of a multi-resolution analysis and of hierarchical multilevel bases which are useful in applications, see e.g. [15].

4) As shown in Sect. 6, quadrangulations can also be refined *locally* which is of particular importance for numerical solution of boundary-value problems.

The paper is organized as follows. In the following section we introduce some notation and state several lemmas needed later. In Sections 3 and 4 we analyze properties of $\mathcal{S}^r_{3r}(\diamondplus)$ for the cases r odd and r even, respectively. Section 5 is devoted to the proof of the following theorem which is the main result of the paper:

Theorem 1.1. *Fix $r \geq 1$ and $0 \leq m \leq 3r$. There exists a linear quasi-interpolation operator Q_m mapping $L_1(\Omega)$ into $S^r_{3r}(\diamondplus)$ such that if f is in the Sobolev space $W^{m+1}_p(\Omega)$ with $1 \leq p \leq \infty$, then*

$$\|D^\alpha_x D^\beta_y (f - Q_m f)\|_{p,\Omega} \leq C \ |\diamondplus|^{m+1-\alpha-\beta} \ |f|_{m+1,p,\Omega}, \tag{1.1}$$

for all $0 \leq \alpha + \beta \leq m$. Here $|\diamondplus|$ is the maximum of the diameters of the triangles in $\diamondplus$. If Ω is convex, then the constant C depends only on r and the smallest angle $\theta_{\diamondplus}$ in $\diamondplus$. If Ω is nonconvex, C also depends on the minimum exterior angle between any two boundary edges of Ω.

We prove this theorem by constructing stable bases for certain super-spline subspaces of $S^r_{3r}(\diamondplus)$. To conclude the paper, in Sect. 6 we discuss quadrangulations and their construction, and present several remarks in Sect. 7.

§2. Preliminaries

The spaces $W^{m+1}_p(\Omega)$ appearing in Theorem 1.1 are the usual Sobolev spaces equipped with the norms

$$\|f\|_{m+1,p,\Omega} = \begin{cases} \left(\displaystyle\sum_{k=0}^{m+1} |f|^p_{k,p,\Omega} \right)^{1/p}, & 1 \leq p < \infty \\ \displaystyle\sum_{k=0}^{m+1} |f|_{k,\infty,\Omega}, & p = \infty, \end{cases}$$

with

$$|f|_{k,p,\Omega} = \begin{cases} \left(\sum_{\nu+\mu=k} \|D_x^\nu D_y^\mu f\|_{p,\Omega}^p \right)^{1/p}, & 1 \le p < \infty \\ \sum_{\nu+\mu=k} \|D_x^\nu D_y^\mu f\|_{\infty,\Omega}, & p = \infty. \end{cases}$$

When $r < m$, the spline $Q_m f$ in Theorem 1.1 does not belong to $W_p^{m+1}(\Omega)$. In this case, the norm on the left-hand side of (1.1) must be modified. If $p = \infty$, we maximize over all triangles, and if $1 \le p < \infty$, we sum over all triangles of Ω to get

$$\|D_x^\alpha D_y^\beta (f - Q_m f)\|_{p,\Omega}^p = \sum_{T_i \in \triangle} \|D_x^\alpha D_y^\beta (f - Q_m f)\|_{p,T_i}^p.$$

In the finite-element literature these are called *mesh-dependent norms.*

Our proof of Theorem 1.1 is based on recent results [14] on *quasi-interpolation operators* of the form

$$Q_m f = \sum_{\xi \in \Gamma}^{N} (\lambda_{\xi,m} f) \phi_\xi, \tag{2.1}$$

where $\{\phi_i\}_{\xi\in\Gamma}$ is a set of locally supported splines and $\lambda_{\xi,m}$ are linear functionals defined on $L_1(\Omega)$.

To state the result of [14] needed here, we recall that given a vertex v in a triangulation, the *star of* v is the union of all triangles which share the vertex v. The *star of order* ℓ is defined recursively as $\mathrm{star}^\ell(v) := \{\cup T : \ T$ shares a vertex with some $\tilde{T} \in \mathrm{star}^{\ell-1}(v)\}$.

Theorem 2.1. [14] *Fix $0 \le m \le d$. Suppose Γ is some finite index set, and let $\{\phi_\xi\}_{\xi\in\Gamma}$ be a set of splines in $\mathcal{S}_d^0(\triangle)$ such that*

- H1) *there exists an integer ℓ such that for each ξ, the support of ϕ_ξ is contained in $\mathrm{star}^\ell(v_\xi)$ for some vertex $v_\xi \in \triangle$;*
- H2) *$K_1 := \max_\xi \ \|\phi_\xi\|_{\infty,\Omega} < \infty$;*
- H3) *$K_2 := \max_T \#(\Sigma_T) < \infty$, where $\Sigma_T := \{\xi : \ T \subset \sigma(\phi_\xi)\}$ and $\sigma(\phi_\xi)$ denotes the support of ϕ_ξ.*

Suppose in addition that there exists a set of linear functionals $\{\lambda_{\xi,m}\}_{\xi\in\Gamma}$ defined on $L_1(\Omega)$ with the property that for all $\xi \in \Gamma$, there is a triangle T_ξ contained in the support of ϕ_ξ with

$$|\lambda_{\xi,m} f| \le \frac{K_3}{A_{T_\xi}^{1/p}} \|f\|_{p,T_\xi} \quad \text{for all } f \in L_p(\Omega) \text{ when } 1 \le p < \infty \tag{2.2}$$

and

$$|\lambda_{\xi,m} f| \le K_3 \|f\|_{\infty,\Omega} \quad \text{for all } f \in L_\infty(\Omega) \text{ when } p = \infty \tag{2.3}$$

for some constant K_3. Finally, suppose that the corresponding quasi-interpolation operator (2.1) reproduces polynomials in the sense that

$$Q_m P = P for all P \in \mathcal{P}_m. \tag{2.4}$$

Then there exists a constant C depending only on r, the constants $K_1, \ldots, K_3$, and the smallest angle $\theta_{\diamondplus}$ of the triangulation such that if $f \in W_p^{m+1}(\Omega)$, then

$$\|D_x^\alpha D_y^\beta (f - Q_m f)\|_{p,\Omega} \leq C|\triangle|^{m+1-\alpha-\beta}|f|_{m+1,p,\Omega} \tag{2.5}$$

for all $0 \leq \alpha + \beta \leq m$ and all $1 \leq p \leq \infty$.

We establish Theorem 1.1 by examining the *super-spline subspace*

$$\mathcal{S}_{3r}^{r,\rho}(\diamondplus) := \{s \in S_{3r}^r(\diamondplus) : s \in C^{\rho(v)}(v),\ v \in V\}, \tag{2.6}$$

where $\rho := \{\rho(v)\}_{v \in V}$ with

$$\rho(v) = \begin{cases} (3r-1)/2, & r \text{ odd}, \\ (3r-2)/2, & r \text{ even and } v \notin V_3, \\ 3r/2, & r \text{ even and } v \in V_3. \end{cases}$$

Here V is the set of all vertices of $\diamondsuit$, and V_3 is the subset of vertices which are shared by exactly three quadrilaterals. We write $s \in C^\nu(v)$ to mean that the derivatives up to order ν of the polynomial pieces $s_T := s|_T$ on the triangles T sharing the vertex v all have the same values at v.

To construct the functions ϕ_i and functionals $\lambda_{i,m}$ needed to define (2.1), following [14] we make use of the well-known Bézier representation for splines. In particular, if $s \in S_d^0(\diamondplus)$ and $T := \langle v_1, v_2, v_3 \rangle$ is a triangle in $\diamondplus$, s can be written in the form

$$s|_T(x,y) = \sum_{i+j+k=d} c_{ijk}^T B_{ijk}^d(x,y),$$

where B_{ijk}^d are the usual Bernstein polynomials of degree d associated with T. As usual we associate the coefficients c_{ijk}^T with the *domain points*

$$\xi_{ijk}^T = \frac{(iv_1 + jv_2 + kv_3)}{d}, \quad i + j + k = d. \tag{2.7}$$

Recall that the *distance of a domain point ξ_{ijk}^T from the vertex v_1* of T is $d - i$, and that its distance from the edge $\langle v_2, v_3 \rangle$ is i, with similar definitions for the other vertices and edges of T. Given an integer $\mu > 0$ and a vertex v,

$$\mathcal{D}_\mu(v) := \{\xi :\ d(\xi, v) \leq \mu\}$$

is called the *disk of radius μ* around v.

For each triangle $T \in \diamondplus$, let $\mathcal{D}_T$ be the set of domain points in T, and let $\mathcal{D}_{\diamondplus} = \cup\{\mathcal{D}_T : T \in \diamondplus\}$, where repeated points (along common edges) are included just once. The set $\mathcal{D}_{\diamondplus}$ is just the set of *Bézier sites* for the *Bézier net* of a spline in $S_d^0(\diamondplus)$. For each point $\xi \in \mathcal{D}_{\diamondplus}$, let γ_ξ be the functional defined on $S_d^0(\diamondplus)$ such that for any spline $s \in S_d^0(\diamondplus)$, $\gamma_\xi s$ is the Bézier coefficient of s associated with the domain point ξ. Given a subspace $\mathcal{S}$ of $\mathcal{S}_d^0(\diamondplus)$, we recall that a subset Γ of $\mathcal{D}_{\diamondplus}$ with the property

$$\big(\gamma_\xi s = 0 \;\text{ for all }\; \xi \in \Gamma\big) \qquad \text{implies} \qquad s \equiv 0 \tag{2.8}$$

is called a *determining set* for $\mathcal{S}$.

To apply Theorem 2.1 to the super-spline space $\mathcal{S}_{3r}^{r,\rho}(\diamondplus)$, following [14] we need to construct a *minimal determining set* Γ for this space and an associated set of splines $\mathcal{B} := \{\phi_\xi\}_{\xi\in\Gamma}$ with the dual property

$$\gamma_\eta \phi_\xi = \delta_{\eta,\xi}, \qquad \text{all } \eta \in \Gamma. \tag{2.9}$$

Then $\dim \mathcal{S}_{3r}^{r,\rho}(\diamondplus) = \#\Gamma$ and $\mathcal{B}$ is a basis for it. We need to choose Γ so that the corresponding set of basis functions $\{\phi_\xi\}_{\xi\in\Gamma}$ possess properties H1)–H3) of Theorem 2.1.

§3. A Minimal Determining Set for $\mathcal{S}_{3r}^{r,\rho}(\diamondplus)$ for r odd

Throughout this section we assume that r is odd. We need some additional notation. Let $\mu = (3r-1)/2$, and suppose $T = \langle w, v_1, v_2\rangle$ is a triangle in $\diamondplus$. Define

$$\begin{aligned}
\mathcal{H}^T(w) &:= \big\{\xi \in \mathcal{D}_T : \operatorname{dist}(\xi, v_1) \ge 2r,\ \operatorname{dist}(\xi, v_2) \ge 2r\big\};\\
\mathcal{B}^T(w) &:= \big\{\xi^T_{3r-i-j,i,j} : j = 0, \ldots, 2(i-\mu+2)+1 \text{ and}\\
&\qquad\qquad i = \mu+2, \ldots, 2r-1\big\};\\
\mathcal{D}^T_\mu(v_1) &:= \big\{\xi \in \mathcal{D}_T : \operatorname{dist}(\xi, v_1) \le \mu\big\};\\
\mathcal{D}^T_\mu(v_2) &:= \big\{\xi \in \mathcal{D}_T : \operatorname{dist}(\xi, v_2) \le \mu\big\};\\
\mathcal{E}^T(e) &:= \big\{\xi \in \mathcal{D}_T : \operatorname{dist}(\xi, e) \le r\big\}\backslash(\mathcal{D}^T_\mu(v_1) \cup \mathcal{D}^T_\mu(v_2)).
\end{aligned} \tag{3.1}$$

These sets do not include all points of $\mathcal{D}_T$. They are disjoint subsets of $\mathcal{D}_T$ except for $\mathcal{H}^T(w)$ and $\mathcal{E}^T(e)$ which intersect in one point.

For $r = 7$, Fig. 1 shows the sets in (3.1) associated with the four triangles making up a single quadrilateral and sharing a vertex w. The points in the sets $\mathcal{H}^T(w)$ are marked with diamonds, points in $\mathcal{B}^T(w)$ with boxes, points in $\mathcal{D}^T_\mu(v_i)$ with circles, and points in $\mathcal{E}^T(e)$ with crosses.

Let $V_\times$ be the set of vertices of $\diamondplus$ which lie at the intersections of diagonals of quadrilaterals.

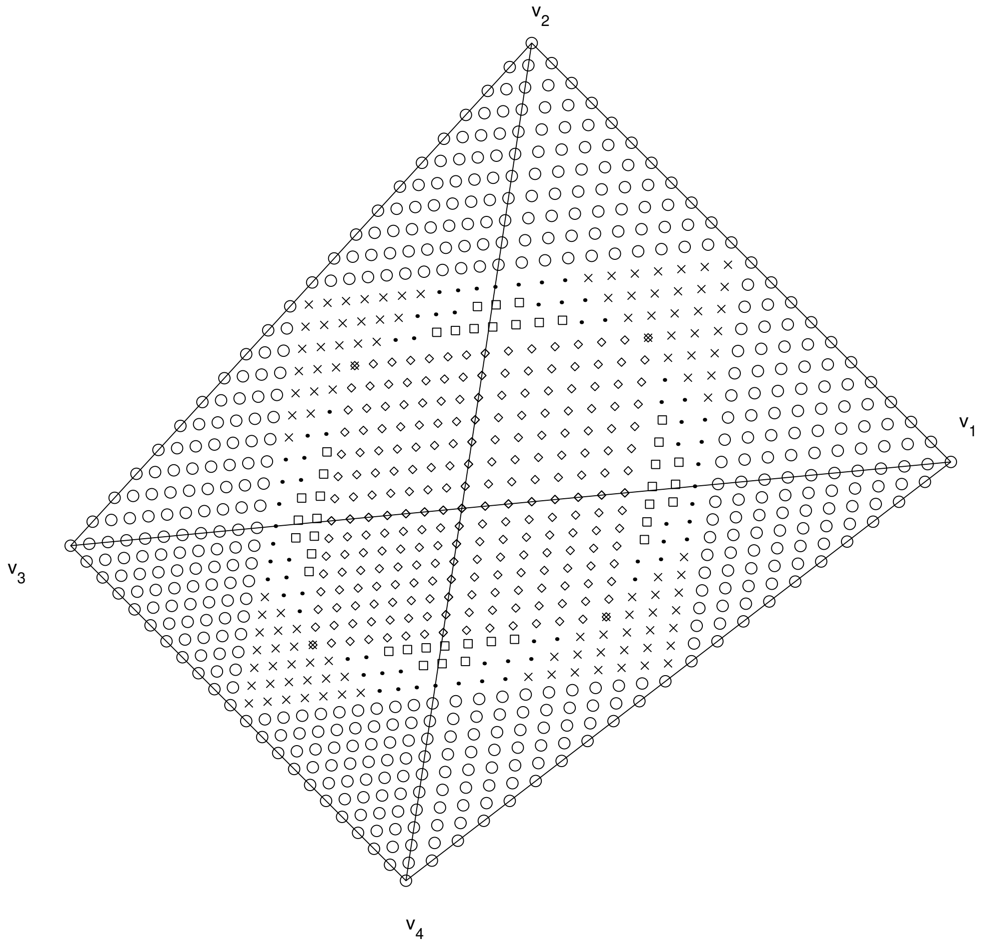

Fig. 1. The Domain Points (3.1) for $r = 7$ and $\mu = 10$.

Theorem 3.1. *Suppose r is odd, and let Γ be the following subset of $\mathcal{D}_{\diamondplus}$:*

1) *for each vertex $v \in \Diamond$, pick a triangle $T \in \diamondplus$ with vertex at v and choose all points in the set $\mathcal{D}^T_\mu(v)$,*
2) *for each edge e not attached to a vertex in $V_\times$, include the set $\mathcal{E}^T(e)$, where T is a triangle sharing the edge e. If e is a boundary edge, there is only one such triangle, while if it is an interior edge, we can work with either of the two triangles sharing it,*
3) *for each $w \in V_\times$ and for each $i = 1, \ldots, 4$, choose all of the points in $\mathcal{B}^{T_i}(w)$, where $T_i = \langle w, v_i, v_{i+1}\rangle$ are the triangles with vertex at w,*
4) *for each $w \in V_\times$, pick a triangle $T = \langle w, v, u\rangle \in \diamondplus$ with vertex at w and choose all of the points in the set $\mathcal{H}^T(w)$ except for the domain points $\xi^T_{r,r,r}$, $\xi^T_{3r,0,0}$, $\xi^T_{r,2r,0}$, $\xi^T_{r,0,2r}$.*

Then Γ is a minimal determining set for $\mathcal{S}^{r,\rho}_{3r}(\diamondplus)$, and there exist a set of splines $\{\phi_\xi\}_{\xi\in\Gamma}$ which satisfy (2.9) and thus are a basis for $\mathcal{S}^{r,\rho}_{3r}(\diamondplus)$. These splines satisfy *properties H1)–H3) of Theorem 2.1.*

Proof: The fact that Γ is a determining set can be verified in the usual way. Indeed, assuming all of the coefficients associated with domain points in Γ are zero, we can show that all coefficients corresponding to $\xi \notin \Gamma$ are also zero using smoothness conditions. To show this for ξ in the disk $\mathcal{D}_\mu(v)$ of radius μ around a vertex v, we can apply the smoothness conditions directly, see e.g. Lemma 6.1 in [14]. For ξ lying in a set of the form $\mathcal{E}^T(e)$, we can again use the smoothness conditions. Next, for ξ in a set of the form $\mathcal{H}^T(w)$ with $w \in V_\times$, we use Lemma 3.2 below. Finally, for all remaining $\xi \notin \Gamma$, we use Lemma 6.2 in [14].

Following [14], we now show that Γ is a minimal determining set by constructing a dual basis satisfying (2.9). Given $\xi \in \Gamma$, we construct $\phi_\xi \in \mathcal{S}_{3r}^{r,\rho}(\diamondplus)$ by choosing the coefficient corresponding to ξ to be 1 while setting the coefficients corresponding to all other $\eta \in \Gamma$ to zero. We then use the above mentioned lemmas to show that all remaining coefficients corresponding to points in $\mathcal{D}_{\diamondplus}$ are uniquely defined and are bounded by a constant K depending only on r and $\theta_{\diamondplus}$ (cf. the proof of Lemma 6.1 in [14]). It follows that H2) of Theorem 2.1 holds. We now show that H1) holds with $\ell = 2$ by identifying the support sets $\sigma(\phi_\xi)$ of the ϕ_ξ. There are five cases depending on where ξ lies.

Case 1: Suppose that $\xi \in \mathcal{D}_\mu^T(v)$ for a vertex $v \in V$. In this case it is easy to see that the coefficients of ϕ_ξ are nonzero only for ξ which lie in one of the quadrilaterals that share the vertex v. Thus, $\sigma(\phi_\xi)$ is the union of these quadrilaterals.

Case 2: Suppose that $\xi \in \mathcal{E}^T(e)$ for some boundary edge of $\diamondplus$. Then the coefficients of ϕ_ξ are nonzero only for ξ which lie in the quadrilateral q containing e. It follows that this quadrilateral is the support set of ϕ_ξ.

Case 3: Suppose that $\xi \in \mathcal{E}^T(e)$ for some interior edge of $\diamondplus$ shared by two quadrilaterals q and q'. Then the coefficients of ϕ_ξ are nonzero only for ξ which lie in either q or q'. Thus the support of ϕ_ξ is $q \cup q'$.

Case 4: Suppose $\xi \in \mathcal{B}^T(w)$ for some $w \in V_\times$, where $T = \langle w, v_1, v_2\rangle$ is one of the triangles of a quadrilateral q with center vertex w (cf. Fig. 1). Then it is easy to see that the coefficients of ϕ_ξ are nonzero only if ξ lies in one of the two triangles $T = \langle w, v_1, v_2\rangle$ or $T = \langle w, v_4, v_1\rangle$. We conclude that the support of ϕ_ξ is q.

Case 5: Suppose $\xi \in \mathcal{H}^T(w)$ for some $w \in V_\times$, where q is a quadrilateral q with center vertex w (cf. Fig. 1). In this case the only coefficients of ϕ_ξ which are nonzero are associated with domain points which lie in one of the four sets $\mathcal{H}^{T_i}(w)$ surrounding w. The fact that these nonzero coefficients are well-defined and uniformly bounded by a constant K depending only on r and $\theta_{\diamondplus}$ follows from Lemma 3.2 below. Again the support of ϕ_ξ is q.

It remains to verify that the ϕ_ξ's satisfy H3) of Theorem 2.1. Given a triangle T contained in a quadrilateral q, we examine the set $\Sigma_T := \{\xi : \ T \subset \sigma(\phi_\xi)\}$. In view of its support properties, ϕ_ξ can only be nonzero at a point of T if ξ is in a quadrilateral which shares a vertex with q. The number of such quadrilaterals is clearly bounded by a constant C depending only on r and the smallest angle $\theta_{\diamondplus}$ of

$\diamondplus$. Since there are 4 triangles per quadrilateral and at most $\binom{3r+2}{2}$ domain points per triangle, we conclude that

$$\#(\Sigma_T) \leq K_2 := 4C\binom{3r+2}{2}.$$

This completes the proof of the theorem. $\square$

The following lemma was used in the above proof. Let q be a quadrilateral with center w associated with four vertices v_i, $i = 1, \ldots, 4$ as shown in Fig. 1. Let $T_i := \langle w, v_i, v_{i+1}\rangle$ for $i = 1, \ldots, 4$, and set

$$\mathcal{H} = \bigcup_{i=1}^{4} \mathcal{H}^{T_i}(w).$$

Lemma 3.2. *Suppose $s \in \mathcal{S}_{3r}^r(\triangle_q)$, where $\triangle_q$ is the triangulation consisting of the four triangles $\{T_i\}_{i=1}^4$ making up q. Then the coefficients of s associated with domain points in $\mathcal{H}$ are uniquely determined by those associated with the domain points*

$$\tilde{H} := \mathcal{H}^{T_1} \cup \{\xi_{r,r,r}^{T_2}, \xi_{r,r,r}^{T_3}, \xi_{r,r,r}^{T_4}\} \setminus \{\xi_{r,2r,0}^{T_1}, \xi_{3r,0,0}^{T_1}, \xi_{r,0,2r}^{T_1}\}.$$

Proof: Suppose we are given the coefficients of s for $\xi \in \tilde{H}$. We claim that all other coefficients of s corresponding to points in $\mathcal{H}$ can be computed from the C^r smoothness conditions, see Lemma 6.2 in [14]. This can be done in the following order: compute the coefficients corresponding to the points $\xi_{r,2r,0}^{T_1}$, $\xi_{r,0,2r}^{T_1}$, and as many coefficients of s on T_2 and T_4 as possible; compute the coefficient corresponding to $\xi_{r,0,2r}^{T_2}$; compute the coefficient corresponding to $\xi_{3r,0,0}^{T_1}$; compute the remaining coefficients corresponding to domain points in T_2, T_3, T_4. Since w is a singular vertex (i.e., the intersection of two straight lines), no incompatibilities arise. $\square$

§4. A Minimal Determining set for $\mathcal{S}_{3r}^{r,\rho}(\diamondplus)$ for r even

Throughout this section we assume that r is even. Suppose $T = \langle w, v_1, v_2\rangle$ is a triangle in $\diamondplus$ and that $e := \langle v_1, v_2\rangle$. Choosing $\mu = (3r-2)/2$, we again make use of the sets $\mathcal{H}^T(w)$, $\mathcal{D}_\mu^T(v_1)$, and $\mathcal{D}_\mu^T(v_2)$, defined in (3.1). However, we now replace the sets $\mathcal{B}^T(w)$ and $\mathcal{E}^T(e)$ by

$$\begin{aligned} \widetilde{\mathcal{B}}^T(w) &:= \{\xi_{3r-i-j,i,j}^T : j = 0, \ldots, 2(i-\mu+2) \text{ and} \\ &\qquad\qquad i = \mu+2, \ldots, 2r-1\}; \\ \widetilde{\mathcal{E}}^T(e) &:= \{\xi \in \mathcal{D}_T : \operatorname{dist}(\xi, e) \leq r\} \backslash (\mathcal{D}_{\mu+1}^T(v_1) \cup \mathcal{D}_{\mu+1}^T(v_2)). \end{aligned} \tag{4.1}$$

As in the odd case, these are disjoint subsets of $\mathcal{D}_T$ except for $\mathcal{H}^T(w)$ and $\widetilde{\mathcal{E}}^T(e)$ which intersect in one point.

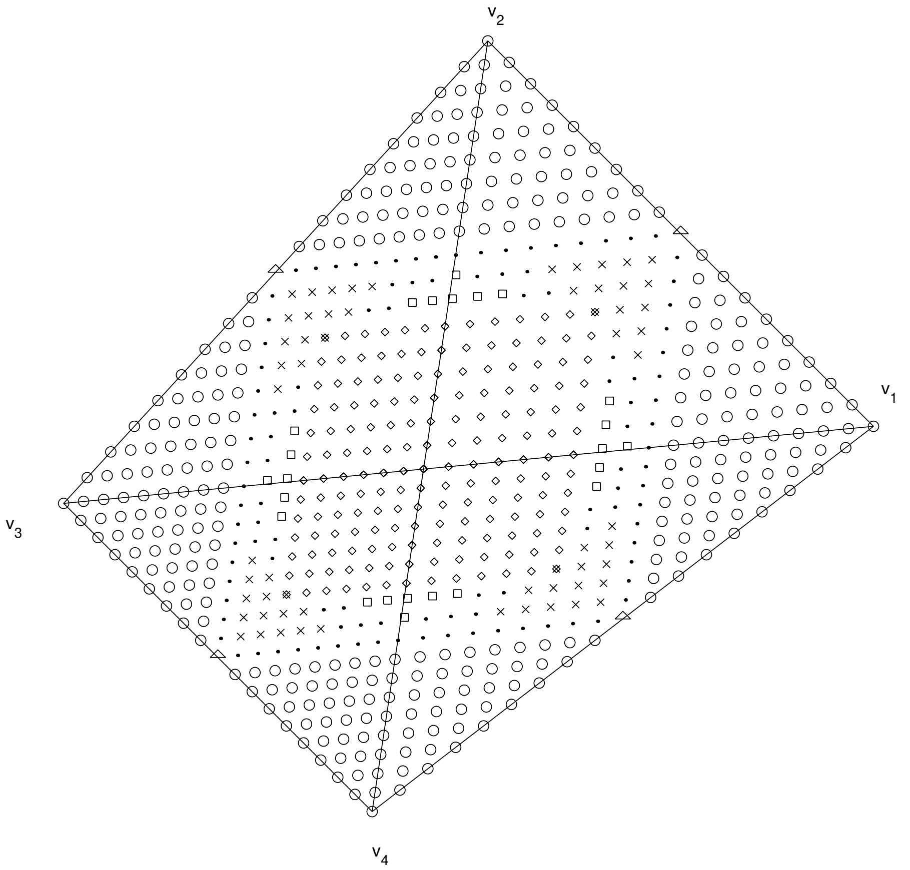

Fig. 2. The Domain Points (4.1) for $r = 6$ and $\mu = 8$.

For $r = 6$, Fig. 2 shows the sets $\mathcal{H}^T(w)$, $\widetilde{\mathcal{B}}^T(w)$, $\mathcal{D}^T_\mu(v_1)$, $\mathcal{D}^T_\mu(v_2)$, and $\widetilde{\mathcal{E}}^T(e)$ associated with the four triangles making up a single quadrilateral. The points in the sets $\mathcal{H}^T(w)$ are marked with diamonds, points in $\widetilde{\mathcal{B}}^T(w)$ with boxes, points in $\mathcal{D}^T_\mu(v_i)$ with circles, and points in $\widetilde{\mathcal{E}}^T(e)$ with crosses.

As in the odd case, in order to apply Theorem 2.1, we need to construct a minimal determining set Γ and an associated set of splines $\mathcal{B} := \{\phi_\xi\}_{\xi\in\Gamma}$ with the dual property (2.9).

Theorem 4.1. *Suppose r is even, and let Γ be the following subset of $\mathcal{D}_{\diamondplus}$:*

1) *for each vertex $v \in \diamondsuit$, pick a triangle $T \in \diamondplus$ with vertex at v and choose all points in $\mathcal{D}^T_\mu(v)$,*
2) *for each edge e not attached to a vertex in $V_\times$, include the set $\widetilde{\mathcal{E}}^T(e)$, where T is a triangle which shares the edge e. If e is a boundary edge, there is only one such triangle, while if it is an interior edge, we can work with either of the two triangles sharing it,*
3) *for each $w \in V_\times$ and for each $i = 1, \ldots, 4$, choose all of the points in $\widetilde{\mathcal{B}}^{T_i}(w)$, where $T_i = \langle w, v_i, v_{i+1}\rangle$ are the triangles with vertex at w,*

4) for each $w \in V_\times$, pick a triangle $T = \langle w, v, u \rangle \in \diamondplus$ with vertex at w and choose all of the points in the set $\mathcal{H}^T(w)$ except for domain points $\xi^T_{r,r,r}$, $\xi^T_{3r,0,0}$, $\xi^T_{r,2r,0}$, $\xi^T_{r,0,2r}$,

5) choose the center domain point on each edge of $\Diamond$,

6) for each boundary vertex $v \in \Diamond$ choose the domain points listed in Lemma 4.2 below,

7) for each interior vertex $v \in \Diamond$ where exactly three quadrilaterals meet, choose the domain points listed in Lemma 4.3 below,

8) for each interior vertex $v \in \Diamond$ where more than three quadrilaterals meet, choose the domain points listed in Lemma 4.4 below. Then Γ is a minimal determining set for $\mathcal{S}^{r,\rho}_{3r}(\diamondplus)$, and there exist a set of splines $\{\phi_\xi\}_{\xi\in\Gamma}$ which satisfy (2.9) and thus are a basis for $\mathcal{S}^{r,\rho}_{3r}(\diamondplus)$. These splines satisfy properties H1)–H3) of Theorem 2.1.

Proof: The proof is very similar to that of Theorem 3.1, and is based on the same lemmas mentioned there plus Lemmas 4.2, 4.3, and 4.4 below. Using these lemmas, it is easy to check that Γ is a determining set. Then we can show that it is a minimal determining set by constructing a dual basis satisfying (2.9) and hypotheses H1)–H3). Given $\xi \in \Gamma$, we construct ϕ_ξ as before by setting the coefficient corresponding to ξ to 1, and the coefficients corresponding to all other $\eta \in \Gamma$ to 0. As in Theorem 3.1, the ϕ_ξ have supports on a single quadrilateral, on two neighboring quadrilaterals, or on the collection of quadrilaterals which share a vertex $v \in V$. $\square$

In the remainder of this section we present three lemmas which are needed for the proof of Theorem 4.1. Given a spline s in $\mathcal{S}^r_d(\mathrm{star}(v))$, we call $\Gamma_v \subset \mathcal{D}_{\mu+1}(v)$ a *minimal determining set* for s on $\mathcal{D}_{\mu+1}(v)$ provided we can arbitrarily choose the coefficients of s for $\xi \in \Gamma_v$, and then uniquely solve for all other coefficients of s corresponding to ξ in $\mathcal{D}_{\mu+1}(v)$.

Given a vertex $v \in \Diamond$, we say that v has *index* n if there are n quadrilaterals attached to v. In this case we label the vertices on the boundary of $\mathrm{star}(v)$ as $v_1, w_1, v_2, w_2, \cdots, w_{n-1}, v_n, w_n, v_{n+1}$ in counterclockwise order, where we identify $v_{n+1} := v_1$ if v is an interior vertex. We denote the triangles of $\diamondplus$ which share the vertex v by $T_i = \langle v, v_i, w_i \rangle$ and $T'_i = \langle v, w_i, v_{i+1} \rangle$, $i = 1, \cdots, n$.

The first lemma is a rephrasing of Lemma 3.1 of [10].

Lemma 4.2. *Let r be even, and set $\mu = (3r-2)/2$. Suppose v is a boundary vertex of $\diamondplus$ of index n, and let Γ_v consist of the domain points*

1) $\xi^{T_1}_{ijk}$, $\quad i + j + k = d$ with $i \ge d - \mu - 1$,

2) $\xi^{T_i}_{d-\mu-1,j,\mu+1-j}$, for $j = 0, \ldots, \mu - r$ and $i = 2, \cdots, n$

3) $\xi^{T'_i}_{d-\mu-1,j,\mu+1-j}$, for $j = 0, \ldots, \mu - r$ and $i = 1, \cdots, n$.

Then Γ_v is a minimal determining set for any $s \in \mathcal{S}^{r,\mu}_d(\mathrm{star}(v))$ on $\mathcal{D}_{\mu+1}(v)$.

Our next two lemmas deal with interior vertices of $\diamondplus$.

Lemma 4.3. *Let r be even, and set $\mu = (3r-2)/2$. Suppose v is an interior vertex of $\diamondsuit$ with index $n = 3$. If $\angle w_1 v v_3 \geq 180°$, let Γ_v consist of the following set of domain points:*

1) $\xi^{T_1}_{ijk}$, $\quad i+j+k = d$ *with* $i \geq d-\mu$,

2) $\xi^{T_i}_{d-\mu-1,\mu+1,0}$, $\qquad i = 1, 2, 3$,

3) $\xi^{T_1}_{d-\mu-1,\mu+1-j,j}$, $\; j = 1, \cdots, \mu-1$.

If $\angle w_1 v v_3 < 180°$, replace T_1 by T_1' in 1), and replace 3) by

3′) $\xi^{T_1'}_{d-\mu-1,j,\mu+1-j}$, $\; j = 1, \cdots, \mu-1$.

Then Γ_v forms a minimal determining set for any $s \in \mathcal{S}^{r,\mu+1}_d(star(v))$ with $d \geq \mu+1$ on $\mathcal{D}_{\mu+1}(v)$.

Proof: We consider only the case where $\angle w_1 v v_3 \geq 180°$ as the other case is similar. On $\mathcal{D}_{\mu+1}(v)$ the smoothness conditions for $s \in \mathcal{S}^{r,\mu+1}_d(\mathrm{star}(v))$ are the same as those for $s \in \mathcal{S}^{r,\mu+1}_{\mu+1}(\mathrm{star}(v)) = \mathcal{P}_{\mu+1}$. This space has dimension $D := \binom{\mu+3}{2}$, and since Γ_v contains exactly D points, to show that Γ_v is a minimal determining set it suffices to show that it is a determining set. Suppose the coefficients of $s \in \mathcal{P}_{\mu+1}$ corresponding to domain points in items 1), 3), and 2)$(i=1)$ are set to zero. Then assuming s is written in Bernstein-Bézier form relative to the triangle $T_1 = \langle v, v_1, w_1 \rangle$, it follows that $s = a_1 B^{\mu+1}_{0,1,\mu} + a_2 B^{\mu+1}_{0,0,\mu+1}$. Now the coefficients corresponding to domain points in the remaining part of 2) are zero if and only if $s(v_2) = s(v_3) = 0$, i.e.,

$$\begin{pmatrix} b_2 b_3^{\mu} & b_3^{\mu+1} \\ \tilde{b}_2 \tilde{b}_3^{\mu} & \tilde{b}_3^{\mu+1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where (b_1, b_2, b_3) and $(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3)$ are the barycentric coordinates of v_2 and v_3 relative to the triangle $\langle v, v_1, w_1 \rangle$, respectively. By the geometry, $b_2 < 0$, $b_3 > 1$, $\tilde{b}_2 \leq 0$, and $\tilde{b}_3 < 0$, which implies that the determinant of the above linear system satisfies

$$|\det| = |b_3 \tilde{b}_3|^{\mu} (b_2 \tilde{b}_3 - \tilde{b}_2 b_3) \geq |b_3 \tilde{b}_3|^{\mu} b_2 \tilde{b}_3 \neq 0,$$

and we conclude that $a_1 = a_2 = 0$. This completes the proof. $\square$

Lemma 4.4. *Let r be even and set $\mu = (3r-2)/2$ and $\ell = r/2$. Suppose that v is an interior vertex of $\diamondsuit$ of index $n \geq 4$ with $\angle v_1 v v_3 \leq 180°$. Let Γ_v consist of the domain points*

1) $\xi^{T_n'}_{ijk}$, $\qquad i+j+k = d$ *with* $i \geq d-\mu-1$,

2) $\xi^{T_i}_{d-\mu-1,\mu+1-j,j}$, $\qquad j = 0, \cdots, \ell-1$ *and* $i = 3, \ldots, n$,

3) $\xi^{T_i'}_{d-\mu-1,\mu+1-j,j}$, $\qquad j = 0, \cdots, \ell-1$ *and* $i = 3, \cdots, n-1$,

4) $\xi^{T_2}_{d-\mu-1,\mu+1-j,j}$, $\qquad j = 0, \cdots, r-2$.

Then Γ_v forms a minimal determining set for any $s \in \mathcal{S}_d^{r,\mu}(star(v))$ with $d \geq \mu+1$, on $\mathcal{D}_{\mu+1}(v)$.

Proof: Since on $\mathcal{D}_{\mu+1}(v)$ the smoothness conditions for $s \in \mathcal{S}_d^{r,\mu}(\mathrm{star}(v))$ are the same as those for $s \in \mathcal{S}_{\mu+1}^{r,\mu}(\mathrm{star}(v))$, it suffices to consider s in the latter space. By Lemma 3.2 of [10], this space has dimension $D := \binom{\mu+2}{2} + 2n[\binom{\mu+2-r}{2} - \binom{\mu+1-r}{2}] + \sigma$, with $\sigma = \sum_{j=\mu-r+1}^{\mu-r+2}(r+j+1-je)_+$, where e is the number of edges in $\diamondsuit$ with different slopes attached to v. Since $e \geq 4$ and $\mu = 3\ell-1$, it is easy to check that $\sigma = 0$ and thus $D = \binom{\mu+2}{2} + 2\ell n$. Since Γ_v contains exactly D points, to show that Γ_v is a minimal determining set it suffices to show that it is a determining set.

Suppose the coefficients of $s \in \mathcal{S}_{\mu+1}^{r,\mu}(\mathrm{star}(v))$ corresponding to domain points in Γ_v are set to zero. By the C^r smoothness conditions, the only possible nonzero coefficients of s are those corresponding to the domain points

$$
\begin{aligned}
&\xi^{T_2'}_{d-\mu-1,\mu+1-j,j}, \qquad && j = \ell-1,\dots,0 \\
&\xi^{T_2}_{d-\mu-1,j,\mu+1-j}, \qquad && j = 1,\dots,\ell+1.
\end{aligned}
$$

and

$$
\begin{aligned}
&\xi^{T_1}_{d-\mu-1,j,\mu+1-j}, \qquad && j = \ell-1,\dots,0 \\
&\xi^{T_1'}_{d-\mu-1,\mu+1-j,j}, \qquad && j = 1,\dots,\ell+1,
\end{aligned}
$$

We number the coefficients corresponding to the first set as $a_1,\dots,a_{r+1}$ and those corresponding to the second set as $b_1,\dots,b_{r+1}$.

Suppose now that $v_2 = \beta_1 v_3 + \gamma_1 w_2$, $v_2 = \beta_2 v_1 + \gamma_2 w_1$, and $w_2 = \alpha_3 v + \beta_3 w_1 + \gamma_3 v_2$. Then by the smoothness conditions, the coefficients a_i and b_i satisfy

$$
\begin{aligned}
a_{\ell+j} &= \sum_{k=0}^{j} \binom{j}{k} \beta_1^{j-k}\gamma_1^k a_{\ell-j+k}, \quad && j = 1,\dots,\ell+1, \\
0 &= \sum_{k=0}^{\ell} \binom{j}{k} \beta_1^{j-k}\gamma_1^k a_{\ell-j+k}, \quad && j = \ell+2,\dots,2\ell,
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
b_{\ell+j} &= \sum_{k=0}^{j} \binom{j}{k} \beta_2^{j-k}\gamma_2^k a_{\ell-j+k}, \quad && j = 1,\dots,\ell+1, \\
0 &= \sum_{k=0}^{\ell} \binom{j}{k} \beta_2^{j-k}\gamma_2^k a_{\ell-j+k}, \quad && j = \ell+2,\dots,2\ell,
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
a_{2\ell} &= \beta_3^{2\ell} b_{2\ell} + 2\ell\beta_3^{2\ell-1}\gamma_3 b_{2\ell+1} \\
a_{2\ell+1} &= \beta_3^{2\ell-1} b_{2\ell+1}
\end{aligned}
\tag{4.4}
$$

where for convenience we have set $a_0 = b_0 = a_{-1} = b_{-1} = 0$.

Setting $x_k = \beta_1^{\ell-k}\gamma_1^k a_k$, we can write the last $\ell+1$ equations of (4.2) as

$$a_{2\ell} = \sum_{k=1}^{\ell} \binom{\ell}{k} x_k \tag{4.5}$$

$$a_{2\ell+1} = \gamma_1 \sum_{k=1}^{\ell} \binom{\ell+1}{k+1} x_k \tag{4.6}$$

$$0 = \sum_{k=1}^{\ell} \binom{\ell+\nu+1}{k+\nu+1} x_k, \quad \nu = 1, \cdots, \ell-1. \tag{4.7}$$

Now using the fact that $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, combining equations in (4.7), we can convert those equations to the triangular system

$$0 = \sum_{k=1}^{\ell-\nu+1} \binom{\ell+2}{k+\nu+1} x_k, \qquad \nu = 1, \cdots, \ell-1.$$

Solving these equations for $x_2, \ldots, x_\ell$ in terms of x_1 gives

$$x_k = (-1)^{k+1} \binom{\ell+k}{\ell+1} x_1, \qquad k = 2, \ldots, \ell. \tag{4.8}$$

Substituting these formulae into (4.5) and (4.6), we see that

$$a_{2\ell} = \sum_{k=1}^{\ell} \binom{\ell}{k} (-1)^{k+1} \binom{\ell+k}{\ell+1} x_1 = (-1)^{\ell+1} \ell x_1$$

and

$$a_{2\ell+1} = \gamma_1 \sum_{k=1}^{\ell} \binom{\ell+1}{k+1} (-1)^{k+1} \binom{\ell+k}{\ell+1} x_1 = (-1)^{\ell+1} \gamma_1 x_1.$$

In terms of β_1, γ_1, and a_1,

$$a_{2\ell} = (-1)^{\ell+1} \ell \beta_1^{\ell-1} \gamma_1 a_1, \qquad a_{2\ell+1} = (-1)^{\ell+1} \beta_1^{\ell-1} \gamma_1^2 a_1.$$

Similarly, we get

$$b_{2\ell} = (-1)^{\ell+1} \ell \beta_2^{\ell-1} \gamma_2 b_1, \qquad b_{2\ell+1} = (-1)^{\ell+1} \beta_2^{\ell-1} \gamma_2^2 b_1.$$

Inserting these formulae in (4.4), we get the linear system

$$\begin{aligned} \beta_1^{\ell-1}\gamma_1 a_1 &= \beta_3^{2\ell}\beta_2^{\ell-1}\gamma_2 b_1 + 2\beta_3^{2\ell-1}\gamma_3\beta_2^{\ell-1}\gamma_2^2 b_1 \\ \beta_1^{\ell-1}\gamma_1^2 a_1 &= \beta_3^{2\ell-1}\beta_2^{\ell-1}\gamma_2^2 b_1. \end{aligned}$$

The determinant of this linear system is

$$
\begin{aligned}
D &= -\beta_1^{\ell-1}\beta_2^{\ell-1}\beta_3^{2\ell-1}\gamma_1\gamma_2 \det\begin{bmatrix}\gamma_1 & \gamma_2\\ 1 & \beta_3+2\gamma_2\gamma_3\end{bmatrix}\\
&= -\beta_1^{\ell-1}\beta_2^{\ell-1}\beta_3^{2\ell-1}\gamma_1\gamma_2(2\gamma_1\gamma_2\gamma_3+\beta_3\gamma_1-\gamma_2).
\end{aligned}
$$

It follows from the geometry and the hypothesis $\angle v_1vv_3 \le 180°$ that $\beta_i < 0$ and $\gamma_i > 0$. Moreover, the fact that

$$
\begin{aligned}
\beta_1 v_3 &= v_2 - \gamma_1 w_2\\
&= v_2 - \gamma_1(\alpha_3 v + \beta_3 w_1 + \gamma_3 v_2)\\
&= (1-\gamma_1\gamma_3)(\beta_2 v_1 + \gamma_2 w_1) - \gamma_1\alpha_3 v - \gamma_1\beta_3 w_1\\
&= -\gamma_1\alpha_3 v + \beta_2(1-\gamma_1\gamma_3)v_1 + (\gamma_2 - \gamma_1\beta_3 - \gamma_1\gamma_2\gamma_3)w_1
\end{aligned}
$$

implies

$$\gamma_2 - \gamma_1\beta_3 - \gamma_1\gamma_2\gamma_3 < 0.$$

This shows that

$$D > \beta_1^{\ell-1}\beta_2^{\ell-1}\beta_3^{2\ell-1}\gamma_1\gamma_2\gamma_1\gamma_2\gamma_3 > 0,$$

and it follows that $a_1 = b_1 = 0$. Now formulae (4.5) and (4.8) imply that $a_k = b_k = 0$ for all $k = 1, \cdots, 2\ell+1$, and the proof is complete. □

It should be noted that the above proof cannot be used to establish Lemma 4.4 for $n = 3$ since we cannot insure the hypothesis $\angle v_1vv_3 \le 180°$, and thus we cannot conclude that the determinant D in the proof is bounded away from zero. We should also point out that although all three of the above lemmas deal with solving homogeneous equations (in order to show that certain sets are determining sets), in constructing the basis splines ϕ_ξ in the proof of Theorem 4.1, we have to solve nonhomogeneous systems with the same matrices. However, in all three lemmas, the systems are not only nonsingular, but the determinants are in fact bounded away from zero by a constant depending only on r and the smallest angle $\theta_{\diamondplus}$ in the quadrangulation $\diamondplus$ (cf. the arguments in [14]).

§5. Proof of Theorem 1.1

We are now in a position to apply Theorem 2.1 to establish our main theorem using a quasi-interpolant Q_m of the form (2.1). Let $\{\phi_\xi\}_{\xi\in\Gamma}$ be the locally supported basis functions for $\mathcal{S}_{3r}^{r,\rho}(\diamondplus)$ constructed in the previous sections for r odd and even, respectively. We have already shown that these basis functions satisfy hypotheseses H1)–H3) of Theorem 2.1.

To define Q_m, we now introduce corresponding linear functionals. Given $\xi \in \Gamma$, let T_ξ be a triangle in which ξ lies. Then for any function $f \in L_1(\Omega)$, we define

$$\lambda_{\xi,m} f := \gamma_\xi(F_{m,B_{T_\xi}} f),$$

where $F_{m,B_{T_\xi}}f$ is the averaged Taylor polynomial associated with f (cf. [14]) and the largest disk B_{T_ξ} contained in T_ξ. Here γ_ξ is the functional defined on $S_d^0(\oplus)$ such that for any spline $s \in S_d^0(\oplus)$, $\gamma_\xi s$ is the Bézier coefficient of s associated with the domain point ξ.

Clearly, $\lambda_{\xi,m}$ is a linear functional, and the value of $\lambda_{\xi,m}f$ depends only on values of f on the triangle T_ξ. It was shown in [14] that there exists a constant K_3 depending only on r and $\theta_{\oplus}$ such that

$$|\lambda_{\xi,m}f| \le K_3 \|f\|_{p,T_\xi},$$

i.e., hypothesis H4) of Theorem 2.1 is satisfied.

To check H5), we have to show that Q_m reproduces polynomials of degree m. Suppose T is a triangle in $\oplus$, and suppose $f \in \mathcal{P}_m$. Then for each $\xi \in \Gamma$, by Lemma 4.4 in [14], $F_{m,B_{T_\xi}}f = f$ on T_ξ. But then the $\lambda_{\xi,m}f$ are just the Bézier coefficients of f for all $\xi \in \Gamma$. Since Γ is a determining set, we conclude that all of the coefficients of $Q_m f$ agree with those of f, and hence $Q_m f = f$. When $m = 3r$, Q_{3r} not only reproduces $\mathcal{P}_{3r}$, but also all splines $s \in \mathcal{S}_{3r}^{r,\rho}(\oplus)$.

We have now verified that Q satisfies all of the hypotheses of Theorem 2.1, and our main result Theorem 1.1 follows immediately.

§6. Quadrangulations

For the sake of completeness, we begin with a precise definition of a quadrangulation. A collection $\diamondsuit := \{q_i\}$ of quadrilaterals is said to be a *quadrangulation* of a connected polygonal domain Ω in $\mathbb{R}^2$ provided that

1) $\Omega = \bigcup q_i$,
2) the intersection of any two quadrilaterals is either empty, a single point, or a common edge,
3) for any two quadrilaterals q_1, q_n, there is a sequence of quadrilaterals $q_1, \dots, q_n$ in $\diamondsuit$ such that each pair q_i, q_{i+1} share exactly one edge with each other.

We call $\diamondsuit$ *convex* if all quadrilaterals are convex, and we call it *nondegenerate* if none of the quadrilaterals is a triangle.

Given any collection $V := \{v_i\}_{i=1}^N$ and a polygonal domain Ω whose boundary vertices are $v_{i_1}, \dots, v_{i_n}$ and which contains all of the points of V, there are many triangulations $\triangle$ of Ω such that the vertices of all of the triangles lie in the set V. The situation is somewhat different for quadrangulations. Indeed, it is easy to see that Ω admits a quadrangulation if and only if the number of vertices n on the boundary of Ω is even. Some algorithms for constructing quadrangulations associated with a given set of vertices have been discussed in [5], although they are not guaranteed to produce nondegenerate convex quadrangulations, even if Ω is convex. We now present two simple methods for creating nondegenerate convex quadrangulations based on subdividing a given triangulation.

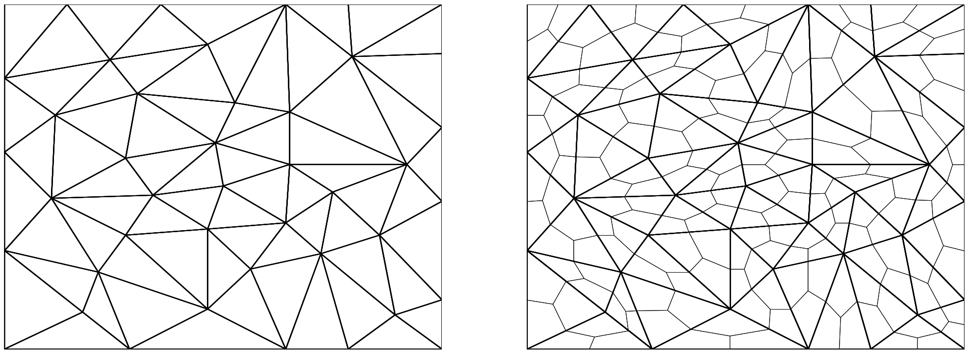

Fig. 3. Use of Method 1 to construct a quadrangulation from a triangulation.

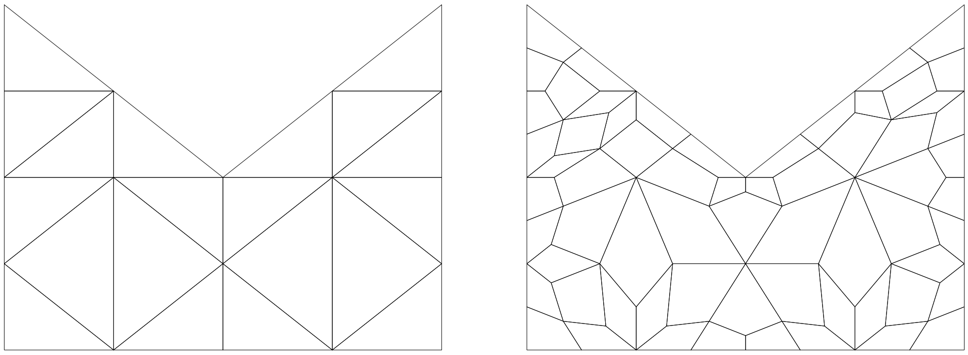

Fig. 4. Use of Method 2 to construct a quadrangulation from a triangulation.

Method 1. *Suppose $\triangle$ is a triangulation of a polygon Ω. Then subdivide each triangle $T \in \triangle$ by connecting its centroid to the midpoints of its edges with straight lines.*

Clearly this method produces a nondegenerate quadrangulation $\diamondsuit$ of Ω consisting of convex quadrilaterals. A typical example is shown in Fig. 3. Each quadrilateral has one vertex at a centroid of a triangle $T \in \triangle$, one vertex at a vertex of T, and two vertices at midpoints of edges of T.

Method 2. *Suppose $\triangle$ is a triangulation of a polygon Ω. Then*

1) *Subdivide each triangle $T \in \triangle$ by connecting its incenter to its three vertices with straight lines.*
2) *Remove all interior edges of $\triangle$.*
3) *Modify the remaining boundary triangles as follows:*
   a) *Suppose v is a boundary vertex of $\triangle$ with at least two triangles attached. For each interior edge $e = \langle v, v' \rangle$ of $\triangle$, let T_1 and T_2 be the two triangles sharing e, and let w_1 and w_2 be their incenters. Remove the line segments $\langle v, w_1 \rangle$ and $\langle v, w_2 \rangle$ and choose $0 < \alpha < 1/2$ so that if we connect both w_1 and w_2 to $v_\alpha := v + \alpha(v' - v)$, the quadrilateral with vertices at*

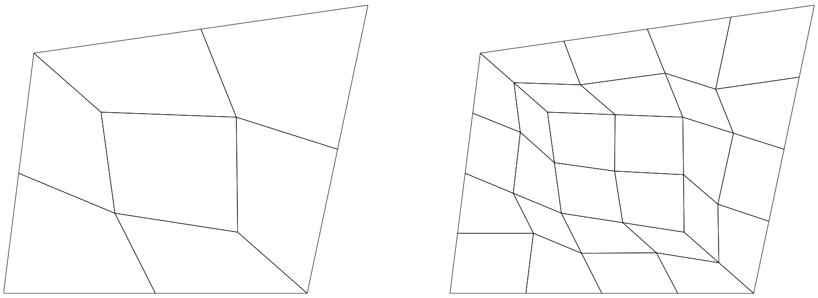

Fig. 5. A quadrangulation and its dyadic refinement.

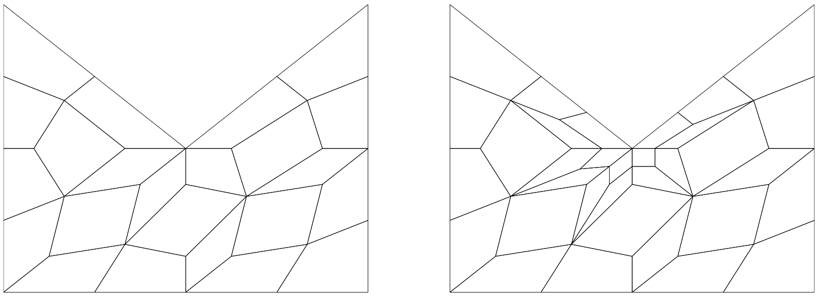

Fig. 6. A quadrangulation and its local refinement at the reentrant corner.

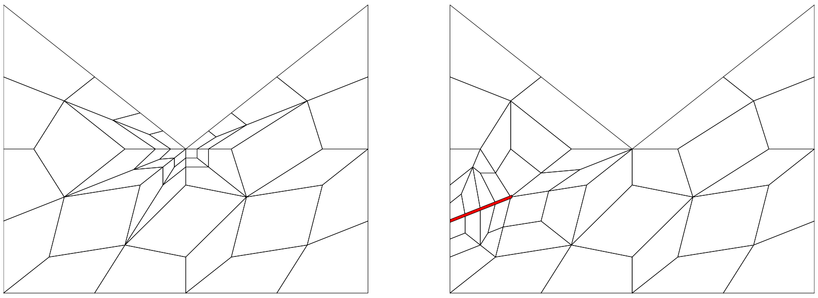

Fig. 7. Two steps of refinement at the reentrant corner and along a crack.

v_α, w_1, v', w_2 is convex. Next connect v_α to v and connect the incenters of the two boundary triangles with vertex at v to the midpoints of their boundary edges.

b) *Suppose v is a boundary vertex of $\triangle$ with only one triangle T attached. Then remove the line segment $\langle v, w\rangle$, where w is the incenter of T.*

Fig. 4 shows an example of the use of Method 2 to convert a given triangulation to a quadrangulation. If $\diamondsuit$ is a quadrangulation created in this way, then the associated triangulation $\diamondsuit\!\!\!\!+$ obtained by inserting the diagonals into each quadrilateral turns out to be the standard 12-triangle Powell-Sabin subtriangulation of $\triangle$ based on incenters and midpoints of edges.

It is clear that a nondegenerate convex quadrangulation can be *dyadically refined* by connecting the midpoints of the four edges of each quadrilateral to the intersection of its two diagonals. Fig. 5 shows a quadrangulation and the result of one level of refinement. If $\diamondplus'$ is a dyadic refinement of $\diamondplus$, then the spline space $\mathcal{S}_d^r(\diamondplus)$ is contained in the spline space $\mathcal{S}_d^r(\diamondplus')$. This leads to hierarchical multilevel bases for the spline spaces associated with refined triangulated quadrangulations, see e.g. [15].

Finally, we note that a nondegenerate convex quadrangulation can also be refined locally, cf. Figs. 6 and 7. Local refinement is absolutely essential in solving boundary value problems since in most applications, solutions of boundary value problems have singularities arising from reentrant corners and cracks in the domain Ω. Local refinement allows the effective approximation of such solutions.

§7. Remarks

Remark 7.1. Spline spaces defined on triangulated quadrangulations have been studied in several papers. In particular, finite elements in $S_3^1(\diamondplus)$ were constructed in [9] and [16]. The approximation properties of $S_3^1(\diamondplus)$ in the usual L_2 norm was studied in [6], while the approximation properties in L_∞ and its application in scattered data interpolation were considered in [12]. In [13], we studied the approximation properties of $S_6^2(\diamondplus)$ in the L_∞ norm. Finite elements spanning a certain subspace of $S_{3r}^r(\diamondplus)$ for odd integer r and $S_{3r+1}^r(\diamondplus)$ for even integer r were also constructed recently in [11], where L_∞ approximation results can also be found.

Remark 7.2. It is interesting to compare the dimensions of spline spaces based on triangulated quadrangulations and those based on standard triangulations. Suppose we are given a set of points in $\mathbb{R}^2$ which admit a nondegenerate convex quadrangulation $\Diamond$ with no holes. Let V_I and V_B be the number of interior and boundary vertices of the quadrangulation $\Diamond$. Suppose $\triangle$ is a triangulation based on the same interior and boundary vertices. We denote the corresponding Clough-Tocher and Powell-Sabin refinements of $\triangle$ by $\triangle_{CT}$ and $\triangle_{PS}$, respectively. Assuming $\triangle$ contains N triangles, then $\triangle_{CT}$ has $3N$ triangles, $\diamondplus$ has $2N$ triangles, and $\triangle_{PS}$ has $6N$ triangles. It is known that

$$\begin{aligned}
\dim\,(S_5^{1,2}(\triangle)) &= 6(V_I+V_B)+3V_I+2V_B-3\\
\dim\,(S_3^1(\triangle_{CT})) &= 3(V_I+V_B)+3V_I+2V_B-3\\
\dim\,(S_3^1(\diamondplus)) &= 3(V_I+V_B)+2V_I+3V_B/2-2\\
\dim\,(S_2^1(\triangle_{PS})) &= 3(V_I+V_B).
\end{aligned}$$

Based on these figures, we believe that $S_3^1(\diamondplus)$ is the best choice for applications since $S_5^{1,2}(\triangle)$ and $S_3^1(\triangle_{CT})$ both have significantly larger dimensions, while $S_2^1(\triangle_{PS})$ has many more triangles (and a lower approximation order). Similarly, we believe $S_6^2(\diamondplus)$ is a better choice for applications than $S_8^2(\triangle)$, $S_6^2(\triangle_{CT})$, or $S_5^2(\triangle_{PS})$ – see the dimension formulae and remarks in [12,13].

Remark 7.3. By changing the way in which the linear functionals $\lambda_{\xi,m}$ are defined, we can create an alternative quasi-interpolant $\tilde{Q}_m$ defined on $C(\Omega)$ which interpolates at the vertices of $\triangle$. We have the following variant of Theorem 1.1.

Theorem 7.4. *Fix integers $r \geq 1$ and m, where $2/p - 1 \leq m \leq 3r$ if $p > 1$ and $2 \leq m \leq 3r$ if $p = 1$. Then there exists a linear operator $\tilde{Q}_m$ mapping $C(\Omega)$ into $S^r_{3r}(\triangle)$ such that $\tilde{Q}_m f(v) = f(v)$ for all vertices $v \in \triangle$. Moreover, $\tilde{Q}_m$ has full approximation power in the sense that for any f in the Sobolev space $W^{m+1}_p(\Omega)$ with $1 \leq p \leq \infty$,*

$$\|D^\alpha_x D^\beta_y (f - \tilde{Q}_m f)\|_{p,\Omega} \leq C\ |\triangle|^{m+1-\alpha-\beta}\ |f|_{m+1,p,\Omega}, \tag{7.1}$$

for all $0 \leq \alpha + \beta \leq m$. If Ω is convex, then the constant C depends only on r and the smallest angle $\theta_\triangle$ in $\triangle$. If Ω is nonconvex, C also depends on the minimum exterior angle between any two boundary edges of Ω.

Proof: We use the functions ϕ_ξ constructed in the previous two sections to construct $\tilde{Q}_m$ just as in (2.1), but replace $\lambda_{i,m}$ by linear functionals $\tilde{\lambda}_{i,m}$ which are based on an interpolating polynomial instead of the averaged Taylor polynomial. Given a triangle T in $\triangle$, let $I_{T,m}$ be the interpolation operator mapping a function f defined on T to the unique polynomial of degree m which interpolates f at the domain points ξ^T_{ijk}, all $i + j + k = m$. For each $\xi \in \Gamma$, let T_ξ be the triangle containing ξ. Then for $f \in C(\Omega)$, we define

$$\tilde{\lambda}_{\xi,m} f := \gamma_\xi (I_{T_\xi,m} f).$$

Here, γ_ξ is the functional on $S^0_d(\triangle)$ defined in Sect. 2 which picks off the Bézier coefficient associated with the domain point ξ. With these linear functionals, it is not hard to see that $\tilde{Q}_m$ reproduces polynomials of degree m. Moreover, using the fact that the minimal determining sets Γ constructed above always contain all of the domain points located at vertices of $\triangle$, it is also clear that $\tilde{Q}_m$ interpolates at those vertices.

To establish the approximation power, we observe that $|\lambda_{\xi,m} f| \leq C\|f\|_{\infty,T_\xi}$, where C is a constant only dependent on r. Now the Sobolev embedding theorem asserts that if $1 \leq p < \infty$,

$$\|f\|_{\infty,T} \leq \frac{K_4}{{A_T}^{1/p}} \left(\|f\|_{p,T} + |T|^{m+1} |f|_{m+1,p,T} \right).$$

for some constant K_4 depending only on r. This shows that for $m > 2/p - 1$ if $p > 1$ and for $m \geq 2$ if $p = 1$,

$$|\tilde{\lambda}_{\xi,m} f| \leq \frac{K_3}{{A_{T_\xi}}^{1/p}} \left(\|f\|_{p,T_\xi} + |\triangle|^{m+1} |f|_{m+1,p,T_\xi} \right).$$

Now using the fact that the interpolation operator $I_{T,m}$ has full (local) approximation power, the rest of the proof proceeds in exactly the same way as before. $\square$

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