SIMPLICES IN THIN SUBSETS OF EUCLIDEAN SPACES

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Abstract. Let Δ be a non-degenerate simplex on k vertices. We prove that there exits a threshold $s_k < k$ such that any set $A \subseteq \mathbb{R}^k$ of Hausdorff dimension $dim\, A \geq s_k$ necessarily contains a similar copy of the simplex Δ.

1. Introduction.

A classical problem of geometric Ramsey theory is to show that a sufficiently large sets contain a given geometric conguration. The underlying settings can be the Euclidean space, the integer lattice of vector spaces over finite fields. By a geometric configuration we understand the family of finite point sets obtained from a given finite set $F \subseteq \mathbb{R}^k$ via translations, rotations and dilations.
If largeness means positive Lebesgue density, then it is known that large sets in $\mathbb{R}^k$ contain a translated and rotated copy of all sufficiently large dilates of any non-degenerate simplex Δ with k vertices [2]. However if by largeness one understands only large Hausdorff dimension $s < k$, then this question is less understood, in fact the only affirmative result in this direction is given by Iosevich-Liu [5].

In the other direction, a construction due to Keleti [6] shows that there exists set $A \subseteq \mathbb{R}$ of full Hausdorff dimension which do not contain any non-trivial 3-term arithmetic progression. In two dimensions an example due to Falconer [3] and Maga [8] shows that there exists set $A \subseteq \mathbb{R}^2$ of Hausdorff dimension 2, which do not contain the vertices of an equilateral triangle, or more generally a non-trivial similar copy of a given non-degenerate triangle. It seems plausible that examples of such sets exist in all dimensions [4].

The aim of this short note is to show that however measurable sets $A \subseteq \mathbb{R}^k$ of sufficiently large Hausdorff dimension $s < k$ contain a similar copy of any given non-degenerate k-simplex whose eccentricity is controlled. Our arguments make use of and show some similarity to those of Lyall-Magyar [7] and we extend out results to bounded degree distance graphs. For the special case of a path (or chain) similar but somewhat stronger results were obtained in [1].

2. Main results.

Let $V = \{v_1, \ldots, v_k\} \subseteq \mathbb{R}^k$ be a non-degenerate k-simplex, a set of k vertices which are in *general position* spanning a $k-1$-dimensional affine subspace. For $1 \leq j \leq k$ let $r_j(V)$ be the distance of one a vertex v_j to the affine subspace spanned by the remaining vertices $v_i,\ i \neq j$ and define $r(V) := \min_{1 \leq j \leq k} r_j(V)$. Let $d(V)$ denote the diameter of the simplex, which is also the maximum distance between two vertices. Then the quantity $\delta(V) := r(V)/d(V)$, which is positive if and only if V is non-degenerate, measures how close the simplex V is to being degenerate. We say that a simplex V' is *similar* to V, if $V' = x + \lambda \cdot U(V)$ for some $x \in \mathbb{R}^k$, $\lambda > 0$ and $U \in SO(k)$, that is if V' is obtained from V by a translation, dilation and rotation.

Theorem 1. *Let $k \in \mathbb{N}$, $\delta > 0$. There exists $s_0 = s_0(k, \delta) < k$ such that if E is a compact subset of $\mathbb{R}^k$ of Hausdorff dimension $dim\, E \geq s_0$, then E contains vertices of a simplex V' similar to V, for any non-degenerate k-simplex V with $\delta(V) \geq \delta$.*

Note that the dimension condition is sharp for $k = 2$ as a construction due to Maga [8] shows the existence of a set $E \subseteq \mathbb{R}^2$ with $dim(E) = 2$ which does not contain any equilateral triangle or more generally a similar copy of any given triangle.

A distance graph is a connected finite graph embedded in Euclidean space, with a set of vertices $V = \{v_0, v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$ and a set of edges $E \subseteq \{(i, j);\ 0 \leq i < j \leq n\}$. We say that a graph $\Gamma = (V, E)$ has degree at most k if $|V_j| \leq k$ for all $1 \leq j \leq n$, where $V_j = |\{v_i :\ (i, j) \in E\}|$. The graph Γ is called *proper* if the sets $V_j \cup \{v_j\}$ are in general position. Let $r(\Gamma)$ be the minimum of the distances from the vertices v_j to the corresponding affine subspace spanned by the sets V_j and note that $r(\Gamma) > 0$ if Γ is proper. Let $d(\Gamma)$ denote length of the longest edge of Γ and let $\delta(\Gamma) := r(\Gamma)/d(\Gamma)$.

We say that a distance graph $\Gamma' = (V', E)$ is *isometric* to Γ, and write $\Gamma' \simeq \Gamma$ if there is a one-one and onto mapping $\phi : V \to V'$ so that $|\phi(v_i) - \phi(v_j)| = |v_i - v_j|$ for all $(i, j) \in E$. One may picture Γ' obtained from Γ by a translation followed by rotating the edges around the vertices, if possible. By $\lambda \cdot \Gamma$ we mean the dilate of the distance graph Γ by a factor $\lambda > 0$ and we say that Γ' is *similar* to Γ if Γ' is isometric to $\lambda \cdot \Gamma$.

Theorem 2. *Let $\delta > 0$, $n \geq 1$, $1 \leq k < d$ and let E be a compact subset of $\mathbb{R}^k$ of Hausdorff dimension $s < d$. There exists $s_0 = s_0(n, d, \delta) < d$ such if $s \geq s_0$ then E contains a distance graph Γ' similar to Γ, for any proper distance graph $\Gamma = (V, E)$ of degree at most k, with $V \subseteq \mathbb{R}^d$, $|V| = n$ and $\delta(\Gamma) \geq \delta$.*

Note that Theorem 2 implies Theorem 1 as a non-degenerate simplex is a proper distance graph of degree at most $k-1$.

3. Proof of Theorem 1.

Let $E \subseteq B(0,1)$ be a compact subset of the unit ball $B(0,1)$ in $\mathbb{R}^k$ of Hausdorff dimension $s < k$. It is well-known that there is a probability measure μ supported on E such that $\mu(B(x,r)) \le C_\mu r^s$ for all balls $B(x,r)$. The following observation shows that we may take $C_\mu = 4$ for our purposes. [1]

Lemma 1. *There exists a set $E' \subseteq B(0,1)$ of the form $E' = \rho^{-1}(F-u)$ for some $\rho > 0$, $u \in \mathbb{R}^k$ and $F \subseteq E$, and a probability measure μ' supported on E' which satisfies*

$$\mu'(B(x,r) \le 4r^s, \quad \textit{for all} \quad x \in \mathbb{R}^k,\ r > 0. \tag{3.1}$$

Proof. Let $K := \inf(S)$, where

$$S := \{C \in \mathbb{R} :\ \mu(B(x,r)) \le Cr^s, \ \ \forall\ B(x,r)\}.$$

By Frostman's lemma [9] we have that $S \ne \emptyset$, $K > 0$, moreover

$$\mu(B(x,r)) \le 2K\, r^s,$$

for all balls $B(x,r)$. There exists a ball $Q = B(v,\rho)$ or radius ρ such that $\mu(Q) \ge \frac{1}{2}K\rho^s$. We translate E so Q is centered at the origin, set $F = E \cap Q$ and denote by μ_F the induced probability measure on F

$$\mu_F(A) = \frac{\mu(A\cap F)}{\mu(F)}.$$

Note that for all balls $B = B(x,r)$,

$$\mu_F(B) \le \frac{2K\, r^s}{\frac{1}{2}K\rho^s} = 4\left(\frac{r}{\rho}\right)^s.$$

Finally we dene the probability measure μ', by $\mu'(A) := \mu_F(\rho A)$. It is supported on $E' = \rho^{-1}F \subseteq B(0,1)$ and satisfies

$$\mu'(B(x,r)) = \mu_F(B(\rho x, \rho r)) \le 4r^s.$$

□

Clearly E contains a similar copy of V if the same holds for E', thus one can pass from E to E' and hence assuming that (3.1) holds, in proving our main results. Given $\varepsilon > 0$ let $\psi_\varepsilon(x) = \varepsilon^{-k}\psi(x/\varepsilon) \ge 0$, where $\psi \ge 0$ is a Schwarz function whose Fourier transform, $\widehat{\psi}$, is a compactly supported smooth function, satisfying $\widehat{\psi}(0) = 1$ and $0 \le \widehat{\psi} \le 1$.

We define $\mu_\varepsilon := \mu * \psi_\varepsilon$. Note that μ_ε is a continuous function satisfying $\|\mu_\varepsilon\|_\infty \le C\varepsilon^{s-k}$ with an absolute constant $C = C_\psi > 0$, by Lemma 1.

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[1] We'd like to thank Giorgis Petridis for bringing this observation to our attention.

Let $V = \{v_1, \dots, v_k\}$ be a given a non-degenerate simplex and note that in proving Theorem 1 we may assume that $d(V) = 1$ hence $\delta(V) = r(V)$. A simplex $V' = \{x_1, \dots, x_k\}$ is isometric to V if for every $1 \leq j \leq k$ one has that $x_j \in S_{x_1,\dots,x_{j-1}}$, where

$$S_{x_1,\dots,x_{j-1}} = \{x \in \mathbb{R}^k : \ |x - x_i| = |v_j - v_i|, \quad 1 \leq i < j\}$$

is a sphere of dimension $k + 1 - j$, of radius $r_j = r_j(V) \geq r(V) > 0$. Let $\sigma_{x_1,\dots,x_{j-1}}$ denote its normalized surface area measure.

Given $0 < \lambda, \varepsilon \leq 1$ define the multi-linear expression,

$$T_{\lambda V}(\mu_\varepsilon) := \tag{3.2}$$
$$\int \mu_\varepsilon(x)\mu_\varepsilon(x - \lambda x_1) \cdots \mu_\varepsilon(x - \lambda x_k)\, d\sigma(x_1)\, d\sigma_{x_1}(x_2) \dots d\sigma_{x_1,\dots,x_{k-1}}(x_k)\, dx.$$

We have the following crucial upper bound

Lemma 2. *There exists a constant $C_k > 0$, depending only on k, such that*

$$|T_{\lambda V}(\mu_{2\varepsilon}) - T_{\lambda V}(\mu_\varepsilon)| \leq \ C_k\, r(V)^{-\frac{1}{2}} \ \lambda^{-\frac{1}{2}} \varepsilon^{(k-\frac{1}{2})(s-k)+\frac{1}{4}}. \tag{3.3}$$

As an immediate corollary we have that

Lemma 3. *Let $k - \frac{1}{4k} \leq s < k$. There exists*

$$T_{\lambda V}(\mu) := \lim_{\varepsilon \to 0} T_{\lambda V}(\mu_\varepsilon), \tag{3.4}$$

moreover

$$|T_{\lambda V}(\mu) - T_{\lambda V}(\mu_\varepsilon)| \leq \ C_k\, r(V)^{-\frac{1}{2}} \ \lambda^{-\frac{1}{2}} \varepsilon^{(k-\frac{1}{2})(s-k)+\frac{1}{4}}. \tag{3.5}$$

Indeed, the left side of (3.4) can be written as telescopic sum:

$$\sum_{j \geq 1} T_{\lambda V}(\mu_{2\varepsilon_j}) - T_{\lambda V}(\mu_{\varepsilon_j}) \quad \textit{with} \quad \varepsilon_j = 2^{-j}\varepsilon.$$

Proof of Lemma 2. Write $\Delta\mu_\varepsilon := \mu_{2\varepsilon} - \mu_\varepsilon$, then

$$\prod_{j=1}^{n} \mu_{2\varepsilon}(x - \lambda x_j) - \prod_{j=1}^{n} \mu_\varepsilon(x - \lambda x_j) = \sum_{j=1}^{k} \Delta_j(\mu_\varepsilon),$$

where

$$\Delta_j(\mu_\varepsilon) = \prod_{i \neq j} \mu_{\varepsilon_{ij}}(x - \lambda x_i)\, \Delta\mu_\varepsilon(x - \lambda x_j), \tag{3.6}$$

where $\varepsilon_{ij} = 2\varepsilon$ for $i < j$ and $\varepsilon_{ij} = \varepsilon$ for $i > j$. Since the arguments below are the same for all $1 \leq j \leq k$, assume $j = k$ for simplicity of notations. Writing $f *_\lambda g(x) := \int f(x - \lambda y) g(y)\, dy$, and using $\|\mu_\varepsilon\|_\infty \leq 4\varepsilon^{s-k}$, we have for $\Delta T(\mu_\varepsilon) := T_{\lambda V}(\mu_\varepsilon) - T_{\lambda V}(\mu_{2\varepsilon})$,

$$|\Delta T(\mu_\varepsilon)| \lesssim \varepsilon^{(k-2)(s-d)} \int \left| \int \mu_\varepsilon(x) \ \Delta\mu_\varepsilon *_\lambda \sigma_{x_1,\dots,x_{k-1}}(x)\, dx \right| d\omega(x_1, \dots, x_{k-1}) \tag{3.7}$$

where $d\omega(x_1,\dots,x_{k-1}) = d\sigma(x_1)\dots d\sigma_{x_1,\dots,x_{k-2}}(x_{k-1})$. The inner integral is of the form

$$|\langle \mu_\varepsilon, \Delta_\varepsilon \mu *_\lambda \sigma_{x_1,\dots,x_{k-1}}\rangle| \lesssim \varepsilon^{s-d}\, \|\Delta\mu_\varepsilon *_\lambda \sigma_{x_1,\dots,x_{k-1}}\|_2,$$

thus by Cauchy-Schwarz and Placherel's identity

$$|\Delta_k T(\mu_\varepsilon)|^2 \lesssim \varepsilon^{2(k-1)(s-d)} \int |\widehat{\Delta\mu_\varepsilon}(\xi)|^2\, I_\lambda(\xi)\, d\xi,$$

where

$$I_\lambda(\xi) = \int |\hat{\sigma}_{x_1,\dots,x_{k-1}}(\lambda\xi)|^2\, d\omega(x_1,\dots,x_{k-1}).$$

Since $S_{x_1,\dots,x_{k-1}}$ is a 1-dimensional circle of radius $r_k \geq r(V) > 0$, contained in an affine subspace orthogonal to $M_{x_1,\dots,x_{k-1}} = Span\{x_1,\dots,x_{k-1}\}$, we have that

$$|\hat{\sigma}_{x_1,\dots,x_{k-1}}(\lambda\xi)|^2 \lesssim (1 + r(V)\lambda\ dist(\xi, M_{x_1,\dots,x_{k-1}}))^{-1}.$$

Since the measure $\omega(x_1,\dots,x_{k-1})$ is invariant with respect to that change of variables $(x_1,\dots,x_{k-1}) \to (Ux_1,\dots,Ux_{k-1})$ for any rotation $U \in SO(k)$, one estimates

$$\begin{aligned}
I_\lambda(\xi) &\lesssim \int\!\!\int (1 + r(V)\lambda\ dist(\xi, M_{Ux_1,\dots,Ux_{k-1}}))^{-1}\, d\omega(x_1,\dots,x_{k-1})\, dU \\
&= \int\!\!\int (1 + r(V)\lambda\ dist(U\xi, M_{x_1,\dots,x_{k-1}}))^{-1}\, d\omega(x_1,\dots,x_{k-1})\, dU \\
&= \int\!\!\int (1 + r(V)\lambda\, |\xi|\ dist(\eta, M_{x_1,\dots,x_{k-1}}))^{-1}\, d\omega(x_1,\dots,x_{k-1})\, d\sigma_{k-1}(\eta) \\
&\lesssim (1 + r(V)\,\lambda\, |\xi|)^{-1},
\end{aligned}$$

where we have written $\eta := |\xi|^{-1}U\xi$ and σ_{k-1} denotes the surface area measure on the unit sphere $S^{k-1} \subseteq \mathbb{R}^k$.

Note that $\widehat{\Delta\mu_\varepsilon}(\xi) = \hat{\mu}(\xi)(\hat{\psi}(2\varepsilon\xi) - \hat{\psi}(\varepsilon\xi))$, which is supported on $|\xi| \lesssim \varepsilon^{-1}$ and is essentially supported on $|\xi| \approx \varepsilon^{-1}$. Indeed, writing

$$\begin{aligned}
J :&= \int |\widehat{\Delta\mu_\varepsilon}(\xi)|^2\, I_\lambda(\xi)\, d\xi \\
&= \int_{|\xi| \leq \varepsilon^{-1/2}} |\widehat{\Delta\mu_\varepsilon}(\xi)|^2\, I_\lambda(\xi)\, d\xi + \int_{\varepsilon^{-1/2} \leq |\xi| \lesssim \varepsilon^{-1}} |\widehat{\Delta\mu_\varepsilon}(\xi)|^2\, I_\lambda(\xi)\, d\xi =: J_1 + J_2.
\end{aligned}$$

Using $|\hat{\psi}(2\varepsilon\xi) - \hat{\psi}(\varepsilon\xi)| \lesssim \varepsilon^{1/2}$ for $|\xi| \leq \varepsilon^{-1/2}$, we estimate

$$J_1 \lesssim \varepsilon^{\frac{1}{2}} \int |\widehat{\mu}(\xi)|^2\, (\hat{\psi}(2\varepsilon\xi) + \hat{\psi}(\varepsilon\xi))\, d\xi \lesssim \varepsilon^{\frac{1}{2}+s-k},$$

as

$$\int |\hat{\mu}(\xi)|^2 \hat{\psi}(\varepsilon\xi)\, d\xi = \int \mu_\varepsilon(x)\, d\mu(x) \lesssim \varepsilon^{s-k}.$$

On the other hand, as $I_\lambda(\xi) \lesssim \varepsilon^{1/2} r(V)^{-1}\lambda^{-1}$ for $|\xi| \geq \varepsilon^{-1/2}$ we have

$$J_2 \lesssim \varepsilon^{1/2} r(V)^{-1}\lambda^{-1} \int |\hat{\mu}(\xi)|^2 \hat{\phi}(\varepsilon\xi)\, d\xi \lesssim r(V)^{-1}\lambda^{-1}\varepsilon^{\frac{1}{2}+s-d},$$

where we have written $\hat{\phi}(\xi) = (\hat{\psi}(2\xi) - \hat{\psi}(\xi))^2$. Plugging this estimates into (3.9) we obtain

$$|\Delta T(\mu_\varepsilon)|^2 \lesssim r(V)^{-1}\lambda^{-1}\varepsilon^{\frac{1}{2}+(2k-1)(s-d)},$$

and (3.5) follows. $\square$

The support of μ_ε is not compact, however as it is a rapidly decreasing function it can be made to be supported in small neighborhood of the support of μ without changing our main estimates. Let $\phi_\varepsilon(x) := \phi(c\,\varepsilon^{-1/2}x)$ with some small absolute constant $c > 0$, where $0 \leq \phi(x) \leq 1$ is a smooth cut-off, which equals to one for $|x| \leq 1/2$ and is zero for $|x| \geq 2$. Define $\tilde{\psi}_\varepsilon = \psi_\varepsilon\, \phi_\varepsilon$ and $\tilde{\mu}_\varepsilon = \mu * \tilde{\psi}_\varepsilon$. It is easy to see that $\tilde{\mu}_\varepsilon \leq \mu_\varepsilon$ and $\int \tilde{\mu}_\varepsilon \geq 1/2$, if $c > 0$ is chosen sufficiently small. Using the trivial upper bound, for $k - \frac{1}{2k} \leq s < k$ we have

$$|T_{\lambda\Delta}(\mu_\varepsilon) - T_{\lambda\Delta}(\tilde{\mu}_\varepsilon)| \leq C_k\, \|\mu_\varepsilon\|_\infty^k\, \|\mu_\varepsilon - \tilde{\mu}_\varepsilon\|_\infty \leq C_k\, \varepsilon^{1/2},$$

it follows that estimate (3.5) remains true with μ_ε replaced with $\tilde{\mu}_\varepsilon$.

Let $f_\varepsilon := c\,\varepsilon^{k-s}\tilde{\mu}_\varepsilon$, where $c = c_\psi > 0$ is a constant so that $0 \leq f_\varepsilon \leq 1$ and $\int f_\varepsilon\, dx = c\,\varepsilon^{k-s}$. Let $\alpha := c\,\varepsilon^{k-s}$ and note that the set $A_\varepsilon := \{x :\ f_\varepsilon(x) \geq \alpha/2\}$ has measure $|A_\varepsilon| \geq \alpha/2$. We apply Theorem 2 (ii) together with the more precise lower bound (18) in [7] for the set A_ε. This gives that there exists and interval I of length $|I| \geq \exp\left(-\varepsilon^{-C_k(d-s)}\right)$, such that for all $\lambda \in I$, one has $|T_{\lambda V}(A_\varepsilon))| \geq\ c\,\alpha^{k+1} = c\,\varepsilon^{(k+1)(k-s)}$, where

$$T_{\lambda V}(A_\varepsilon) = \int \mathbf{1}_{A_\varepsilon}(x)\mathbf{1}_{A_\varepsilon}(x)\ldots\mathbf{1}_{A_\varepsilon}(x)\, d\sigma(x_1)\ldots d\sigma_{x_1,\ldots,x_{k-1}}(x_k)\, dx.$$

Since

$$T_{\lambda\Delta}(\tilde{\mu}_\varepsilon) \geq c\,\alpha^{k+1} T_{\lambda v}(A_\varepsilon),$$

we have that

$$T_{\lambda V}(\tilde{\mu}_\varepsilon) \geq c > 0, \tag{3.8}$$

for all $\lambda \in I$, for a constant $c = c(k, \psi, r(V)) > 0$.

Now, let

$$T_V(\tilde{\mu}_\varepsilon) := \int_0^1 \lambda^{1/2}\, T_{\lambda V}(\tilde{\mu}_\varepsilon)\, d\lambda.$$

For $k - \frac{1}{4k} \leq s < k$, by (3.5) we have that

$$|T_{\lambda V}(\mu) - T_{\lambda V}(\tilde{\mu}_\varepsilon)| \leq\ C_k\, r(V)^{-\frac{1}{2}}\, \lambda^{-\frac{1}{2}}\, \varepsilon^{\frac{1}{8}},$$

it follows that

$$\int_0^1 \lambda^{1/2}\, |T_{\lambda V}(\mu) - T_{\lambda V}(\tilde{\mu}_\varepsilon)|\, d\lambda \leq\, C_k\, r(V)^{-\frac{1}{2}}\, \varepsilon^{\frac{1}{8}}, \tag{3.9}$$

and in particular $\int_0^1 \lambda^{1/2}\, T_{\lambda V}(\mu)\, d\lambda < \infty$. On the other hand by (3.8), one has

$$\int_0^1 \lambda^{1/2}\, T_{\lambda V}(\tilde{\mu}_\varepsilon)\, d\lambda \geq \exp\left(-\varepsilon^{-C_k(k-s)}\right). \tag{3.10}$$

Now fix a small $\varepsilon = \varepsilon_{k,\delta} > 0$ and the choose $s = s(\varepsilon,\delta) < k$, noting that $r(V) \geq \delta$, such that

$$C_k\, \delta^{-\frac{1}{2}}\, \varepsilon^{\frac{1}{8}} < \frac{1}{2}\, \exp\left(-\varepsilon^{-C_k(k-s)}\right),$$

which ensures that

$$\int_0^1 \lambda^{1/2}\, T_{\lambda V}(\mu)\, d\lambda > 0,$$

thus there exist $\lambda > 0$ such that $T_{\lambda V}(\mu) > 0$. Fix such a λ, and assume indirectly that $E^k = E \times \ldots \times E$ does not contain any simplex isometric to λV, i.e. any point of the compact configuration space $S_{\lambda V} \subseteq \mathbb{R}^{k^2}$ of such simplices. By compactness, this implies that there is some $\eta > 0$ such that the η-neighborhood of E^k also does not contain any simplex isometric to λV. As the support of $\tilde{\mu}_\varepsilon$ is contained in the $C_k\varepsilon^{1/2}$-neighborhood of E, as $E = \mathit{supp}\, \mu$, it follows that $T_{\lambda V}(\tilde{\mu}_\varepsilon) = 0$ for all $\varepsilon < c_k\, \eta^2$ and hence $T_{\lambda V}(\mu) = 0$, contradicting our choice of λ. This proves Theorem 1.

4. The configuration space of isometric distance graphs.

Let $\Gamma_0 = (V_0, E)$ be a fixed proper distance graph, with vertex set $V_0 = \{v_0 = 0, v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$ of degree $k < d$. Let $t_{ij} = |v_i - v_j|^2$ for $(i,j) \in E$. A distance graph $\Gamma = (V, E)$ with $V = \{x_0 = 0, x_1, \ldots, x_n\}$ is isometric to Γ_0 if and only if $\mathbf{x} = (x_1, \ldots, x_n) \in S_{\Gamma_0}$, where

$$S_{\Gamma_0} = \{(x_1, \ldots, x_n) \in \mathbb{R}^{dn};\ |x_i - x_j|^2 = t_{ij},\ \forall\ \ 0 \leq i < j \leq n,\ (i,j) \in E\}$$

We call the algebraic set S_{Γ_0} the *configuration space* of isometric copies of the Γ_0. Note that S_{Γ_0} is the zero set of the family $\mathcal{F} = \{f_{ij};\ (i,j) \in E\}$, $f_{ij}(\mathbf{x}) = |x_i - x_j|^2 - t_{ij}$, thus it is a special case of the general situation described in Section 5.

If $\Gamma \simeq \Gamma_0$ with vertex set $V = \{x_0 = 0, x_1, \ldots, x_n\}$ is proper then $\mathbf{x} = (x_1, \ldots, x_n)$ is a non-singular point of S_{Γ_0}. Indeed, for a fixed $1 \leq j \leq n$ let Γ_j be the distance graph obtained from Γ by removing the vertex x_j together with all edges emanating from it. By induction we may assume that $\mathbf{x}' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ is a non-singular point i.e the gradient vectors $\nabla_{\mathbf{x}'} f_{ik}(\mathbf{x})$, $(i,k) \in E$, $i \neq j, k \neq j$ are linearly independent. Since Γ is proper the gradient vectors $\nabla_{x_j} f_{ij}(\mathbf{x}) = 2(x_i - x_j)$, $(i,j) \in E$ are also linearly independent hence $\mathbf{x}$ is a non-singular point.

In fact we have shown that the partition of coordinates $\mathbf{x} = (y, z)$ with $y = x_j$ and $z = \mathbf{x}'$ is admissible and hence (6.4) holds.

Let $r_0 = r(\Gamma_0) > 0$. It is clear that if $\Gamma \simeq \Gamma_0$ and $|x_j - v_j| \leq \eta_0$ for all $1 \leq j \leq n$, for a sufficiently small $\eta = \eta(r_0) > 0$, then Γ is proper and $r(\Gamma) \geq r_0/2$. for given $1 \leq j \leq n$, let $X_j := \{x_i \in V;\ (i,j) \in E\}$ and define

$$S_{X_j} := \{x \in \mathbb{R}^d;\ |x - x_i|^2 = t_{ij},\ \textit{for all}\ \ x_i \in X_i\}.$$

As explained in Section 6, S_{X_j} is a sphere of dimension $d - |X_j| \geq 1$ with radius $r(X_j) \geq r_0/2$. Let σ_{X_j} denote the surface area measure on S_{X_j} and write $\nu_{X_j} := \phi_j\, \sigma_{X_j}$ where ϕ_j is a smooth cut-off function supported in an η-neighborhood of v_j with $\phi_j(v_j) = 1$.

Write $\mathbf{x} = (x_1, \ldots, x_n)$, $\phi(\mathbf{x}) := \prod_{j=1}^n \phi_j(x_j)$, then by (6.4) and (6.5), one has

$$\int g(\mathbf{x})\, \phi(\mathbf{x})\, d\omega_{\mathcal{F}}(\mathbf{x}) = c_j(\Gamma_0) \int \int g(\mathbf{x})\, \phi(\mathbf{x}')\, d\nu_{X_j}(x_j)\, d\omega_{\mathcal{F}_j}(\mathbf{x}'), \tag{4.1}$$

where $\mathbf{x}' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ and $\mathcal{F}_j = \{f_{il};\ (i,l) \in E, I, l \neq j\}$. The constant $c_j(\Gamma_0) > 0$ depends only on r_0, as the radius of $x \in S_{X_j}$ is equal to the distance of v_j to the subspace spanned by its neighbors which is at least $r_0/2$. The constant $c_j(\Gamma_0)$ is the reciprocal of volume of the parallelotope with sides $x_j - x_i$, $(i,j) \in E$ which is easily shown to be at least $c_k r_0^k$.

5. Proof of Theorem 2.

Let $d > k$ and again, without loss of generality, assume that $d(\Gamma) = 1$ and hence $\delta(\Gamma) = r(\Gamma)$. Given $\lambda, \varepsilon > 0$ define the multi-linear expression,

$$T_{\lambda\Gamma_0}(\mu_\varepsilon) \ := \tag{5.1}$$
$$\int \cdots \int \mu_\varepsilon(x)\mu_\varepsilon(x - \lambda x_1) \cdots \mu_\varepsilon(x - \lambda x_n)\, \phi(x_1, \ldots, x_n) d\omega_{\mathcal{F}}(x_1, \ldots, x_n)\, dx.$$

Given a proper distance graph $\Gamma_0 = (V, E)$ on $|V| = n$ vertices of degree $k < n$ one has the following upper bound;

Lemma 4. *There exists a constant $C = C_{n,d,k}(r_0) > 0$ such that*

$$|T_{\lambda\Gamma_0}(\mu_) 2\varepsilon - T_{\lambda\Gamma_0}(\mu_\varepsilon)| \leq \ C\, \lambda^{-1/2}\, \varepsilon^{(n+\frac{1}{2})(s-d)+\frac{1}{4}}. \tag{5.2}$$

This implies again that in dimesnions $d - \frac{1}{4n+2} \leq s \leq d$, there exists the limit $T_{\lambda\Gamma_0}(\mu) := \lim_{\varepsilon \to 0} T_{\lambda\Gamma_0}(\mu_\varepsilon)$. Also, the lower bound (3.8) holds for distance graphs of degree k, as it was shown for a large class of graphs, the so-called k-degenerate distance graphs, see [7]. Thus one has

$$T_{\lambda\Gamma_0}(\tilde{\mu}_\varepsilon) \geq c > 0, \tag{5.3}$$

with a constant $c = c(n, \psi, r_0)$. Then one may argue exactly as in Section 3, to prove that there exists a $\lambda > 0$ for which $T_{\lambda\Gamma_0}(\mu) > 0$ and Theorem 2 follows from the compactness of the configuration space $S_{\lambda\Gamma_0} \subseteq \mathbb{R}^{dn}$. It remains to prove Lemma 4.

Proof of Lemma 4. Write $\Delta T(\mu_\varepsilon) := T_{\lambda\Gamma_0}(\mu_\varepsilon) - T_{\lambda\Gamma_0}(\mu_{2\varepsilon})$. Then we have $\Delta T(\mu_\varepsilon) = \sum_{j=1} \Delta_j T(\mu_\varepsilon)$, where $\Delta_j T(\mu_\varepsilon)$ is given by (5.1) with $\mu_\varepsilon(x - \lambda x_j)$ replaced by $\Delta\mu_\varepsilon(x - \lambda x_j)$ given in (3.8), and $\mu_\varepsilon(x - \lambda x_i)$ by $\mu_{2\varepsilon}(x - \lambda x_j)$ for $i > j$. Then by (4.1) we have the analogue of estimate (3.9)

$$|\Delta T(\mu_\varepsilon)| \lesssim \varepsilon^{(n-1)(s-d)} \int \left| \int \mu_\varepsilon(x)\ \Delta\mu_\varepsilon *_\lambda \nu_{X_j}(x)\, dx \right| \phi(\mathbf{x}')\, d\omega_{\mathcal{F}_j}(\mathbf{x}'), \tag{5.4}$$

where $\phi(\mathbf{x}') = \prod_{i \neq j} \phi(x_j)$. Thus by Cauchy-Schwarz and Plancherel,

$$|\Delta_j T^\varepsilon(\mu)|^2 \lesssim \varepsilon^{2n(s-d)} \int |\widehat{\Delta_\varepsilon \mu}(\xi)|^2\, I^j_\lambda(\xi)\, d\xi,$$

where

$$I^j_\lambda(\xi) = \int |\hat{\nu}_{X_j}(\lambda\xi)|^2\, \phi(\mathbf{x}')\, d\omega_{\mathcal{F}_j}(\mathbf{x}').$$

Recall that on the support of $\phi(\mathbf{x}')$ S_{X_j} is a sphere of dimension at least 1 and of radius $r \geq r_0/2 > 0$, contained in an affine subspace orthogonal to $Span\, X_j$. Thus,

$$|\hat{\nu}_{X_j}(\lambda\xi)|^2 \lesssim (1 + r_0\lambda\, dist(\xi, Span\, X_j))^{-1}.$$

Let $U : \mathbb{R}^d \to \mathbb{R}^d$ be a rotation and for $\mathbf{x}' = (x_i)_{i \neq j}$ write $U\mathbf{x}' = (Ux_i)_{i \neq j}$. As explained in Section 6, the measure $\omega_{\mathcal{F}_j}$ is invariant under the transformation $\mathbf{x}' \to U\mathbf{x}'$, hence

$$\begin{aligned} I_\lambda(\xi) &\lesssim \int\int (1 + r_0\lambda\ dist(\xi, Span\, UX_j))^{-1}\, d\omega_{\mathcal{F}_j}(\mathbf{x}')\, dU \\ &= \int\int (1 + r_0\lambda\, |\xi|\ dist(\eta, Span\, X_j))^{-1}\, d\sigma_{d-1}(\eta)\, d\omega\mathcal{F}_j(\mathbf{x}') \\ &\lesssim (1 + r_0\, \lambda\, |\xi|)^{-1}, \end{aligned}$$

where we have written again $\eta := |\xi|^{-1}U\xi \in S^{d-1}$.
Then we argue as in Lemma 2, noting that $\widehat{\Delta\mu_\varepsilon}(\xi)$ is essentially supported on $|\xi| \approx \varepsilon^{-1}$ we have that

$$|\Delta T(\mu_\varepsilon)|^2 \lesssim r_0^{-1}\lambda^{-1}\varepsilon^{2n(s-d)+\frac{1}{2}} \int |\hat{\mu}(\xi)|^2 \hat{\phi}(\varepsilon\xi)\, d\xi \lesssim r_0^{-1}\lambda^{-1}\varepsilon^{(2n+1)(s-d)+\frac{1}{2}},$$

with $\tilde{\mu}_\varepsilon = \mu_\varepsilon$ or $\tilde{\mu}_\varepsilon = \mu * \phi_\varepsilon$. This proves Lemma 4. $\square$

6. Measures on real algebraic sets.

Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a family of polynomials $f_i : \mathbb{R}^d \to \mathbb{R}$. We will describe certain measures supported on the algebraic set

$$S_{\mathcal{F}} := \{x \in \mathbb{R}^d : \ f_1(x) = \ldots = f_n(x) = 0\}. \tag{6.1}$$

A point $x \in S_{\mathcal{F}}$ is called *non-singular* if the gradient vectors $\nabla f_1(x), \ldots, \nabla f_n(x)$ are linearly independent, and let $S^0_{\mathcal{F}}$ denote the set of non-singular points. It is well-known and is easy to see, that if $S^0_{\mathcal{F}} \neq \emptyset$ then it is a relative open, dense subset of $S_{\mathcal{F}}$, and moreover it is an $d - n$-dimensional sub-manifold of $\mathbb{R}^d$. If $x \in S^0_{\mathcal{F}}$ then there exists a set of coordinates, $J = \{j_1, \ldots, j_n\}$, with $1 \leq j_1 < \ldots < j_n \leq d$, such that

$$j_{\mathcal{F},J}(x) := det \left(\frac{\partial f_i}{\partial x_j}(x) \right)_{1 \leq i \leq n, j \in J} \neq 0. \tag{6.2}$$

Accordingly, we will call a set of coordinates J *admissible*, if (6.2) holds for at least one point $x \in S^0_{\mathcal{F}}$, and will denote by $S_{\mathcal{F},J}$ the set of such points. For a given set of coordinates x_J let $\nabla_{x_J} f(x) := (\partial_{x_j} f(x))_{j \in J}$, and note that J is admissible if and only if the gradient vectors $\nabla_{x_J} f_1(x), \ldots, \nabla_{x_J} f_n(x)$, are linearly independent at at least one point $x \in S_{\mathcal{F}}$. It is clear that $S_{\mathcal{F},J}$ is a relative open and dense subset of $S_{\mathcal{F}}$ and is a also $d - n$-dimensional sub-manifold, moreover

$$S^0_{\mathcal{F}} = \bigcup_{J \ admissible} S_{\mathcal{F},J}.$$

We define a measure, near a point $x_0 \in S_{\mathcal{F},J}$ as follows. For simplicity of notation assume that $J = \{1, \ldots, n\}$ and let $\Phi(x) := (f_1, \ldots, f_n, x_{n+1}, \ldots, x_d)$. Then $\Phi : U \to V$ is a diffeomorphism on some open set $x_0 \in U \subseteq \mathbb{R}^d$ to its image $V = \Phi(U)$, moreover $S_{\mathcal{F}} = \Phi^{-1}(V \cap \mathbb{R}^{d-n})$. Indeed, $x \in S_{\mathcal{F}} \cap U$ if and only if $\Phi(x) = (0, \ldots, 0, x_{n+1}, \ldots, x_d) \in V$. Let $I = \{n+1, \ldots, d\}$ and write $x_I := (x_{n+1}, \ldots, x_d)$. Let $\Psi(x_I) = \Phi^{-1}(0, x_I)$ and in local coordinates x_I define the measure $\omega_{\mathcal{F}}$ via

$$\int g \, d\omega_{\mathcal{F}} := \int g(\Psi(x_I)) \ Jac^{-1}_{\Phi}(\Psi(x_I)) \, dx_I, \tag{6.3}$$

for a continuous function g supported on U. Note that $Jac_{\Phi}(x) = j_{\mathcal{F},J}(x)$, i.e. the Jacobian of the mapping Φ at $x \in U$ is equal to the expression given in (6.2), and that the measure $d\omega_{\mathcal{F}}$ is supported on $S_{\mathcal{F}}$. Define the local coordinates $y_j = f_j(x)$ for $1 \leq j \leq n$ and $y_j = x_j$ for $n < j \leq d$. Then

$$dy_1 \wedge \ldots \wedge dy_d = df_1 \wedge \ldots \wedge df_n \wedge dx_{n+1} \wedge \ldots \wedge dx_d = Jac_{\Phi}(x) \, dx_1 \wedge \ldots \wedge dx_d,$$

thus

$$dx_1 \wedge \ldots \wedge dx_d = Jac_{\Phi}(x)^{-1} df_1 \wedge \ldots \wedge df_n \wedge dx_{n+1} \wedge \ldots \wedge dx_d = df_1 \wedge \ldots \wedge df_n \wedge d\omega_{\mathcal{F}}.$$

This shows that the measure $d\omega_{\mathcal{F}}$ (given as a differential $d-n$-form on $S_{\mathcal{F}} \cap U$) is independent of the choice of local coordinates x_I. Then $\omega_{\mathcal{F}}$ is defined on $S^0_{\mathcal{F}}$ and moreover the set $S^0_{\mathcal{F}} \backslash S_{\mathcal{F},J}$ is of measure zero with respect to ω_F, as it is a proper analytic subset on $\mathbb{R}^{d-n}$ in any other admissible local coordinates.

Let $x = (z,y)$ be a partition of coordinates in $\mathbb{R}^d$, with $y = x_{J_2}$, $z = X_{J_1}$, and assume that for $i = 1,\dots,m$ the functions f_i depend only on the z-variables. We say that the partition of coordinates is *admissible*, if there is a point $x = (z,y) \in S_{\mathcal{F}}$ such that both the gradient vectors $\nabla_z f_1(x),\dots,\nabla_z f_m(x)$ and the vectors $\nabla_y f_{m+1}(x),\dots,\nabla_y f_n(x)$ for a linearly independent system. Partition the system $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ with $\mathcal{F}_1 = \{f_1,\dots,f_m\}$ and $\mathcal{F}_2 = \{f_{m+1},\dots,f_n\}$. Then there is set $J_1' \subseteq J_1$ for which

$$j_{\mathcal{F}_1,J_1'}(z) := \det\left(\frac{\partial f_i}{\partial x_j}(z)\right)_{1\le i\le m,\, j\in J_1'} \neq 0,$$

and also a set $J_2' \subseteq J_2$ such that

$$j_{\mathcal{F}_2,J_2'}(z,y) := \det\left(\frac{\partial f_i}{\partial x_j}(z,y)\right)_{m+1\le i\le n,\, j\in J_2'} \neq 0.$$

Since $\nabla_y f_i \equiv 0$ for $1 \le i \le m$, it follows that the set of coordinates $J' = J_1' \cup J_2'$ is admissible, moreover

$$j_{\mathcal{F},J'}(y,z) = j_{\mathcal{F}_1,J_1'}(z)\, j_{\mathcal{F}_2,J_2'}(y,z).$$

For fixed z, let $f_{i,z}(y) := f_i(z,y)$ and let $\mathcal{F}_{2,z} = \{f_{m+1,z},\dots,f_{n,z}\}$. Then clearly $j_{\mathcal{F}_2,J_2'}(y,z) = j_{\mathcal{F}_{2,z},J_2'}(y)$ as it only involves partial derivatives with respect to the y-variables. Thus we have an analogue of Fubini's theorem, namely

$$\int g(x)\, d\omega_{\mathcal{F}}(x) \;=\; \int\!\!\int g(z,y)\, d\omega_{\mathcal{F}_{2,z}}(y)\, d\omega_{\mathcal{F}_1}(z). \tag{6.4}$$

Consider now algebraic sets given as the intersection of spheres. Let $x_1,\dots,x_m \in \mathbb{R}^d$, $t_1,\dots,t_m > 0$ and $\mathcal{F} = \{f_1,\dots,f_m\}$ where $f_i(x) = |x - x_i|^2 - t_i$ for $i = 1,\dots,m$. Then $S_{\mathcal{F}}$ is the intersection of spheres centered at the points x_i of radius $r_i = t_i^{1/2}$. If the set of points $X = \{x_1,\dots,x_m\}$ is in general position (i.e they span an $m-1$-dimensional affine subspace), then a point $x \in S_{\mathcal{F}}$ is non-singular if $x \notin \mathit{span}\, X$, i.e if x cannot be written as linear combination of $x_1,\dots,x_m$. Indeed, since $\nabla f_i(x) = 2(x - x_i)$ we have that

$$\sum_{i=1}^m a_i \nabla f_i(x) = 0 \iff \sum_{i=1}^m a_i\, x = \sum_{i=1}^m a_i x_i,$$

which implies $\sum_{i=1}^m a_i = 0$ and $\sum_{i=1}^m a_i x_i = 0$. By replacing the equations $|x - x_i|^2 = t_i$ with $|x - x_1|^2 - |x - x_i|^2 = t_1 - t_i$, which is of the form

$x \cdot (x_1 - x_i) = c_i$, for $i = 2, \ldots, m$, it follows that $S_{\mathcal{F}}$ is the intersection of sphere with an $n-1$-codimensional affine subspace Y, perpendicular to the affine subspace spanned by the points x_i. Thus $S_{\mathcal{F}}$ is an m-codimensional sphere of $\mathbb{R}^d$ if $S_{\mathcal{F}}$ has one point $x \notin span\{x_1, \ldots, x_m\}$ and all of its points are non-singular. Let x' be the orthogonal projection of x to $spanX$. If $y \in Y$ is a point with $|y - x'| = |x - x'|$ then by the Pythagorean theorem we have that $|y - x_i| = |x - x_i|$ and hence $y \in S_{\mathcal{F}}$. It follows that $S_{\mathcal{F}}$ is a sphere centered at x' and contained in Y.

Let $T = T_X$ be the inner product matrix with entries $t_{ij} := (x-x_i)\cdot(x-x_j)$ for $x \in S_{\mathcal{F}}$. Since $(x-x_i)\cdot(x-x_j) = 1/2(t_i+t_j-|x_i-x_j|^2)$ the matrix T is independent of x. We will show that $d\omega_{\mathcal{F}} = c_T \, d\sigma_{S_{\mathcal{F}}}$ where $d\sigma_{S_{\mathcal{F}}}$ denotes the surface area measure on the sphere $S_{\mathcal{F}}$ and $c_T = 2^{-m} det(T)^{-1/2} > 0$, i.e for a function $g \in C_0(\mathbb{R}^d)$,

$$\int_{S_{\mathcal{F}}} g(x) \, d\omega_{\mathcal{F}}(x) = c_T \int_{S_{\mathcal{F}}} g(x) \, d\sigma_{S_{\mathcal{F}}}(x). \tag{6.5}$$

Let $x \in S_{\mathcal{F}}$ be fixed and let $e_1, \ldots, e_d$ be an orthonormal basis so that the tangent space $T_x S_{\mathcal{F}} = Span\{e_{m+1}, \ldots, e_d\}$ and moreover we have that $Span\{\nabla f_1, \ldots, \nabla f_m\} = Span\{e_1, \ldots, e_m\}$. Let $x_1, \ldots, x_n$ be the corresponding coordinates on $\mathbb{R}^d$ and note that in these coordinates the surface area measure, as a $d-m$-form at x, is

$$d\sigma_{S_{\mathcal{F}}}(x) = dx_{m+1} \wedge \ldots \wedge dx_d.$$

On the other hand, in local coordinates $x_I = (x_{m+1}, \ldots, x_d)$, it is easy to see form (6.2)-(6.3) that $j_{\mathcal{F},J}(x) = 2^m \, vol(x - x_1, \ldots, x - x_m)$ and hence

$$d\omega_{\mathcal{F}}(x) = 2^{-m} vol(x - x_1, \ldots, x - x_m)^{-1} \, dx_{m+1} \wedge \ldots \wedge dx_d,$$

where $vol(x - x_1, \ldots, x - x_m)$ is the volume of the parallelotope with side vectors $x - x_j$. Finally, it is a well-known fact from linear algebra that

$$vol(x - x_1, \ldots, x - x_m)^2 = det\,(T),$$

i.e. the volume of a parallelotope is the square root of the Gram matrix formed by the inner products of its side vectors.

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