DISCRETE RADON TRANSFORMS AND APPLICATIONS TO ERGODIC THEORY

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Abstract. We prove L^p boundedness of certain non-translation-invariant discrete maximal Radon transforms and discrete singular Radon transforms. We also prove maximal, pointwise, and L^p ergodic theorems for certain families of non-commuting operators.

Contents

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1. Introduction

In this paper we are concerned with L^p estimates for discrete operators in certain non-translation-invariant settings, and the applications of such estimates to ergodic theorems for certain families of non-commuting operators. We describe first the type of operators we consider in the translation-invariant setting. Assume $P : \mathbb{Z}^{d_1} \to \mathbb{Z}^{d_2}$ is a polynomial mapping and $K : \mathbb{R}^{d_1} \setminus B(1) \to \mathbb{C}$ is a Calderón–Zygmund kernel (see (1.3) and (1.4) for precise definitions). For (compactly supported) functions $f : \mathbb{Z}^{d_2} \to \mathbb{C}$ we define the maximal operator

$$\widetilde{M}(f)(m) = \sup_{r>0} \Big| \frac{1}{|B(r) \cap \mathbb{Z}^{d_1}|} \sum_{n \in B(r) \cap \mathbb{Z}^{d_1}} f(m - P(n)) \Big|,$$

and the singular integral operator

$$\widetilde{T}(f)(m) = \sum_{n \in \mathbb{Z}^{d_1} \setminus \{0\}} K(n) f(m - P(n)).$$

The maximal operator $\widetilde{M}(f)$ was considered by Bourgain [3], [4], [5], who showed that

$$\|\widetilde{M}(f)\|_{L^p(\mathbb{Z}^{d_2})} \leq C_p \|f\|_{L^p(\mathbb{Z}^{d_2})},\ p \in (1, \infty] \text{ if } d_1 = d_2 = 1. \tag{1.1}$$

Maximal inequalities such as (1.1) have applications to pointwise and L^p, $p \in (1, \infty)$, ergodic theorems, see [3], [4], and [5]. A typical theorem is the following: assume $P : \mathbb{Z} \to \mathbb{Z}$ is a polynomial mapping, (X, μ) is a finite measure space, and $T : X \to X$ is a measure-preserving invertible transformation. For $F \in L^p(X)$, $p \in (1, \infty)$, let

$$\widetilde{A}_r(F)(x) = \frac{1}{2r+1} \sum_{|n| \leq r} F(T^{P(n)} x) \text{ for any } r \in \mathbb{Z}_+.$$

Then there is a function $F_* \in L^p(X)$ with the property that

$$\lim_{r \to \infty} \widetilde{A}_r(F) = F_* \text{ almost everywhere and in } L^p.$$

In addition, $F_* = \mu(X)^{-1} \int_X F(x)\, d\mu$ if T^q is ergodic for $q = 1, 2, \ldots$.

The related singular integral operator $\widetilde{T}(f)$ was considered first by Stein and Wainger [15]. Following earlier work of [1], [15], and [17], Ionescu and Wainger [8] proved that

$$\|\widetilde{T}(f)\|_{L^p(\mathbb{Z}^{d_2})} \leq C_p \|f\|_{L^p(\mathbb{Z}^{d_2})},\ p \in (1, \infty). \tag{1.2}$$

A more complete description of the results leading to the bound (1.2) can be found in the introduction of [8].

In this paper we start the systematic study of the suitable analogues of the operators $\widetilde{M}$ and $\widetilde{T}$ in discrete settings which are not translation-invariant.[1] As before, the maximal function estimate has applications to ergodic theorems involving families of non-commuting operators.

Motivated by models involving actions of nilpotent groups, we consider a special class of non-translation-invariant Radon transforms, called the "quasi-translation" invariant Radon transforms. Assume $d, d' \geq 1$ and $P : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^{d'}$ is a polynomial mapping, For any $r > 0$ let $B(r)$ denote the ball $\{x \in \mathbb{R}^d : |x| < r\}$. Let $K : \mathbb{R}^d \setminus B(1) \to \mathbb{C}$ denote a Calderón–Zygmund kernel, i.e.

$$|x|^d |K(x)| + |x|^{d+1} |\nabla K(x)| \leq 1,\ |x| \geq 1, \tag{1.3}$$

and

$$\Big| \int_{|x| \in [1,N]} K(x)\, dx \Big| \leq 1 \text{ for any } N \geq 1. \tag{1.4}$$

For (compactly supported) functions $f : \mathbb{Z}^d \times \mathbb{Z}^{d'} \to \mathbb{C}$ we define the discrete maximal Radon transform

$$M(f)(m_1, m_2) = \sup_{r>0} \Big| \frac{1}{|B(r) \cap \mathbb{Z}^d|} \sum_{n \in B(r) \cap \mathbb{Z}^d} f(m_1 - n, m_2 - P(m_1, n)) \Big|, \tag{1.5}$$

and the discrete singular Radon transform

$$T(f)(m_1, m_2) = \sum_{n \in \mathbb{Z}^d \setminus \{0\}} K(n) f(m_1 - n, m_2 - P(m_1, n)). \tag{1.6}$$

The operator T was considered by Stein and Wainger [16], who proved that

$$\|T\|_{L^2(\mathbb{Z}^d \times \mathbb{Z}^{d'}) \to L^2(\mathbb{Z}^d \times \mathbb{Z}^{d'})} \leq C. \tag{1.7}$$

In this paper we prove estimates like (1.7) in the full range of exponents p for both the singular integral operator T and the maximal operator M, in the special case in which

$$\text{the polynomial } P \text{ has degree at most 2.} \tag{1.8}$$

Theorem 1.1. *Assuming* (1.8)*, the discrete maximal Radon transform M extends to a bounded (subadditive) operator on $L^p(\mathbb{Z}^d \times \mathbb{Z}^{d'})$, $p \in (1, \infty]$,with*

$$\|M\|_{L^p(\mathbb{Z}^d \times \mathbb{Z}^{d'}) \to L^p(\mathbb{Z}^d \times \mathbb{Z}^{d'})} \leq C_p.$$

The constant C_p depends only on the exponent p and the dimension d.

Theorem 1.2. *Assuming* (1.8)*, the discrete singular Radon transform T extends to a bounded operator on $L^p(\mathbb{Z}^d \times \mathbb{Z}^{d'})$, $p \in (1, \infty)$, with*

$$\|T\|_{L^p(\mathbb{Z}^d \times \mathbb{Z}^{d'}) \to L^p(\mathbb{Z}^d \times \mathbb{Z}^{d'})} \leq C_p.$$

The constant C_p depends only on the exponent p and the dimension d.

[1] Such operators, called Radon transforms, have been studied extensively in the continuous setting, see [6] and the references therein.

See also Theorems 2.1, 2.2, 2.3, 2.4, and 5.2 for equivalent versions of Theorems 1.1 and 1.2 in the setting of nilpotent groups. In the special case $d = d' = 1$, $P(m_1, n) = n^2$, Theorem 1.1 gives

$$\Big|\Big|\sup_{r>0} \frac{1}{|B(r)\cap\mathbb{Z}|}\sum_{|n|\leq r}\big|f(m_1-n, m_2-n^2)\big|\Big|\Big|_{L^p(\mathbb{Z}^2)} \leq C_p\|f\|_{L^p(\mathbb{Z}^2)}, \tag{1.9}$$

for any $p \in (1,\infty]$ and $f \in L^p(\mathbb{Z}^2)$. We consider functions f of the form $f(m_1, m_2) = g(m_2)\cdot \mathbf{1}_{[-M,M]}(m_1)$; by letting $M \to \infty$ it follows from (1.9) that

$$\Big|\Big|\sup_{r>0} \frac{1}{|B(r)\cap\mathbb{Z}|}\sum_{|n|\leq r}\big|g(m-n^2)\big|\Big|\Big|_{L^p(\mathbb{Z})} \leq C_p\|g\|_{L^p(\mathbb{Z})},$$

which is Bourgain's theorem [5] in the case $P(n) = n^2$.

We state now our main ergodic theorem. Let (X, μ) denote a finite measure space, and let $T_1, \ldots, T_d, S_1, \ldots, S_{d'}$ denote a family of measure-preserving invertible transformations on X satisfying the commutator relations

$$[T_i, S_j] = [S_i, S_j] = I, \quad [[T_i, T_j], T_k] = I \text{ for all } i, j, k. \tag{1.10}$$

Here I denotes the identity transformation, and $[T, S] = T^{-1}S^{-1}TS$ the commutator of T and S. For a polynomial mapping

$$Q = (Q_1, \ldots, Q_{d'}) : \mathbb{Z}^d \to \mathbb{Z}^{d'} \text{ of degree at most 2}, \tag{1.11}$$

and $F \in L^p(X)$, $p \in (1,\infty)$, we define the averages

$$A_r(F)(x) = \frac{1}{|B(r)\cap\mathbb{Z}^d|}\sum_{n=(n_1,\ldots,n_d)\in B(r)\cap\mathbb{Z}^d} F\left(T_1^{n_1}\ldots T_d^{n_d}S_1^{Q_1(n)}\ldots S_{d'}^{Q_{d'}(n)}x\right). \tag{1.12}$$

Theorem 1.3. *Assume $T_1, \ldots, T_d, S_1, \ldots, S_{d'}$ satisfy* (1.10) *and Q is as in* (1.11). *Then for every $F \in L^p(X)$, $p \in (1,\infty)$, there exists $F_* \in L^p(X)$ such that*

$$\lim_{r\to\infty} A_r(F) = F_* \text{ almost everywhere and in } L^p. \tag{1.13}$$

Moreover, if the family of transformations $\{T_i^q, S_k^q : 1 \leq i \leq d,\ 1 \leq k \leq d'\}$ is ergodic for every integer $q \geq 1$, then

$$F_* = \frac{1}{\mu(X)}\int_X F\,d\mu. \tag{1.14}$$

See also Theorem 5.1 for an equivalent version formulated in terms of the action of a discrete nilpotent group of step 2.

It would be desirable to remove the restrictions on the degrees of the polynomials P and Q in (1.8) and (1.11), and allow more general commutator relations in (1.10).[2] These two issues are related. In this paper we exploit the restriction (1.8)

[2]A possible setting for the pointwise ergodic theorem would be that of polynomial sequences in nilpotent groups, compare with [2] and [9].

to connect the Radon transforms M and T to certain group translation-invariant Radon transforms on discrete nilpotent groups of step 2. We then analyze the resulting Radon transforms using Fourier analysis techniques. The analogue of this construction for higher degree polynomials P leads to nilpotent Lie groups of higher step, for which it is not clear whether the Fourier transform method can be applied. We hope to return to this in the future.

We describe now some of the ingredients in the proofs of Theorems 1.1, 1.2, and 1.3. In section 2 we use a transference principle and reduce Theorems 1.1 and 1.2 to Lemmas 2.3 and 2.4 on the discrete nilpotent group $\mathbb{G}_0^{\#}$.

In section 3 we prove four technical lemmas concerning oscillatory integrals on $L^2(Z_q^d)$ and $L^2(\mathbb{Z}^d)$. These bounds correspond to estimates for fixed θ after using the Fourier transform in the central variable of the group $\mathbb{G}_0^{\#}$. We remark that natural scalar-valued objects, such as the Gauss sums, become operator-valued objects in our non-commutative setting. For example, the bound $||\mathcal{S}^{a/q}||_{L^2(Z_q^d)\to L^2(Z_q^d)} \leq q^{-1/2}$ in Lemma 3.1 is the natural analogue of the standard scalar bound on Gauss sums $|S^{a/q}| \leq Cq^{-1/2}$.

In section 4 we prove Lemma 2.3 (which implies Theorem 1.1). In subsection 4.1 we prove certain strong L^2 bounds (see Lemma 4.1); the proof of these L^2 bounds is based on a variant of the "circle method", adapted to our non-translation-invariant setting. In subsection 4.2 we prove a restricted L^p bound, $p > 1$, with a logarithmic loss. The idea of using such restricted L^p estimates as an ingredient for proving the full L^p estimates originates in Bourgain's paper [5]. Finally, in subsection 4.3 we prove Lemma 2.3, by combining the strong L^2 bounds in subsection 4.1, and the restricted L^p bounds in subsection 4.2.

In section 5 we prove Theorem 1.3. First we restate Theorem 1.3 in terms of actions of discrete nilpotent groups of step 2, see Theorem 5.1. Then we use a maximal ergodic theorem, which follows by transference from Theorem 1.1, to reduce matters to proving almost everywhere convergence for functions F in a dense subset of $L^p(X)$. For this we adapt a limiting argument of Bourgain [5].

In section 6 we prove Lemma 2.4 (which implies Theorem 1.2). In subsection 6.1 we prove strong L^2 bounds, using only Plancherel theorem and the fixed θ estimates in section 3. In subsection 6.2 we recall (without proofs) a partition of the integers and a square function estimate used by Ionescu and Wainger [8]. In subsection 6.3 we complete the proof of Lemma 2.4. First we reduce matters to proving a suitable square function estimate for a more standard oscillatory singular integral operator (see Lemma 6.6). Then we use the equivalence between square function estimates and weighted inequalities (cf. [7, Chapter V]) to further reduce to proving a weighted inequality for an (essentially standard) oscillatory singular integral operator. This weighted inequality is proved in section 7.

In section 7, which is self-contained, we collect several estimates related to the real-variable theory on the group $\mathbb{G}_0^\#$. We prove weighted L^p estimates for maximal averages and oscillatory singular integrals, in which the relevant underlying balls have eccentricity $N \gg 1$. The main issue is to prove these L^p bounds with only logarithmic losses of the type $(\ln N)^C$. These logarithmic losses can then be combined with the gains of $N^{-\overline{c}}$ in the L^2 estimates in Lemmas 4.1 and 6.1 to obtain the theorems in the full range of exponents p. The proofs in this section are essentially standard real-variable proofs (compare with [14]); we provide all the details for the sake of completeness.

2. Preliminary reductions: a transference principle

In this section we reduce Theorems 1.1 and 1.2 to Lemmas 2.3 and 2.4 on the discrete free group $\mathbb{G}_0^\#$ defined below. This is based on the "method of transference". Since the polynomial mapping P in Theorems 1.1 and 1.2 has degree at most 2 (see (1.8)), we can write

$$P(m_1, n) = R(n, m_1 - n) + A(m_1 - n) + B(m_1), \tag{2.1}$$

for some polynomial mappings $A, B : \mathbb{Z}^d \to \mathbb{Z}^{d'}$, and a bilinear mapping $R : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^{d'}$. The representation (2.1) follows simply by setting $B(m) = P(m, m)$, $A(m) = P(m, 0) - P(m, m)$, and $R(m, m') = P(m + m', m) + P(m', m') - P(m + m', m + m') - P(m', 0)$. Since $R(m, 0) = R(0, m') = 0$ for any $m, m' \in \mathbb{Z}^d$, it follows from (1.8) that R is bilinear.

The definitions (1.5) and (1.6) show that

$$\begin{cases} M(f)(m_1, m_2) = \widetilde{M}(f_A)(m_1, m_2 - B(m_1)); \\ T(f)(m_1, m_2) = \widetilde{T}(f_A)(m_1, m_2 - B(m_1)), \end{cases}$$

where $f_A(m_1, m_2) = f(m_1, m_2 - A(m_1))$ and $\widetilde{M}$, $\widetilde{T}$ are defined in the same way as M, T by replacing $P(m_1, n)$ with $R(n, m_1 - n)$. Therefore, in proving Theorems 1.1 and 1.2 we may assume $P(m_1, n) = R(n, m_1 - n)$, where R is a bilinear mapping. In this case, the operators M and T can be viewed as group translation-invariant operators on certain nilpotent Lie groups, which we define below.

Assume $d, d' \geq 1$ are integers and $R : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d'}$ is a bilinear map. We define the nilpotent Lie group

$$\mathbb{G} = \{(x, s) \in \mathbb{R}^d \times \mathbb{R}^{d'} : (x, s) \cdot (y, t) = (x + y, s + t + R(x, y))\}, \tag{2.2}$$

with the standard unimodular Haar measure $dx\, ds$. In addition, if

$$R(\mathbb{Z}^d \times \mathbb{Z}^d) \subseteq \mathbb{Z}^{d'}, \tag{2.3}$$

then the set

$$\mathbb{G}^\# = \mathbb{Z}^d \times \mathbb{Z}^{d'} \subseteq \mathbb{G} \tag{2.4}$$

is a discrete subgroup of $\mathbb{G}$, equipped with the counting Haar measure.

For any (bounded compactly supported) function $F : \mathbb{G} \to \mathbb{C}$ we define the discrete maximal Radon transform

$$\mathcal{M}(F)(x,s) = \sup_{r>0} \Big| \frac{1}{|B(r) \cap \mathbb{Z}^d|} \sum_{n \in B(r)\cap\mathbb{Z}^d} F((n,0)^{-1} \cdot (x,s)) \Big|, \tag{2.5}$$

and the discrete singular Radon transform

$$\mathcal{T}(F)(x,s) = \sum_{n \in \mathbb{Z}^d \setminus \{0\}} K(n) F((n,0)^{-1} \cdot (x,s)). \tag{2.6}$$

Assuming (2.3), for (compactly supported) functions $f : \mathbb{G}^\# \to \mathbb{C}$, we define

$$\mathcal{M}^\#(f)(m,u) = \sup_{r>0} \Big| \frac{1}{|B(r) \cap \mathbb{Z}^d|} \sum_{n \in B(r)\cap\mathbb{Z}^d} f((n,0)^{-1} \cdot (m,u)) \Big| \tag{2.7}$$

and

$$\mathcal{T}^\#(f)(m,u) = \sum_{n \in \mathbb{Z}^d \setminus \{0\}} K(n) f((n,0)^{-1} \cdot (m,u)). \tag{2.8}$$

In view of (2.1), Theorems 1.1 and 1.2 follow from Theorems 2.1 and 2.2 below.

Theorem 2.1. *Assume that $R : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^{d'}$ is a bilinear map satisfying* (2.3). *Then the discrete maximal Radon transform $\mathcal{M}^\#$ extends to a bounded (subadditive) operator on $L^p(\mathbb{G}^\#)$, $p \in (1,\infty]$, with*

$$\|\mathcal{M}^\#(f)\|_{L^p(\mathbb{G}^\#)} \leq C_p \|f\|_{L^p(\mathbb{G}^\#)}.$$

The constant C_p depends only on the exponent p and the dimension d.

Theorem 2.2. *Assume that $R : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^{d'}$ is a bilinear map satisfying* (2.3). *Then the discrete singular Radon transform $\mathcal{T}^\#$ extends to a bounded operator on $L^p(\mathbb{G}^\#)$, $p \in (1,\infty)$, with*

$$\|\mathcal{T}^\#(f)\|_{L^p(\mathbb{G}^\#)} \leq C_p \|f\|_{L^p(\mathbb{G}^\#)}.$$

The constant C_p depends only on the exponent p and the dimension d.

Theorems 2.1 and 2.2 can be restated as theorems on the Lie group $\mathbb{G}$.

Theorem 2.3. *Assume that $R : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^{d'}$ is a bilinear map. Then the discrete maximal Radon transform $\mathcal{M}$ extends to a bounded (subadditive) operator on $L^p(\mathbb{G})$, $p \in (1,\infty]$, with*

$$||\mathcal{M}(F)||_{L^p(\mathbb{G})} \leq C_p ||F||_{L^p(\mathbb{G})}.$$

The constant C_p may depend only on the exponent p and the dimension d.

Theorem 2.4. *Assume that $R : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^{d'}$ is a bilinear map. Then the discrete singular Radon transform $\mathcal{T}$ extends to a bounded operator on $L^p(\mathbb{G})$, $p \in (1, \infty)$, with*

$$||\mathcal{T}(F)||_{L^p(\mathbb{G})} \leq C_p ||F||_{L^p(\mathbb{G})}.$$

The constant C_p may depend only on the exponent p and the dimension d.

Assuming (2.3), we justify now the equivalence of Theorems 2.3 and 2.1 and Theorems 2.4 and 2.2. We notice that the map $\Phi : \mathbb{G}^\# \times [0,1)^d \times [0,1)^{d'} \to \mathbb{G}$,

$$\Phi((m,u),(\mu,\alpha)) = (m,u) \cdot (\mu,\alpha) = (m+\mu, u+\alpha+R(m,\mu))$$

establishes a measure preserving bijection between $\mathbb{G}^\# \times [0,1)^d \times [0,1)^{d'}$ and $\mathbb{G}$. For any (compactly supported) function $f : \mathbb{G}^\# \to \mathbb{C}$ we define

$$F : \mathbb{G} \to \mathbb{C},\ F(\Phi((m,u),(\mu,\alpha))) = f(m,u).$$

The definitions show that for any $(\mu,\alpha) \in [0,1)^d \times [0,1)^{d'}$

$$\mathcal{M}^\#(f)(m,u) = \mathcal{M}(F)(\Phi((m,u),(\mu,\alpha)));$$
$$\mathcal{T}^\#(f)(m,u) = \mathcal{T}(F)(\Phi((m,u),(\mu,\alpha))).$$

Thus Theorem 2.3 implies Theorem 2.1 and Theorem 2.4 implies Theorem 2.2.

For the converse, assume $F : \mathbb{G} \to \mathbb{C}$ is given. For any $(\mu,\alpha) \in [0,1)^d \times [0,1)^{d'}$ we define

$$f_{(\mu,\alpha)} : \mathbb{G}^\# \to \mathbb{C},\ f_{(\mu,\alpha)}(m,u) = F(\Phi((m,u),(\mu,\alpha))).$$

The definitions show that

$$\mathcal{M}(F)(\Phi((m,u),(\mu,\alpha))) = \mathcal{M}^\#(f_{(\mu,\alpha)})(m,u);$$
$$\mathcal{T}(F)(\Phi((m,u),(\mu,\alpha))) = \mathcal{T}^\#(f_{(\mu,\alpha)})(m,u),$$

so Theorem 2.1 implies Theorem 2.3 and Theorem 2.2 implies Theorem 2.4.

We further reduce Theorems 2.3 and 2.4 to a special "universal" case. We define the bilinear map $R_0 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d^2}$,

$$R_0(x,y) = \sum_{l_1,l_2=1}^{d} x_{l_1} y_{l_2} e_{l_1 l_2}, \tag{2.9}$$

where $\{e_{l_1 l_2} : l_1, l_2 = 1 \ldots d\}$ denotes the standard orthonormal basis of R^{d^2}. Using the bilinear map R_0 we define the nilpotent Lie group $\mathbb{G}_0$ as in (2.2). For any (bounded compactly supported) function $F : \mathbb{G}_0 \to \mathbb{C}$ we define $\mathcal{M}_0(F)$ and $\mathcal{T}_0(F)$ as in (2.5) and (2.6).

Lemma 2.1. *The discrete maximal Radon transform $\mathcal{M}_0$ extends to a bounded operator on $L^p(\mathbb{G}_0)$, $p \in (1, \infty]$.*

Lemma 2.2. *The discrete singular Radon transform $\mathcal{T}_0$ extends to a bounded operator on $L^p(\mathbb{G}_0)$, $p \in (1, \infty)$.*

We show now that Lemmas 2.1 and 2.2 imply Theorems 2.3 and 2.4 respectively. Assume that the bilinear map R in the definition of the group $\mathbb{G}$ is

$$R(x,y)=\sum_{l_1,l_2=1}^{d} x_{l_1}y_{l_2}v_{l_1l_2},$$

for some vectors $v_{l_1l_2}\in\mathbb{R}^{d'}$. We define the linear map $L:\mathbb{R}^{d^2}\to\mathbb{R}^{d'}$ by $L(e_{l_1l_2})=v_{l_1l_2}$ (so $L(R_0(x,y))=R(x,y)$ for any $x,y\in\mathbb{R}^d$) and the group morphism

$$\widetilde{L}:\mathbb{G}_0\to\mathbb{G},\ \widetilde{L}(x,s)=(x,L(s)).$$

We define the isometric representation π of $\mathbb{G}_0$ on $L^p(\mathbb{G})$, $p\in[1,\infty]$, by

$$\pi(g_0)(F)(g)=F(\widetilde{L}(g_0^{-1})\cdot g),\ g_0\in\mathbb{G}_0,\ F\in L^p(\mathbb{G}),\ g\in\mathbb{G}.$$

For $r>0$ we define the generalized measures μ_r and ν_r on $C_c(\mathbb{G}_0)$ by

$$\mu_r(F_0)=\frac{1}{|B(r)\cap\mathbb{Z}^d|}\sum_{n\in B(r)\cap\mathbb{Z}^d}F_0(n,0);$$
$$\nu_r(F_0)=\sum_{n\in B(r)\cap\mathbb{Z}^d\setminus\{0\}}K(n)F_0(n,0).$$

Clearly, for any (bounded compactly supported) function $F_0:\mathbb{G}_0\to\mathbb{C}$,

$$\mathcal{M}_0(F_0)(g_0)=\sup_{r>0}|F_0*\mu_r(g_0)|;$$
$$\mathcal{T}_0(F_0)(g_0)=\lim_{r\to\infty}F_0*\nu_r(g_0).$$

Also, the definitions show that for any (bounded compactly supported) function $F:\mathbb{G}\to\mathbb{C}$

$$\mathcal{M}(F)(g)=\sup_{r>0}\Big|\int_{G_0}[\pi(g_0)(F)](g)\,d\mu_r(g_0)\Big|;$$
$$\mathcal{T}(F)(g)=\lim_{r\to\infty}\int_{G_0}[\pi(g_0)(F)](g)\,d\nu_r(g_0).$$

By [12, Proposition 5.1], Theorems 2.3 and 2.4 follow from Lemmas 2.1 and 2.2 respectively.

Finally, we define the discrete subgroup $\mathbb{G}_0^{\#}=\mathbb{Z}^d\times\mathbb{Z}^{d^2}\subseteq\mathbb{G}_0$. Then we define the operators $\mathcal{M}_0^{\#}$ and $\mathcal{T}_0^{\#}$ as in (2.7) and (2.8),

$$\mathcal{M}_0^{\#}(f)(m,u)=\sup_{r>0}\Big|\frac{1}{|B(r)\cap\mathbb{Z}^d|}\sum_{n\in B(r)\cap\mathbb{Z}^d}f((n,0)^{-1}\cdot(m,u))\Big|$$

and

$$\mathcal{T}_0^{\#}(f)(m,u)=\sum_{n\in\mathbb{Z}^d\setminus\{0\}}K(n)f((n,0)^{-1}\cdot(m,u)),$$

for (compactly supported) functions $f : \mathbb{G}_0^\# \to \mathbb{C}$. In view of the equivalence discussed earlier (since R_0 clearly satisfies (2.3)), it suffices to prove the following two lemmas.

Lemma 2.3. *The discrete maximal Radon transform $\mathcal{M}_0^\#$ extends to a bounded operator on $L^p(\mathbb{G}_0^\#)$, $p \in (1,2]$.*

Lemma 2.4. *The discrete singular Radon transform $\mathcal{T}_0^\#$ extends to a bounded operator on $L^p(\mathbb{G}_0^\#)$, $p \in [2,\infty)$.*

We remark that in Lemma 2.4 it suffices to prove the estimate for $p \in [2,\infty)$. Indeed, assume $p \in (1,2]$, $p' = p/(p-1) \in [2,\infty)$, and let $\widetilde{K}(n,v) = K(n)\mathbf{1}_{\{0\}}(v)$, $\widetilde{K} : \mathbb{G}_0^\# \to \mathbb{C}$. Then $\mathcal{T}_0^\#(f) = f * \widetilde{K}$ and, by duality,

$$\|\mathcal{T}_0^\#\|_{L^p(\mathbb{G}_0^\#)\to L^p(\mathbb{G}_0^\#)} = \sup_{\|f\|_{L^{p'}(\mathbb{G}_0^\#)}=1} \Big|\Big| \int_{\mathbb{G}_0^\#} f(h\cdot g)\widetilde{K}(h)\,dh \Big|\Big|_{L_g^{p'}(\mathbb{G}_0^\#)}. \tag{2.10}$$

We define now the “dual” group $\mathbb{G}'^\#_0$,

$$\mathbb{G}'^\#_0 = \{(m,u) \in \mathbb{Z}^d \times \mathbb{R}^{d^2} : (m,u)\cdot(n,v) = (m+n, u+v+R_0'(m,n))\},$$

where $R_0'(m,n) = R_0(n,m) = \sum_{l_1,l_2=1}^d m_{l_1} n_{l_2} e_{l_2 l_1}$. The right-hand side of (2.10) is equal to

$$\sup_{\|f\|_{L^{p'}(\mathbb{G}'^\#_0)}=1} \Big|\Big| \int_{\mathbb{G}'^\#_0} f(g\cdot h)\widetilde{K}(h)\,dh \Big|\Big|_{L_g^{p'}(\mathbb{G}'^\#_0)} = \sup_{\|f\|_{L^{p'}(\mathbb{G}'^\#_0)}=1} \|f *_{\mathbb{G}'^\#_0} \widetilde{K}\|_{L^{p'}(\mathbb{G}'^\#_0)}. \tag{2.11}$$

We use now the bijection $\mathbb{G}_0^\# \leftrightarrow \mathbb{G}'^\#_0$, $\Big(m, \sum_{l_1,l_2} u_{l_1 l_2} e_{l_1 l_2}\Big) \leftrightarrow \Big(m, \sum_{l_1,l_2} u_{l_1 l_2} e_{l_2 l_1}\Big)$. Since $p' \in [2,\infty)$, it follows from Lemma 2.4 that

$$\|f *_{\mathbb{G}'^\#_0} \widetilde{K}\|_{L^{p'}(\mathbb{G}'^\#_0)} \le C_{p'} \|f\|_{L^{p'}(\mathbb{G}'^\#_0)}.$$

Using (2.10) and (2.11), it follows that $\|\mathcal{T}_0^\#\|_{L^p(\mathbb{G}_0^\#)\to L^p(\mathbb{G}_0^\#)} \le C_p$, as desired.

3. Oscillatory integrals on $L^2(Z_q^d)$ and $L^2(\mathbb{Z}^d)$

In this section we prove four lemmas concerning oscillatory integrals on L^2. The bounds in these lemmas depend on a fixed parameter θ in the Fourier space corresponding to taking the Fourier transform in the central variable of the group $\mathbb{G}_0^\#$. In Lemma 3.1, $\theta = a/q$ (the Gauss sum operator). In Lemma 3.2, θ is close to a/q, q large. In Lemma 3.3, θ is close to a/q, q small. Finally, Lemma 3.4 is an estimate for a singular integral. The main issue in all these lemmas is to have a quantitative gain over the trivial $L^2 \to L^2$ estimates with bound 1. Lemmas of this type have been used in [10] and [16].

We assume throughout this section that $d' = d^2$, and $\mathbb{G}_0^{\#}$ is the discrete nilpotent group defined in section 2. For any $\mu \geq 1$ let $Z_\mu = \mathbb{Z} \cap [1, \mu]$. If $a = (a_{l_1 l_2})_{l_1, l_2 = 1, \ldots, d} \in \mathbb{Z}^{d'}$ is vector and $q \geq 1$ is an integer, then we denote by (a, q) the greatest common divisor of a and q, i.e. the largest integer $q' \geq 1$ that divides q and all the components $a_{l_1 l_2}$. Any number in $\mathbb{Q}^{d'}$ can be written uniquely in the form

$$a/q,\ q \in \{1, 2, \ldots\},\ a \in \mathbb{Z}^{d'},\ (a, q) = 1. \tag{3.1}$$

A number as in (3.1) will be called an *irreducible d'-fraction.* For any irreducible d'-fraction a/q and $g : Z_q^d \to \mathbb{C}$ we consider the (Gauss sum) operator

$$\mathcal{S}^{a/q}(g)(m) = q^{-d} \sum_{n \in Z_q^d} g(n) e^{-2\pi i R_0(m-n,n)\cdot a/q}. \tag{3.2}$$

Lemma 3.1 (Gauss sum estimate)**.** *With the notation above,*

$$||\mathcal{S}^{a/q}(g)||_{L^2(Z_q^d)} \leq q^{-1/2} ||g||_{L^2(Z_q^d)}. \tag{3.3}$$

Proof of Lemma 3.1. We consider the operator $\mathcal{S}^{a/q}(\mathcal{S}^{a/q})^*$; the kernel of this operator is

$$L(m,n) = q^{-2d} \sum_{w \in Z_q^d} e^{-2\pi i R_0(m-n,w)\cdot a/q} = q^{-2d} \prod_{l_2=1}^{d} \delta_q\Big(\sum_{l_1=1}^{d} (m_{l_1} - n_{l_1}) \cdot a_{l_1 l_2}\Big), \tag{3.4}$$

where $\delta_q : \mathbb{Z} \to \{0, q\}$,

$$\delta_q(m) = \begin{cases} q \text{ if } m/q \in \mathbb{Z}; \\ 0 \text{ if } m/q \notin \mathbb{Z}. \end{cases} \tag{3.5}$$

We have to show that $\sum_{m \in Z_q^d} |L(m,n)|$ and $\sum_{n \in Z_q^d} |L(m,n)|$ are bounded uniformly by q^{-1}. In view of (3.4), it suffices to prove that the number of solutions $(m_1, \ldots, m_d) \in Z_q^d$ of the system

$$\sum_{l_1=1}^{d} m_{l_1} a_{l_1 l_2} = 0 \,(\mathrm{mod}\ q) \text{ for any } l_2 = 1, \ldots, d, \tag{3.6}$$

is at most q^{d-1}.

Assume $q = p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$ is the unique decomposition of q as a product of powers of distinct primes. Any integer m can be written uniquely in the form

$$m = \sum_{i=1}^{k} m^i \cdot (q/p_i^{\alpha_i}) \,(\mathrm{mod}\ q),\ \ m^i \in Z_{p_i^{\alpha_i}}. \tag{3.7}$$

We write $a_{l_1l_2}$ and m_{l_1} as in (3.7). Since the primes p_i are distinct, the system (3.6) is equivalent to the system

$$\sum_{l_1=1}^{d} m_{l_1}^i a_{l_1l_2}^i = 0\,(\mathrm{mod}\ p_i^{\alpha_i}) \text{ for any } l_2 = 1,\ldots,d \text{ and } i = 1,\ldots,k. \tag{3.8}$$

We use now the fact that a/q is an irreducible d'-fraction. Thus for any $i = 1,\ldots,k$ there are some $l_1(i), l_2(i) \in 1,\ldots,d$ with the property that $(a_{l_1(i)l_2(i)}, p_i) = 1$. For any $i = 1,\ldots,k$ we consider only the equation in the system (3.8) corresponding to $l_2 = l_2(i)$. Since $a_{l_1(i)l_2(i)}$ is invertible in the ring $\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}$, for any i fixed the system (3.8) can have at most $[p_i^{\alpha_i}]^{d-1}$ solutions $(m_1^i,\ldots,m_d^i) \in Z^d_{p_i^{\alpha_i}}$. The lemma follows. □

Assume now that $j \geq 0$ is an integer and $\Phi_j : \mathbb{R}^d \to \mathbb{C}$ is a function supported in the set $\{x : |x| \leq 2^{j+1}\}$ such that

$$2^{dj}|\Phi_j(x)| + 2^{(d+1)j}|\nabla\Phi_j(x)| \leq 1,\ x \in \mathbb{R}^d. \tag{3.9}$$

For $\theta \in \mathbb{R}^{d'}$ and (compactly supported) functions $g : \mathbb{Z}^d \to \mathbb{C}$ we define

$$\mathcal{U}_j^\theta(g)(m) = \sum_{n\in\mathbb{Z}^d} \Phi_j(m-n)g(n)e^{-2\pi i R_0(m-n,n)\cdot\theta}. \tag{3.10}$$

We prove two L^2 bounds for the operators $\mathcal{U}_j^\theta(g)$.

Lemma 3.2 (Minor arcs). *Assume that a/q is an irreducible d'-fraction, $\delta > 0$, and $\theta \in \mathbb{R}^{d'}$. Assume also that there are some indices $k_1, k_2 \in \{1,\ldots,d\}$ with the property that*

$$\begin{cases} a_{k_1k_2}/q = \overline{a}_{k_1k_2}/\overline{q}_{k_1k_2},\ (\overline{a}_{k_1k_2}, \overline{q}_{k_1k_2}) = 1; \\ 2^{\delta j} \leq \overline{q}_{k_1k_2} \leq 2^{(2-\delta)j} \text{ and } |\theta_{k_1k_2} - \overline{a}_{k_1k_2}/\overline{q}_{k_1k_2}| \leq 2^{-2j}. \end{cases} \tag{3.11}$$

Then

$$||\mathcal{U}_j^\theta(g)||_{L^2(\mathbb{Z}^d)} \leq C2^{-\delta' j}||g||_{L^2(\mathbb{Z}^d)},\ \delta' > 0. \tag{3.12}$$

Proof of Lemma 3.2. Clearly, we may assume that $j \geq C$. The kernel of the operator $\mathcal{U}_j^\theta(\mathcal{U}_j^\theta)^*$ is

$$L_j^\theta(m,n) = \sum_{w\in\mathbb{Z}^d} \Phi_j(m-w)\overline{\Phi}_j(n-w)e^{-2\pi i R_0(m-n,w)\cdot\theta}. \tag{3.13}$$

Notice that the kernel L_j^θ is supported in the set $\{(m,n) : |m-n| \leq 2^{j+2}\}$ and the sum in (3.13) is taken over $|w-m| \leq 2^{j+1}$. Let $A_{l_2}(m) = \sum_{l_1=1}^d m_{l_1}\theta_{l_1l_2}$. We write $w = (w_{k_2}, w')$. It follows from (3.13) that

$$|L_j^\theta(m,n)| \leq \sum_{w'\in\mathbb{Z}^{d-1}} \Big| \sum_{w_{k_2}\in\mathbb{Z}} \Phi_j(m-(w_{k_2},w'))\overline{\Phi}_j(n-(w_{k_2},w'))e^{-2\pi i w_{k_2}\cdot A_{k_2}(m-n)} \Big|. \tag{3.14}$$

By summation by parts, it is easy to see that

$$\Big|\sum_{v\in\mathbb{Z}} e^{-2\pi i v\xi} h(v)\Big| \le C\rho(\xi)^{-1}||h'||_{L^1},$$

for any $h\in C^1(\mathbb{R})$, where $\rho(\xi)$ denotes the distance from the real number ξ to $\mathbb{Z}$. Using (3.9), it follows that

$$|L_j^\theta(m,n)| \le C2^{-dj}\mathbf{1}_{[0,2^{j+2}]}(|m-n|)[1+2^j\rho(A_{k_2}(m-n))]^{-1}. \tag{3.15}$$

We estimate $\sum_{n\in\mathbb{Z}^d}|L_j^\theta(m,n)|$ and $\sum_{m\in\mathbb{Z}^d}|L_j^\theta(m,n)|$. We write $m=(m_{k_1},m')$ and $n=(n_{k_1},n')$. Using (3.15),

$$\sum_{n\in\mathbb{Z}^d}|L_j^\theta(m,n)|+\sum_{m\in\mathbb{Z}^d}|L_j^\theta(m,n)| \le C2^{-j}\sup_{\mu\in\mathbb{R}}\sum_{v=-2^{j+2}}^{2^{j+2}}[1+2^j\rho(\theta_{k_1k_2}v+\mu)]^{-1}. \tag{3.16}$$

Thus, for (3.12), it suffices to prove that for some constants $C\ge 1$ and $\delta'>0$,

$$\#\{v\in[-2^{j+2},2^{j+2}]\cap\mathbb{Z} : \rho(\theta_{k_1k_2}v+\mu)\le C^{-1}2^{-(1-\delta')j}\}\le C2^{(1-\delta')j} \tag{3.17}$$

for any $\mu\in\mathbb{R}$ and $j\ge C$. Since $|\theta_{k_1k_2}-\overline{a}_{k_1k_2}/\overline{q}_{k_1k_2}|\le 2^{-2j}$ (see (3.11)), we may replace $\theta_{k_1k_2}$ with $\overline{a}_{k_1k_2}/\overline{q}_{k_1k_2}$ in (3.17). We have two cases: if $\overline{q}_{k_1k_2}\ge 2^{j+4}$, then the set of points $\{\overline{a}_{k_1k_2}v/\overline{q}_{k_1k_2} : v\in[-2^{j+2},2^{j+2}]\cap\mathbb{Z}\}$ is a subset of the set $\{b/\overline{q}_{k_1k_2} : b\in\mathbb{Z}\}$ and $\overline{a}_{k_1k_2}v/\overline{q}_{k_1k_2}-\overline{a}_{k_1k_2}v'/\overline{q}_{k_1k_2}\notin\mathbb{Z}$ if $v\ne v'\in[-2^{j+2},2^{j+2}]\cap\mathbb{Z}$. Using (3.11), $\overline{q}_{k_1k_2}\le 2^{(2-\delta)j}$. Thus the number of points in $\{b/\overline{q}_{k_1k_2} : b\in\mathbb{Z}/(\overline{q}_{k_1k_2}\mathbb{Z})\}$ that lie in an interval of length $C^{-1}2^{-(1-\delta')j}$ is at most $\overline{q}_{k_1k_2}C^{-1}2^{-(1-\delta')j}+1\le C2^{(1-\delta')j}$, as desired.

Assume now that $\overline{q}_{k_1k_2}\le 2^{j+4}$. We divide the interval $[-2^{j+2},2^{j+2}]$ into at most $C2^j/\overline{q}_{k_1k_2}$ intervals J of length $\le\overline{q}_{k_1k_2}/2$. By the same argument as before,

$$\#\{v\in J\cap\mathbb{Z} : \rho(\overline{a}_{k_1k_2}v/\overline{q}_{k_1k_2}+\mu)\le C^{-1}2^{-(1-\delta')j}\}\le\overline{q}_{k_1k_2}C^{-1}2^{-(1-\delta')j}+1,$$

for any of these intervals J and any $\mu\in\mathbb{R}$. The bound (3.17) follows since $2^{\delta j}\le\overline{q}_{k_1k_2}$, see (3.11). □

Lemma 3.3 (Major arcs)**.** *Assume that a/q is an irreducible d'-fraction, $\theta\in\mathbb{R}^{d'}$,*

$$q\le 2^{j/4}\ \text{and}\ |\theta-a/q|\le 2^{-7j/4}. \tag{3.18}$$

Then

$$||\mathcal{U}_j^\theta(g)||_{L^2(\mathbb{Z}^d)}\le Cq^{-1/2}(1+2^{2j}|\theta-a/q|)^{-1/4}||g||_{L^2(\mathbb{Z}^d)}. \tag{3.19}$$

Proof of Lemma 3.3. We may assume $j\ge C$ and let $\theta=a/q+\xi$. Since R_0 is bilinear, we may assume that the functions g and $\mathcal{U}_j^\theta(g)$ are supported in the ball $\{|m|\le C2^j\}$. We write

$$m=qm'+\mu\ \text{ and }\ n=qn'+\nu,$$

with $\mu,\nu\in[Z_q]^d$ and $|m'|,|n'|\leq C2^j/q$, and identify $\mathbb{Z}^d$ with $\mathbb{Z}^d\times[Z_q]^d$ using these maps. Since R_0 is bilinear, it follows from (3.9) and (3.18) that

$$\begin{aligned}&\Phi_j(m-n)e^{-2\pi i R_0(m-n,n)\cdot\theta}\\&=[q^d\Phi_j(q(m'-n'))e^{-2\pi i R_0(m'-n',n')\cdot q^2\xi}]\cdot[q^{-d}e^{-2\pi i R_0(\mu-\nu,\nu)\cdot a/q}]+E(m,n),\end{aligned}\tag{3.20}$$

where $|E(m,n)|\leq C2^{-j/2}2^{-dj}\mathbf{1}_{[0,2^{j+3}]}(|m-n|)$. The operator defined by this error term is bounded on L^2 with bound $C2^{-j/2}$, which suffices. Let $\widetilde{\mathcal{U}}_j^\theta$ denote the operator defined by the first term in (3.20), i.e.

$$\begin{aligned}&\widetilde{\mathcal{U}}_j^\theta(g)(m',\mu)\\&=\sum_{n'\in\mathbb{Z}^d}\sum_{\nu\in[Z_q]^d}g(n',\nu)[q^d\Phi_j(q(m'-n'))e^{-2\pi i R_0(m'-n',n')\cdot q^2\xi}]\cdot[q^{-d}e^{-2\pi i R_0(\mu-\nu,\nu)\cdot a/q}]\\&=\sum_{n'\in\mathbb{Z}^d}\mathcal{S}^{a/q}(g)(n',\mu)\cdot q^d\Phi_j(q(m'-n'))e^{-2\pi i R_0(m'-n',n')\cdot q^2\xi}.\end{aligned}\tag{3.21}$$

In view of Lemma 3.1, for (3.19) it suffices to prove that

$$\begin{aligned}\Big|\Big|\sum_{n'\in\mathbb{Z}^d}g'(n')\cdot q^d\Phi_j(q(m'-n'))e^{-2\pi i R_0(m'-n',n')\cdot q^2\xi}\Big|\Big|_{L^2(\mathbb{Z}^d)}&\\ \leq C(1+2^{2j}|\xi|)^{-1/4}||g'||_{L^2(\mathbb{Z}^d)},&\end{aligned}$$

for any (compactly supported) function $g':\mathbb{Z}^d\to\mathbb{C}$. Using the restriction (3.18), it suffices to prove that

$$||\mathcal{U}_j^\xi(g)||_{L^2(\mathbb{Z}^d)}\leq C(1+2^{2j}|\xi|)^{-1/4}||g||_{L^2(\mathbb{Z}^d)}\text{ if }|\xi|\leq 2^{-5j/4}.\tag{3.22}$$

In proving (3.22) we may assume $|\xi|\geq C2^{-2j}$ (and j large). Fix $k_1,k_2\in\{1,\ldots,d\}$ with the property that $|\xi_{k_1k_2}|\geq C^{-1}|\xi|$. We repeat the $\mathcal{U}_j^\xi(\mathcal{U}_j^\xi)^*$ argument from Lemma 3.2. In view of (3.16), it suffices to prove that

$$2^{-j}\sup_{\mu\in\mathbb{R}}\sum_{v=-2^{j+2}}^{2^{j+2}}[1+2^j\rho(\xi_{k_1k_2}v+\mu)]^{-1}\leq C(2^{2j}|\xi|)^{-1/2},\tag{3.23}$$

provided that $|\xi_{k_1k_2}|\in[2^{-2j},2^{-5j/4}]$ (see (3.22)). The points $\{\xi_{k_1k_2}v+\mu:v\in[-2^{j+2},2^{j+2}]\cap\mathbb{Z}\}$ lie in an interval of length $1/2$. We partition this interval into $C2^j$ subintervals of length 2^{-j}. Each of these subintervals contains at most $C(2^j|\xi_{k1k2}|)^{-1}$ of the points in the set $\{\xi_{k_1k_2}v+\mu:v\in[-2^{j+2},2^{j+2}]\cap\mathbb{Z}\}$. An easy rearrangement argument then shows that the sum in the left-hand side of (3.23) is dominated by

$$C2^{-j}(2^j|\xi_{k1k2}|)^{-1}\sum_{k\in[1,C2^{2j}|\xi_{k1k2}|]\cap\mathbb{Z}}k^{-1},$$

which proves (3.23). □

Our last lemma in this section concerns Calderón–Zygmund kernels. Assume $K_j : \mathbb{R}^d \to \mathbb{C}$, $j \geq 1$, are kernels as in (6.1) and (6.2). For any finite set $I \subseteq \{1, \ldots\}$ we define

$$K^I = \sum_{j \in I} K_j. \tag{3.24}$$

For $\theta \in \mathbb{R}^{d'}$ and (compactly supported) functions $g : \mathbb{Z}^d \to \mathbb{C}$ we define

$$\mathcal{V}_I^\theta(g)(m) = \sum_{n \in \mathbb{Z}^d} K^I(m-n) g(n) e^{-2\pi i R_0(m-n,n)\cdot\theta}. \tag{3.25}$$

Lemma 3.4. *Assume that a/q is an irreducible d'-fraction, $\theta \in \mathbb{R}^{d'}$, and*

$$I \subseteq \{j : q^8 \leq 2^{2j} \leq |\theta - a/q|^{-1}\}. \tag{3.26}$$

Then

$$||\mathcal{V}_I^\theta(g)||_{L^2(\mathbb{Z}^d)} \leq C q^{-1/2} ||g||_{L^2(\mathbb{Z}^d)}. \tag{3.27}$$

Proof of Lemma 3.4. Let $\theta = a/q + \xi$. Since R_0 is bilinear, we may assume that the functions g and $\mathcal{V}_I^\theta(g)$ are supported in the ball $\{m : |m| \leq C|\xi|^{-1/2}\}$. As in Lemma 3.3, we write

$$m = qm' + \mu \ \text{ and } \ n = qn' + \nu,$$

with $\mu, \nu \in [Z_q]^d$ and $|m'|, |n'| \leq C|\xi|^{-1/2}/q$, and identify $\mathbb{Z}^d$ with $\mathbb{Z}^d \times [Z_q]^d$ using these maps. Since R_0 is bilinear, it follows from (3.26) that

$$\begin{aligned} &K^I(m-n)e^{-2\pi i R_0(m-n,n)\cdot\theta} \\ &= [q^d K^I(q(m'-n'))e^{-2\pi i R_0(m'-n',n')\cdot q^2\xi}] \cdot [q^{-d} e^{-2\pi i R_0(\mu-\nu,\nu)\cdot a/q}] + E'(m,n), \end{aligned} \tag{3.28}$$

where $|E'(m,n)| \leq Cq|m-n|^{-d-1/2}\mathbf{1}_{[q^4/2, 2|\xi|^{-1/2}]}(|m-n|)$. The operator defined by this error term in bounded on L^2 with bound Cq^{-1}, which suffices. Let $\widetilde{\mathcal{V}}_I^\theta$ denote the operator defined by the first term in (3.28), i.e.

$$\begin{aligned} &\widetilde{\mathcal{V}}_I^\theta(g)(m', \mu) \\ &= \sum_{n' \in \mathbb{Z}^d} \sum_{\nu \in [Z_q]^d} g(n', \nu)[q^d K^I(q(m'-n'))e^{-2\pi i R_0(m'-n',n')\cdot q^2\xi}] \cdot [q^{-d} e^{-2\pi i R_0(\mu-\nu,\nu)\cdot a/q}] \\ &= \sum_{n' \in \mathbb{Z}^d} \mathcal{S}^{a/q}(g)(n', \mu) \cdot q^d K^I(q(m'-n'))e^{-2\pi i R_0(m'-n',n')\cdot q^2\xi}. \end{aligned}$$

In view of Lemma 3.1, for (3.27) it suffices to prove that

$$\Big|\Big| \sum_{n' \in \mathbb{Z}^d} g'(n') \cdot q^d K^I(q(m'-n'))e^{-2\pi i R_0(m'-n',n')\cdot q^2\xi} \Big|\Big|_{L^2(\mathbb{Z}^d)} \leq C||g'||_{L^2(\mathbb{Z}^d)}, \tag{3.29}$$

for any (compactly supported) function $g' : \mathbb{Z}^d \to \mathbb{C}$.

Since R_0 is bilinear, if $|m'|, |n'| \leq C|\xi|^{-1/2}/q$ then

$$\begin{aligned}|q^d K_j(q(m'-n'))e^{-2\pi i R_0(m'-n',n')\cdot q^2\xi} - q^d K_j(q(m'-n'))|\\ \leq C(2^j|\xi|^{1/2})(2^j/q)^{-d}\mathbf{1}_{[2^{j-1}/q,2^{j+1}/q]}(|m'-n'|).\end{aligned}$$

Thus

$$|q^d K^I(q(m'-n'))e^{-2\pi i R_0(m'-n',n')\cdot q^2\xi} - q^d K^I(q(m'-n'))| \leq E''(m'-n'),$$

where $||E''||_{L^1(\mathbb{Z}^d)} \leq C$. The estimate (3.29) follows from the boundedness of standard singular integrals on $\mathbb{Z}^d$. □

4. The maximal Radon transform

In this section we prove Lemma 2.3. The proof is based on three main ingredients: a strong L^2 bound, a restricted (weak) L^p bound, $p \in (1, 2]$, and an interpolation argument. We assume throughout this section that $d' = d^2$, and $\mathbb{G}_0^\#$ is the discrete nilpotent group defined in section 2.

4.1. L^2 estimates.

The main result in this subsection is Lemma 4.1, which is a quantitative L^2 estimate. The proof of Lemma 4.1 is based on a non-commutative variant of the circle method, in which we divide the Fourier space into major arcs and minor arcs. This partition is achieved using cutoff functions like $\Psi_j^{N,\mathcal{R}}$ defined in (4.6). The minor arcs estimate (4.12) is based on Plancherel theorem and Lemmas 3.2 and 3.3. The major arcs estimate (4.13) is based on the change of variables (4.28), the L^2 boundedness of the standard maximal function on the group $\mathbb{G}_0^\#$, and Lemma 3.1.

In this section we assume $\Omega : \mathbb{R}^d \to [0, 1]$ is a function supported in the set $\{x : |x| \leq 4\}$, and

$$\begin{cases}|\Omega(x)| + |\nabla\Omega(x)| \leq 10 \text{ for any } x \in \mathbb{R}^d;\\ \Omega_j(x) = 2^{-dj}\Omega(x/2^j),\ j = 0, 1, \ldots.\end{cases} \tag{4.1}$$

Clearly, if $\Omega(x) = 1$ in the set $\{x : |x| \leq 1\}$,

$$\mathcal{M}_0^\#(f)(m,u) \leq C \sup_{j\geq 0} \sum_{n\in\mathbb{Z}^d} \Omega_j(n) f((n,0)^{-1}\cdot(m,u)),$$

for any (compactly supported) function $f : \mathbb{G}_0^\# \to [0,\infty)$. For integers $j \geq 0$ and (compactly supported) functions $f : \mathbb{G}_0^\# \to \mathbb{C}$ let

$$\mathcal{M}_j(f)(m,u) = \sum_{n\in\mathbb{Z}^d} \Omega_j(n) f((n,0)^{-1}\cdot(m,u)). \tag{4.2}$$

For Lemma 2.3, it suffices to prove that for any (compactly supported) function $f : \mathbb{G}_0^\# \to \mathbb{C}$,

$$\Big|\Big|\sup_{j\geq 0}|\mathcal{M}_j(f)|\Big|\Big|_{L^p(\mathbb{G}_0^\#)} \leq C_p||f||_{L^p(\mathbb{G}_0^\#)},\ p\in(1,2]. \tag{4.3}$$

For any (compactly supported) function $f : \mathbb{G}_0^\# \to \mathbb{C}$ let $\widehat{f}$ denote its Fourier transform in the central variable, i.e.,

$$\widehat{f}(m,\theta) = \sum_{u\in\mathbb{Z}^{d'}} f(m,u)e^{-2\pi i u\cdot\theta},\ m\in\mathbb{Z}^d,\ \theta\in\mathbb{R}^{d'}. \tag{4.4}$$

Then

$$\widehat{\mathcal{M}_j(f)}(m,\theta) = \sum_{n\in\mathbb{Z}^d}\Omega_j(m-n)\widehat{f}(n,\theta)e^{-2\pi i R_0(m-n,n)\cdot\theta}. \tag{4.5}$$

We use the formula (4.5) and multipliers in the Fourier variable θ to decompose the operators $\mathcal{M}_j$.

Let $\psi : \mathbb{R}^{d'} \to [0,1]$ denote a smooth function supported in the set $\{|\xi|\leq 2\}$ and equal to 1 in the set $\{|\xi|\leq 1\}$. Assume $N\in[1/4,\infty)$, $j\in[0,\infty)\cap\mathbb{Z}$ and $\mathcal{R}\subseteq\mathbb{Q}^{d'}$ is a discrete periodic set (i.e. if $r\in\mathcal{R}$ then $r+a\in\mathcal{R}$ for any $a\in\mathbb{Z}^{d'}$ and $\mathcal{R}\cap[0,1)^{d'}$ is finite). We define

$$\Psi_j^{N,\mathcal{R}}(\theta) = \sum_{r\in\mathcal{R}}\psi(2^{2j}N^{-1}(\theta-r)). \tag{4.6}$$

The function $\Psi_j^{N,\mathcal{R}}$ is periodic in θ (i.e. $\Psi_j^{N,\mathcal{R}}(\theta+a)=\Psi_j^{N,\mathcal{R}}(\theta)$ if $a\in\mathbb{Z}^d$), and supported in the union of the $2N2^{-2j}$-neighborhoods of the points in $\mathcal{R}$. We will always assume that j is sufficiently large (depending on N and $\mathcal{R}$) such that these neighborhoods are disjoint, so $\Psi_j^{N,\mathcal{R}} : \mathbb{R}^{d'}\to[0,1]$. By convention, $\Psi_j^{N,\emptyset}=0$. For (compactly supported) functions $f : \mathbb{G}_0^\# \to \mathbb{C}$ we define $\mathcal{M}_j^{N,\mathcal{R}}(f)$ by

$$\widehat{\mathcal{M}_j^{N,\mathcal{R}}}(f)(m,\theta) = \widehat{\mathcal{M}_j(f)}(m,\theta)\cdot\Psi_j^{N,\mathcal{R}}(\theta). \tag{4.7}$$

Our main lemma in this subsection is the following L^2 estimate:

Lemma 4.1 (Strong L^2 bound)**.** *Assume that $N\in[1/2,\infty)$, $\mathcal{R}_N\subseteq\mathbb{Q}^{d'}$ is a discrete periodic set, and $J_{N,\mathcal{R}_N}\in[0,\infty)$ is a real number with the properties*

$$\begin{cases} \{a/q : q\in[1,N] \text{ and } (a,q)=1\}\subseteq\mathcal{R}_N, \\ 2^{J_{N,\mathcal{R}_N}} \geq [100\max_{a/q\in\mathcal{R}_N \text{ and } (a,q)=1} q]^4. \end{cases} \tag{4.8}$$

Then

$$\Big|\Big|\sup_{j\geq J_{N,\mathcal{R}_N}}|\mathcal{M}_j(f)-\mathcal{M}_j^{N,\mathcal{R}_N}(f)|\Big|\Big|_{L^2(\mathbb{G}_0^\#)} \leq C(N+1)^{-\overline{c}}||f||_{L^2(\mathbb{G}_0^\#)}, \tag{4.9}$$

where $\overline{c}=\overline{c}(d)>0$.

Remark: In section 5, Lemma 5.5, we need to allow slightly more general kernels Ω, that is $\Omega : \mathbb{R}^d \to [0,1]$, supported in the set $\{x : |x| \leq 4\}$, equal to 1 in the set $\{x : |x| \leq 2\}$, and satisfying

$$|\nabla\Omega(x)| \leq A \text{ for any } x \in \mathbb{R}^d,$$

where $A \gg 1$. In this case the bound (4.9) becomes

$$\Big|\Big| \sup_{j \geq J_{N,\mathcal{R}_N}} |\mathcal{M}_j(f) - \mathcal{M}_j^{N,\mathcal{R}_N}(f)|\Big|\Big|_{L^2(\mathbb{G}_0^\#)} \leq A \cdot C(N+1)^{-\bar{c}}||f||_{L^2(\mathbb{G}_0^\#)}.$$

The rest of this subsection is concerned with the proof of Lemma 4.1. The bound (4.3) for $p = 2$ corresponds to the case $N = 1/2$, $\mathcal{R}_{1/2} = \emptyset$, $J_{N,\mathcal{R}_N} = 0$ in Lemma 4.1. The condition (4.8) guarantees that $\Psi_j^{N,\mathcal{R}_N} : \mathbb{R}^{d'} \to [0,1]$ if $j \geq J_{N,\mathcal{R}_N}$. We decompose the operator $\mathcal{M}_j - \mathcal{M}_j^{N,\mathcal{R}_N}$ into the main contribution coming from the "major arcs" (in θ) and an error-type contribution coming from the complement of these major arcs. For integers $j, s \geq 0$ let $\gamma(j,s) = 1$ if $2^s \leq j^{3/2}$, and $\gamma(j,s) = 0$ if $2^s > j^{3/2}$. For (compactly supported) functions $f : \mathbb{G}_0^\# \to \mathbb{C}$ we define $\mathcal{N}_{j,s}^{N,\mathcal{R}_N}(f)$ by

$$\begin{aligned}\widehat{\mathcal{N}_{j,s}^{N,\mathcal{R}_N}}(f)(m,\theta) = \gamma(j,s)\Big[\widehat{\mathcal{M}_j}(f)(m,\theta) - \widehat{\mathcal{M}_j^{N,\mathcal{R}_N}}(f)(m,\theta)\Big]& \\ \Big[\sum_{2^s \leq q < 2^{s+1}} \psi(2^{2j+2}(\theta - a/q))\Big]&,\end{aligned} \tag{4.10}$$

where the sum in the second line of (4.10) is taken over irreducible d'-fractions a/q with $2^s \leq q < 2^{s+1}$. Then we write

$$\mathcal{M}_j(f) - \mathcal{M}_j^{N,\mathcal{R}_N}(f) = \sum_{s \geq 0} \mathcal{N}_{j,s}^{N,\mathcal{R}_N}(f) + \mathcal{E}_j^{N,\mathcal{R}_N}(f). \tag{4.11}$$

This is our basic decomposition. It follows from (4.8) that $\mathcal{N}_{j,s}^{N,\mathcal{R}_N}(f) \equiv 0$ if $2^{s+1} \leq N$. Thus, for Lemma 4.1, it suffices to prove that

$$\Big|\Big|\Big[\sum_{j \geq J_{N,\mathcal{R}_N}} |\mathcal{E}_j^{N,\mathcal{R}_N}(f)|^2\Big]^{1/2}\Big|\Big|_{L^2(\mathbb{G}_0^\#)} \leq C(N+1)^{-\bar{c}}||f||_{L^2(\mathbb{G}_0^\#)}. \tag{4.12}$$

and

$$\Big|\Big| \sup_{j \geq J_{N,\mathcal{R}_N}} |\mathcal{N}_{j,s}^{N,\mathcal{R}_N}(f)|\Big|\Big|_{L^2(\mathbb{G}_0^\#)} \leq C2^{-\bar{c}s}||f||_{L^2(\mathbb{G}_0^\#)} \tag{4.13}$$

if $2^{s+1} \geq N$.

Proof of (4.12) (Minor arcs estimate). Let $s(j)$ denote the largest integer ≥ 0 with the property that $2^{s(j)} \leq j^{3/2}$. Notice that

$$\widehat{\mathcal{E}_j^{N,\mathcal{R}_N}(f)}(m,\theta) = m_j^{N,\mathcal{R}_N}(\theta) \sum_{n \in \mathbb{Z}^d} \Omega_j(m-n)\widehat{f}(n,\theta)e^{-2\pi i R_0(m-n,n)\cdot\theta},$$

with

$$m_j^{N,\mathcal{R}_N}(\theta) = [1 - \Psi_j^{N,\mathcal{R}_N}(\theta)]\Big[1 - \sum_{q \le 2^{s(j)+1}-1} \psi(2^{2j+2}(\theta - a/q))\Big], \tag{4.14}$$

where the sum in (4.14) is taken over irreducible d'-fractions a/q with $q \le 2^{s(j)+1} - 1$. For $\theta \in \mathbb{R}^{d'}$ and (compactly supported) functions $g : \mathbb{Z}^d \to \mathbb{C}$, we define

$$\mathcal{U}_j^\theta(g)(m) = \sum_{n \in \mathbb{Z}^d} \Omega_j(m-n) g(n) e^{-2\pi i R_0(m-n,n)\cdot\theta}. \tag{4.15}$$

By Plancherel theorem

$$\begin{aligned}\Big|\Big|\Big[\sum_{j \ge J_{N,\mathcal{R}_N}} |\mathcal{E}_j^{N,\mathcal{R}_N}(f)|^2\Big]^{1/2}\Big|\Big|^2_{L^2(\mathbb{G}_0^\#)} \\ = \int_{[0,1)^{d'}} \sum_{j \ge J_{N,\mathcal{R}_N}} |m_j^{N,\mathcal{R}_N}(\theta)|^2 ||\mathcal{U}_j^\theta(\widehat{f}(.,\theta))||^2_{L^2(\mathbb{Z}^d)}\, d\theta.\end{aligned}$$

Using Plancherel theorem again, for (4.12) it suffices to prove that

$$\sum_{j \ge J_{N,\mathcal{R}_N}} |m_j^{N,\mathcal{R}_N}(\theta)|^2 ||\mathcal{U}_j^\theta||^2_{L^2(\mathbb{Z}^d)\to L^2(\mathbb{Z}^d)} \le C(N+1)^{-2\overline{c}} \tag{4.16}$$

for any $\theta \in \mathbb{R}^{d'}$ fixed.

By Diriclet's principle, for any $\Lambda \ge 1$ and $\xi \in \mathbb{R}$ there are $q \in Z_\Lambda = \mathbb{Z} \cap [1, \Lambda]$ and $a \in \mathbb{Z}$, $(a,q) = 1$, with the property that $|\xi - a/q| \le 1/(\Lambda q)$. For $\theta \in \mathbb{R}^{d'}$ we apply this to each component $\theta_{l_1l_2}$; thus there are $q_{l_1l_2} \in Z_\Lambda$ and $a_{l_1l_2} \in \mathbb{Z}$, $(a_{l_1l_2}, q_{l_1l_2}) = 1$, with the property that

$$|\theta_{l_1l_2} - a_{l_1l_2}/q_{l_1l_2}| \le C/(\Lambda q_{l_1l_2}). \tag{4.17}$$

Assume that $\theta \in \mathbb{R}^{d'}$ is fixed. For any $j \ge J_{N,\mathcal{R}_N}$ we use the approximation (4.17) with $\Lambda = 2^{(2-\delta)j}$, where $\delta = \delta(d) > 0$ is sufficiently small ($\delta = 1/(10d')$ would work). Thus there are irreducible 1-fractions $a^j_{l_1l_2}/q^j_{l_1l_2}$ such that

$$1 \le q^j_{l_1l_2} \le 2^{(2-\delta)j} \quad \text{and} \quad |\theta_{l_1l_2} - a^j_{l_1l_2}/q^j_{l_1l_2}| \le C/(2^{(2-\delta)j} q^j_{l_1l_2}). \tag{4.18}$$

We fix these irreducible 1-fractions $a^j_{l_1l_2}/q^j_{l_1l_2}$ and partition the set $\mathbb{Z} \cap [J_{N,\mathcal{R}_N}, \infty)$ into two subsets:

$$I_1 = \{j \in \mathbb{Z} \cap [J_{N,\mathcal{R}_N}, \infty) : \max_{l_1,l_2=1,\dots,d} q^j_{l_1l_2} > 2^{j/(6d')}\}$$

and

$$I_2 = \{j \in \mathbb{Z} \cap [J_{N,\mathcal{R}_N}, \infty) : \max_{l_1,l_2=1,\dots,d} q^j_{l_1l_2} \le 2^{j/(6d')}\}.$$

For $j \in I_1$ we use Lemma 3.2:

$$\sum_{j \in I_1} |m_j^{N,\mathcal{R}_N}(\theta)|^2 ||\mathcal{U}_j^\theta||^2_{L^2(\mathbb{Z}^d)\to L^2(\mathbb{Z}^d)} \le \sum_{j \in I_1} 2^{-\delta' j} \le C(N+1)^{-\overline{c}},$$

as desired.

For $j \in I_2$ let a_j/q_j denote the irreducible d'-fraction with the property that $a_j/q_j = (a^j_{l_1l_2}/q^j_{l_1l_2})_{l_1,l_2=1,\dots,d}$. In view of (4.18) and the definition of I_2,

$$1 \le q_j \le 2^{j/6} \quad \text{and} \quad |\theta - a_j/q_j| \le C/2^{(2-\delta)j}. \tag{4.19}$$

An easy argument, using (4.19) shows that if $j, j' \in I_2$ and $j, j' \ge C$ then

$$\text{either } a_j/q_j = a_{j'}/q_{j'} \text{ or } |q_j/q_{j'}| \notin [1/2, 2]. \tag{4.20}$$

We further partition the set I_2,

$$I_2 = \cup_{a/q} I_2^{a/q} \text{ where } I_2^{a/q} = \{j \in I_2 : a_j/q_j = a/q\}. \tag{4.21}$$

For $j \in I_2^{a/q}$ we use Lemma 3.3:

$$\sum_{j\in I_2^{a/q}} |m_j^{N,\mathcal{R}_N}(\theta)|^2 ||\mathcal{U}_j^\theta||^2_{L^2(\mathbb{Z}^d)\to L^2(\mathbb{Z}^d)} \le C \sum_{j\in I_2^{a/q}} q^{-1}(1+2^{2j}|\theta - a/q|)^{-1/2} |m_j^{N,\mathcal{R}_N}(\theta)|^2. \tag{4.22}$$

To estimate the right-hand side of (4.22) we consider two cases: $q \le N$ and $q > N$. If $q \le N$, then, using (4.8), (4.6), and (4.14), $|m_j^{N,\mathcal{R}_N}(\theta)|^2 \le \mathbf{1}_{[1,\infty)}(2^{2j}N^{-1}|\theta - a/q|)$. Thus the right-hand side of (4.22) is dominated by $Cq^{-1}N^{-1/2}$. If $q > N$, then, using (4.14) and the fact that $j \ge 2^{2s(j)/3}$, the right-hand side of (4.22) is dominated by

$$\begin{aligned} &C \sum_{j\in I_2^{a/q}\cap[0,Cq^{2/3}]} q^{-1} \\ &+ C \sum_{j\in I_2^{a/q}\cap[Cq^{2/3},\infty)} q^{-1}(1 + 2^{2j}|\theta - a/q|)^{-1/2} \mathbf{1}_{[1/2,\infty)}(2^{2j}|\theta - a/q|) \le Cq^{-1/3}. \end{aligned}$$

The bound (4.16) follows since the possible denominators q form a lacunary sequence (see (4.20)). This completes the proof of (4.12).

Proof of (4.13) (Major arcs estimate). Clearly, if $j \ge \max(J_{N,\mathcal{R}_N}, 2^{2s/3}, C)$ then

$$\Big[\sum_{2^s\le q<2^{s+1}} \psi(2^{2j+2}(\theta - a/q))\Big](1 - \Psi_j^{N,\mathcal{R}_N}(\theta)) = \sum_{r\in\mathcal{R}'} \psi(2^{2j+2}(\theta - r)),$$

where $\mathcal{R}' = \{a/q \in \mathbb{Q}^{d'} \setminus \mathcal{R}_N : (a, q) = 1 \text{ and } q \in [2^s, 2^{s+1})\}$. We define $\mathcal{M}_j^{1/4,\mathcal{R}'}(f)$ by

$$\widehat{\mathcal{M}_j^{1/4,\mathcal{R}'}}(f)(m,\theta) = \widehat{\mathcal{M}_j(f)}(m,\theta)\Big[\sum_{r\in\mathcal{R}'} \psi(2^{2j+2}(\theta - r))\Big]. \tag{4.23}$$

(compare with (4.7)). Thus, for (4.13), it suffices to prove that if $s \geq 0$ and $\mathcal{R}' \subseteq \{a/q : (a,q) = 1 \text{ and } q \in [2^s, 2^{s+1})\}$ then

$$\Big|\Big| \sup_{j \geq 2^{2s/3}} |\mathcal{M}_j^{1/4,\mathcal{R}'}(f)| \Big|\Big|_{L^2(\mathbb{G}_0^\#)} \leq C2^{-\overline{c}s}||f||_{L^2(\mathbb{G}_0^\#)}.$$

We partition the set $\mathcal{R}' \subseteq \{a/q : (a,q) = 1 \text{ and } q \in [2^s, 2^{s+1})\}$ into at most $C2^{2s/5}$ subsets with the property that each of these subsets contains irreducible d'-fractions with at most $2^{3s/5}$ denominators q. Thus, it suffices to prove that if $s \geq 0$ and

$$\mathcal{R}' \subseteq \{a/q : (a,q) = 1 \text{ and } q \in S\},\ S \subseteq [2^s, 2^{s+1}) \cap \mathbb{Z},\ |S| \leq 2^{3s/5}, \tag{4.24}$$

then

$$\Big|\Big| \sup_{j \geq 2^{2s/3}} |\mathcal{M}_j^{1/4,\mathcal{R}'}(f)| \Big|\Big|_{L^2(\mathbb{G}_0^\#)} \leq C2^{-s/2}||f||_{L^2(\mathbb{G}_0^\#)}. \tag{4.25}$$

In view of the definitions (4.5) and (4.23), and the Fourier inversion formula,

$$\begin{aligned}
&\mathcal{M}_j^{1/4,\mathcal{R}'}(f)(m,u) \\
&= \sum_{(n,v)\in\mathbb{G}_0^\#} f(n,v)\Omega_j(m-n) \int_{[0,1)^{d'}} \Big[\sum_{r\in\mathcal{R}'} \psi(2^{2j+2}(\theta - r))\Big] e^{2\pi i(u-v-R_0(m-n,n))\cdot\theta}\, d\theta \\
&= \sum_{(n,v)\in\mathbb{G}_0^\#} f(n,v)\Omega_j(m-n) \\
&\qquad\qquad \eta_{2^{2j+2}}(u-v-R_0(m-n,n)) \sum_{r\in\mathcal{R}'\cap[0,1)^{d'}} e^{2\pi i(u-v-R_0(m-n,n))\cdot r}
\end{aligned} \tag{4.26}$$

where $\eta(s) = \int_{\mathbb{R}^{d'}} \psi(\xi) e^{2\pi i s\cdot\xi}\, d\xi$ is the Euclidean inverse Fourier transform of ψ, and $\eta_{2^{2j+2}}(s) = 2^{-d'(2j+2)}\eta(s/2^{2j+2})$. We recognize that the formula (4.26) is the convolution on $\mathbb{G}_0^\#$ of the function f and the kernel

$$(m,u) \to \Omega_j(m)\eta_{2^{2j+2}}(u) \sum_{r\in\mathcal{R}'\cap[0,1)^{d'}} e^{2\pi i u\cdot r}.$$

Let $Q = \prod_{q\in S} q$, see (4.24). Since $|S| \leq 2^{3s/5}$,

$$Q \leq 2^{(s+1)2^{3s/5}}. \tag{4.27}$$

To continue, we introduce new coordinates on $\mathbb{G}_0^\#$ adapted to the factor Q. For integers $Q \geq 1$ we define

$$\begin{cases} \Phi_Q : \mathbb{G}_0^\# \times [Z_Q^d \times Z_{Q^2}^{d'}] \to \mathbb{G}_0^\#, \\ \Phi_Q((m',u'),(\mu,\alpha)) = (Qm' + \mu, Q^2u' + \alpha + QR_0(\mu,m')). \end{cases} \tag{4.28}$$

Notice that $\Phi_Q((m',u'),(\mu,\alpha)) = (\mu,\alpha)\cdot(Qm',Q^2u')$ if we regard (μ,α) and (Qm',Q^2u') as elements of $\mathbb{G}_0^\#$. Clearly, the map Φ_Q establishes a bijection between $\mathbb{G}_0^\#\times[Z_Q^d\times Z_{Q^2}^{d'}]$ and $\mathbb{G}_0^\#$. Let $F((n',v'),(\nu,\beta)) = f(\Phi_Q((n',v'),(\nu,\beta)))$ and $G_j((m',u'),(\mu,\alpha)) = \mathcal{M}_j^{1/4,\mathcal{R}'}(f)(\Phi_Q((m',u'),(\mu,\alpha)))$. Since $Qr\in\mathbb{Z}$ for any $r\in\mathcal{R}'$, the formula (4.26) is equivalent to

$$G_j((m',u'),(\mu,\alpha)) = \sum_{(n',v')\in\mathbb{G}_0^\#}\sum_{(\nu,\beta)\in Z_Q^d\times Z_{Q^2}^{d'}} F((n',v'),(\nu,\beta))\Omega_j(Q(m'-n')+E_1)$$
$$\eta_{2^{2j+2}}(Q^2(u'-v'-R_0(m'-n',n'))+E_2)\sum_{r\in\mathcal{R}'\cap[0,1)^{d'}} e^{2\pi i(\alpha-\beta-R_0(\mu-\nu,\nu))\cdot r},$$

where $E_1 = \mu-\nu$ and

$$E_2 = (\alpha-\beta-R_0(\mu-\nu,\nu)) + Q(R_0(\mu,m'-n')-R_0(m'-n',\nu)).$$

In view of (4.25) and (4.27), $2^j\geq 2^{2^{2s/3}}$ and $Q\leq 2^{(s+1)2^{3s/5}}$, thus $C2^j\geq Q^{10}$. Clearly, $|E_1|\leq CQ$ and $|E_2|\leq C2^jQ$ if $|m'-n'|\leq C2^j/Q$. Let

$$\widetilde{G}_j((m',u'),(\mu,\alpha)) = \sum_{(n',v')\in\mathbb{G}_0^\#}\sum_{(\nu,\beta)\in Z_Q^d\times Z_{Q^2}^{d'}} F((n',v'),(\nu,\beta))\Omega_j(Q(m'-n'))$$
$$\eta_{2^{2j+2}}(Q^2(u'-v'-R_0(m'-n',n')))\sum_{r\in\mathcal{R}'\cap[0,1)^{d'}} e^{2\pi i(\alpha-\beta-R_0(\mu-\nu,\nu))\cdot r}. \tag{4.29}$$

In view of the estimates above on $|E_1|$ and $|E_2|$ and the fact that the sum over $r\in\mathcal{R}'\cap[0,1)^{d'}$ in (4.29) has at most $C2^{2s}$ terms, we have

$$|G_j((m',u'),(\mu,\alpha)) - \widetilde{G}_j((m',u'),(\mu,\alpha))|$$
$$\leq C2^{Cs}(Q/2^j)\sum_{(n',v')\in\mathbb{G}_0^\#}\sum_{(\nu,\beta)\in Z_Q^d\times Z_{Q^2}^{d'}} |F((n',v'),(\nu,\beta))|Q^{-d}Q^{-2d'}$$
$$(2^j/Q)^{-d}\mathbf{1}_{[0,C2^j/Q]}(|m'-n'|)\phi_{2^{2j}/Q^2}(u'-v'-R_0(m'-n',n')),$$

where, as in (7.7)

$$\phi(s) = (1+|s|^2)^{-(d'+d+1)/2} \text{ and } \phi_r(s) = r^{-d'}\phi(s/r),\ r\geq 1.$$

Thus,

$$\sum_{j\geq 2^{2s/3}} ||G_j-\widetilde{G}_j||_{L^2(\mathbb{G}_0^\#\times[Z_Q^d\times Z_{Q^2}^{d'}])}\leq C2^{-2^{s/2}}||F||_{L^2(\mathbb{G}_0^\#\times[Z_Q^d\times Z_{Q^2}^{d'}])}.$$

For (4.25) it suffices to prove that

$$\Big|\Big|\sup_{2^j\geq Q}|\widetilde{G}_j|\Big|\Big|_{L^2(\mathbb{G}_0^\#\times[Z_Q^d\times Z_{Q^2}^{d'}])}\leq C2^{-s/2}||F||_{L^2(\mathbb{G}_0^\#\times[Z_Q^d\times Z_{Q^2}^{d'}])}, \tag{4.30}$$

where $\widetilde{G}_j$ is defined in (4.29). For this we notice that the function $\widetilde{G}_j$ is obtained as the composition of the operator

$$\mathcal{A}(f)(\mu,\alpha) = Q^{-d}Q^{-2d'} \sum_{(\nu,\beta)\in Z_Q^d\times Z_{Q^2}^{d'}} f(\nu,\beta) \sum_{r\in\mathcal{R}'\cap[0,1)^{d'}} e^{2\pi i(\alpha-\beta-R_0(\mu-\nu,\nu))\cdot r} \tag{4.31}$$

acting on functions $f : Z_Q^d \times Z_{Q^2}^{d'} \to \mathbb{C}$, followed by an average over a standard ball of radius $\approx 2^j/Q$ in $\mathbb{G}_0^\#$ (with the terminology of section 7). In view (7.11) with $N = 1$, for (4.30) it suffices to prove that

$$||\mathcal{A}(f)||_{L^2(Z_Q^d\times Z_{Q^2}^{d'})} \le C2^{-s/2}||f||_{L^2(Z_Q^d\times Z_{Q^2}^{d'})}. \tag{4.32}$$

For functions $f : Z_Q^d \times Z_{Q^2}^{d'} \to \mathbb{C}$ we define the Fourier transform in the second variable

$$\widetilde{f}(\mu, a/Q^2) = \sum_{\alpha\in\mathbb{Z}_{Q^2}^{d'}} f(\mu,\alpha)e^{-2\pi i\alpha\cdot a/Q^2},\ a \in \mathbb{Z}^{d'}.$$

It is easy to see that

$$||f||_{L^2(Z_Q^d\times Z_{Q^2}^{d'})} = Q^{-d'}\Big(\sum_{\mu\in Z_Q^d}\sum_{a\in\mathbb{Z}_{Q^2}^{d'}} |\widetilde{f}(\mu,a/Q^2)|^2\Big)^{1/2},$$

for any $f : Z_Q^d \times Z_{Q^2}^{d'} \to \mathbb{C}$ (Plancherel's identity). Since $\mathcal{R}' \subseteq \{a/Q^2 : a \in \mathbb{Z}^{d'}\}$ (see (4.24) and the definition of Q), it follows from (4.31) that

$$\widetilde{\mathcal{A}(f)}(\mu, a/Q^2) = \mathbf{1}_{\mathcal{R}'}(a/Q^2)Q^{-d}\sum_{\nu\in Z_Q^d} \widetilde{f}(\nu, a/Q^2)e^{-2\pi iR_0(\mu-\nu,\nu)\cdot a/Q^2}.$$

By Plancherel's identity, for (4.32) it suffices to prove that for any $r \in \mathcal{R}'$ and any $g : Z_q^d \to \mathbb{C}$,

$$\Big|\Big|Q^{-d}\sum_{\nu\in Z_Q^d} g(\nu)e^{-2\pi iR_0(\mu-\nu,\nu)\cdot r}\Big|\Big|_{L^2_\mu(Z_Q^d)} \le C2^{-s/2}||g||_{L^2_\nu(Z_Q^d)}.$$

This follows from Lemma 3.1 and the fact that $r = a/q$, $(a, q) = 1$, $q \in [2^s, 2^{s+1})$ (see (4.24)). This completes the proof of Lemma 4.1.

4.2. **A restricted L^p estimate.** Recall that the operators $\mathcal{M}_j$ were defined in (4.2). In the rest of this section, in addition to (4.1) we assume that $\Omega(x) = 1$ if $|x| \le 2$. In this subsection we prove the following restricted L^p estimate.

Lemma 4.2 (Restricted L^p estimate)**.** *Assume $J \ge 2$ is an integer. Then*

$$\Big|\Big|\sup_{j\in[J+1,2J]}|\mathcal{M}_j(f)|\Big|\Big|_{L^p(\mathbb{G}_0^\#)} \le C_p(\ln J)||f||_{L^p(\mathbb{G}_0^\#)},\ p \in (1, 2]. \tag{4.33}$$

The idea of using restricted L^p estimates like (4.33) together with L^2 bounds to prove the full L^p estimates (4.3) originates in Bourgain's paper [5]. In proving Lemma 4.2 we exploit the positivity of the operators $\mathcal{M}_j$. Let $\widetilde{\Omega}_j:\mathbb{G}_0^\#\to[0,\infty)$ denote the kernel $\widetilde{\Omega}_j(m,u)=\Omega_j(m)\cdot\mathbf{1}_{\{0\}}(u)$, so $\mathcal{M}_j(f)=f*\widetilde{\Omega}_j$, and let $\Omega_j'(h)=\widetilde{\Omega}_j(h^{-1})$. To be able to use the same notation as in the previous section, it is more convenient to prove the maximal inequality

$$\Big|\Big|\sup_{j\in[J+1,2J]}|f*\Omega_j'|\Big|\Big|_{L^p(\mathbb{G}_0^\#)}\leq C_p(\ln J)||f||_{L^p(\mathbb{G}_0^\#)},\ p\in(1,2]. \tag{4.34}$$

The bounds (4.33) and (4.34) are equivalent, in view of the duality argument following the statement of Lemma 2.4. By interpolation, we may assume that $p'=p/(p-1)$ is an integer ≥ 2 and it suffices to prove the $L^p\to L^{p,\infty}$ estimate

$$\Big|\Big|\sup_{j\in[J+1,2J]}|f*\Omega_j'|\Big|\Big|_{L^{p,\infty}(\mathbb{G}_0^\#)}\leq C_p(\ln J)||f||_{L^p(\mathbb{G}_0^\#)},\ p'\in[2,\infty)\cap\mathbb{Z}. \tag{4.35}$$

By duality, the bound (4.35) is equivalent to the inequality

$$\Big|\Big|\sum_{j=J+1}^{2J}f_j*\widetilde{\Omega}_j\Big|\Big|_{L^k(\mathbb{G}_0^\#)}\leq C_k(\ln J)\Big|\Big|\sum_{j=J+1}^{2J}f_j\Big|\Big|_{L^k(\mathbb{G}_0^\#)},$$

where $k=p/(p-1)$ is an integer ≥ 2 and f_j are characteristic functions of disjoint, bounded sets. We may assume $J\geq C_k$ and partition the set $[J+1,2J]\cap\mathbb{Z}$ into at most $C_k(\ln J)$ subsets S with the separation property

$$S\subseteq[J+1,2J]\cap\mathbb{Z}\text{ and if }j\neq j'\in S\text{ then }|j-j'|\geq A_k(\ln J), \tag{4.36}$$

where A_k is a large constant to be fixed later. It suffices to prove that if S is as above and $k\geq 2$ is an integer then

$$\Big|\Big|\sum_{j\in S}f_j*\widetilde{\Omega}_j\Big|\Big|_{L^k(\mathbb{G}_0^\#)}\leq C_k\Big|\Big|\sum_{j\in S}f_j\Big|\Big|_{L^k(\mathbb{G}_0^\#)}, \tag{4.37}$$

where f_j are characteristic functions of disjoint, bounded sets. Let ρ denote the smallest constant $C_k\geq 1$ for which (4.37) holds. By expanding the left-hand side of (4.37),

$$\begin{aligned}\Big|\Big|\sum_{j\in S}f_j*\widetilde{\Omega}_j\Big|\Big|^k_{L^k(\mathbb{G}_0^\#)}&\leq C_k\sum_{j_1<\ldots<j_k}\int_{\mathbb{G}_0^\#}(f_{j_1}*\widetilde{\Omega}_{j_1})\cdot\ldots\cdot(f_{j_k}*\widetilde{\Omega}_{j_k})\,dg\\&+C_k\int_{\mathbb{G}_0^\#}\Big(\sum_{j\in S}f_j*\widetilde{\Omega}_j\Big)^{k-1}dg,\end{aligned} \tag{4.38}$$

since f_j are characteristic functions. The second term in the right-hand side of (4.38) is dominated by $C_k\rho^{k-1}||\sum_{j\in S}f_j||^k_{L^k(\mathbb{G}_0^\#)}$.

To deal with the first term we will prove the bound

$$\Big|\Big|[(f_{j_2} * \widetilde{\Omega}_{j_2}) \cdot \ldots \cdot (f_{j_k} * \widetilde{\Omega}_{j_k})] * (\Omega'_{j_1} - \Omega'_J)\Big|\Big|_{L^2(\mathbb{G}_0^\#)} \le C_k J^{-k} ||f_{j_2} + \ldots + f_{j_k}||_{L^2(\mathbb{G}_0^\#)}, \tag{4.39}$$

provided that $f_{j_2}, \ldots, f_{j_k}$ are characteristic functions of disjoint, bounded sets, $j_1 < \ldots < j_k \in S$, and the constant A_k in (4.36) is sufficiently large. Assuming (4.39), we would have

$$\Big| \int_{\mathbb{G}_0^\#} (f_{j_1} * \widetilde{\Omega}_{j_1}) \cdot \ldots \cdot (f_{j_k} * \widetilde{\Omega}_{j_k})\, dg - \int_{\mathbb{G}_0^\#} (f_{j_1} * \widetilde{\Omega}_J) \cdot \ldots \cdot (f_{j_k} * \widetilde{\Omega}_{j_k})\, dg \Big|$$
$$\le C_k J^{-k} \Big|\Big| \sum_{j\in S} f_j \Big|\Big|^2_{L^2(\mathbb{G}_0^\#)} = C_k J^{-k} \Big|\Big| \sum_{j\in S} f_j \Big|\Big|^k_{L^k(\mathbb{G}_0^\#)},$$

since f_j are characteristic functions of disjoint, bounded sets. Thus the first term in the right-hand side of (4.38) can be estimated by

$$C_k \Big|\Big| \sum_{j\in S} f_j \Big|\Big|^k_{L^k(\mathbb{G}_0^\#)} + C_k \sum_{j_2 < \ldots < j_k} \int_{\mathbb{G}_0^\#} (\sum_{j\in S} f_j * \widetilde{\Omega}_J) \cdot \ldots \cdot (f_{j_k} * \widetilde{\Omega}_{j_k})\, dg. \tag{4.40}$$

Since f_j are characteristic functions of disjoint, bounded sets, $\sum_{j\in S} f_j * \widetilde{\Omega}_J \le C$. Thus the expression in (4.40) can be estimated by $C_k(1+\rho^{k-1})||\sum_{j\in S} f_j||^k_{L^k(\mathbb{G}_0^\#)}$. It follows from (4.38) that $\rho^k \le C_k(1+\rho^{k-1})$, so $\rho \le C_k$ as desired.

It remains to prove the bound (4.39). Clearly, we may assume $J \ge C_k$. We start with a sequence of appropriate constants $B_2 < \ldots < B_k$, which depend only on the constant $\overline{c} > 0$ in Lemma 4.1, and define $N_l = J^{B_l}$ and $\mathcal{R}_{N_l} = \{a/q : q \in [1, N_l] \text{ and } (a,q) = 1\}$, $l = 2, \ldots, k$. By Lemma 4.1,

$$||\mathcal{M}_{j_l}(f_{j_l}) - \mathcal{M}_{j_l}^{N_l, \mathcal{R}_{N_l}}(f_{j_l})||_{L^2(\mathbb{G}_0^\#)} \le C J^{-\overline{c} B_l} ||f_{j_l}||_{L^2(\mathbb{G}_0^\#)},\ l = 2, \ldots, k. \tag{4.41}$$

A computation similar to (4.26) shows that

$$\begin{cases} \mathcal{M}_{j_l}^{N_l, \mathcal{R}_{N_l}}(f_{j_l}) = f_{j_l} * L_{j_l}^{N_l, \mathcal{R}_{N_l}}; \\ L_{j_l}^{N_l, \mathcal{R}_{N_l}}(m,u) = \Omega_{j_l}(m) \eta_{2^{2j_l}/N_l}(u) \sum_{r \in \mathcal{R}_{N_l} \cap [0,1)^{d'}} e^{2\pi i u \cdot r}. \end{cases} \tag{4.42}$$

Since $\mathcal{R}_{N_l}$ has at most $CJ^{(d'+1)B_l}$ elements, $||L_{j_l}^{N_l}||_{L^1(\mathbb{G}_0^\#)} \le CJ^{(d'+1)B_l}$. Thus

$$||\mathcal{M}_{j_l}^{N_l, \mathcal{R}_{N_l}}(f_{j_l})||_{L^\infty(\mathbb{G}_0)} \le C J^{(d'+1)B_l},\ l = 2, \ldots, k, \tag{4.43}$$

since f_{j_l} are characteristic functions of sets. We now estimate the left-hand side of (4.39) by

$$\begin{aligned} &||\mathcal{M}_{j_2}(f_{j_2}) - \mathcal{M}_{j_2}^{N_2, \mathcal{R}_{N_2}}(f_{j_2})||_{L^2} \cdot \ldots \cdot ||\mathcal{M}_{j_k}(f_{j_k})||_{L^\infty} + \ldots \\ &+ ||\mathcal{M}_{j_2}^{N_2, \mathcal{R}_{N_2}}(f_{j_2}))||_{L^\infty} \cdot \ldots \cdot ||\mathcal{M}_{j_k}(f_{j_k}) - \mathcal{M}_{j_k}^{N_k, \mathcal{R}_{N_k}}(f_{j_k})||_{L^2} \\ &+ ||[\mathcal{M}_{j_2}^{N_2, \mathcal{R}_{N_2}}(f_{j_2}) \cdot \ldots \cdot \mathcal{M}_{j_k}^{N_k, \mathcal{R}_{N_k}}(f_{j_k})] * (\Omega'_{j_1} - \Omega'_J)||_{L^2}. \end{aligned} \tag{4.44}$$

By choosing the constants B_l in geometric progression and using (4.41) and (4.43), for (4.39) it remains to control the last term in (4.44). We examine now the formula (4.42) and notice that each kernel $L_{j_l}^{N_l,\mathcal{R}_{N_l}}$ is the sum over r of at most $C_k J^{C_k}$ kernels. For any irreducible d'-fraction a_l/q_l let

$$L_{j_l}^{N_l,a_l/q_l}(m,u) = \Omega_{j_l}(m)\eta_{2^{2j_l}/N_l}(u)e^{2\pi iu\cdot a_l/q_l}. \tag{4.45}$$

To control the last term in (4.44) it suffices to prove the following:

Lemma 4.3. *With the notation above, for any constant* $\widetilde{B}_k$

$$||[(f_{j_2} * L_{j_2}^{N_2,a_2/q_2})\cdot\ldots\cdot(f_{j_k} * L_{j_k}^{N_k,a_k/q_k})] * (\Omega'_{j_1} - \Omega'_J)||_{L^2} \leq C_k J^{-\widetilde{B}_k}||f_{j_2} + \ldots + f_{j_k}||_{L^2}, \tag{4.46}$$

provided that $f_{j_2},\ldots,f_{j_k}$ *are characteristic functions of disjoint, bounded sets,* $N_l \leq J^{\widetilde{B}_k}$, a_l/q_l *are irreducible* d'*-fractions with* $q_l \leq J^{\widetilde{B}_k}$, $l = 2,\ldots,k$, $J < j_1 < j_2 < \ldots < j_k \leq 2J$, *and* $j_2 - j_1 \geq A_k(\ln J)$, A_k *sufficiently large depending on* $\widetilde{B}_k$.

Proof of Lemma 4.3. From the definitions,

$$\begin{aligned}&[(f_{j_2} * L_{j_2}^{N_2,a_2/q_2})\cdot\ldots\cdot(f_{j_k} * L_{j_k}^{N_k,a_k/q_k})] * (\Omega'_{j_1} - \Omega'_J)(g)\\ &= \int_{[\mathbb{G}_0^{\#}]^{k-1}} f_{j_2}(h_2)\cdot\ldots\cdot f_{j_k}(h_k)H(g\cdot h_2^{-1},\ldots,g\cdot h_k^{-1})\,dh_2\ldots dh_k,\end{aligned} \tag{4.47}$$

where

$$\begin{aligned}H(g_2,\ldots,g_k) = \sum_{n\in\mathbb{Z}^d}&(\Omega_{j_1}(n) - \Omega_J(n))\\ &L_{j_2}^{N_2,a_2/q_2}((n,0)\cdot g_2)\cdot\ldots\cdot L_{j_k}^{N_k,a_k/q_k}((n,0)\cdot g_k).\end{aligned} \tag{4.48}$$

Let $g_l = (m_l,u_l)$, $l = 2,\ldots,k$. With ϕ as in (7.7), we show that

$$|H(g_2,\ldots,g_k)| \leq C_k J^{\widetilde{B}_k}[2^{j_1-j_2} + 2^{-J/2}]\prod_{l=2}^k \Omega_{j_l+2}(m_l)\phi_{2^{2j_l}/N_l}(u_l). \tag{4.49}$$

Assuming (4.49), the bound (4.46) follows easily from (4.47) and the fact that f_{j_l} are characteristic functions.

To prove (4.49) let

$$Q = q_2\cdot\ldots\cdot q_k,\ Q \leq J^{(k-1)\widetilde{B}_k}. \tag{4.50}$$

Writing $n = Qn' + \nu$, $n' \in \mathbb{Z}^d$, $\nu \in Z_Q^d$, the formula (4.48) becomes

$$|H(g_2, \dots, g_k)| = \Big| \sum_{n' \in \mathbb{Z}^d} \sum_{\nu \in Z_Q^d} (\Omega_{j_1}(Qn' + \nu) - \Omega_J(Qn' + \nu)) \\ \prod_{l=2}^{k} \Omega_{j_l}(m_l + Qn' + \nu)\eta_{2^{2j_l}/N_l}(u_l + R_0(Qn' + \nu, m_l))e^{2\pi i R_0(\nu, m_l)\cdot a_l/q_l} \Big|. \tag{4.51}$$

We use the bound (4.50) on Q and the observation that $|n'| \leq 1002^{j_1}/Q$ in (4.51). It follows that

$$|\Omega_{j_l}(m_l + Qn' + \nu)\eta_{2^{2j_l}/N_l}(u_l + R_0(Qn' + \nu, m_l)) \\ - \Omega_{j_l}(m_l)\eta_{2^{2j_l}/N_l}(u_l)| \leq CN_l 2^{j_1 - j_l}\Omega_{j_l+2}(m_l)\phi_{2^{2j_l}/N_l}(u_l).$$

Thus, using (4.51)

$$|H(g_2, \dots, g_k)| \leq C_k \prod_{l=2}^{k} \Omega_{j_l+2}(m_l)\phi_{2^{2j_l}/N_l}(u_l) \\ \Big[J^{\widetilde{B}_k} 2^{j_1 - j_2} + \Big| \sum_{n' \in \mathbb{Z}^d} \sum_{\nu \in Z_Q^d} (\Omega_{j_1}(Qn' + \nu) - \Omega_J(Qn' + \nu)) \prod_{l=2}^{k} e^{2\pi i R_0(\nu, m_l)\cdot a_l/q_l} \Big| \Big]. \tag{4.52}$$

We make the simple observations $|\Omega_{j_1}(Qn'+\nu) - \Omega_{j_1}(Qn')| \leq CQ2^{-j_1}\Omega_{j_1+2}(Qn')$, $|\Omega_J(Qn' + \nu) - \Omega_J(Qn')| \leq CQ2^{-J}\Omega_J(Qn')$, since $|\nu| \leq Q$. In addition, since $\int_{\mathbb{R}^d}[\Omega_{j_1}(x') - \Omega_J(x')]\, dx' = 0$, we have

$$Q^d \Big| \sum_{n' \in \mathbb{Z}^d} [\Omega_{j_1}(Qn') - \Omega_J(Qn')] \Big| \leq CQ2^{-J}.$$

The bound (4.49) follows from (4.52). □

4.3. **Proof of Lemma 2.3.** In this subsection we prove the bound (4.3) for any $p > 1$, thus completing the proof of Lemma 2.3. Our main ingredients are the bound (7.11) in section 7, Lemma 4.1, and Lemma 4.2. The bound (4.3) follows by interpolation (see [8, Section 7]) from the following more quantitative estimate.

Lemma 4.4. *Assume $p \in (1, 2]$ is an exponent and $\epsilon = (p - 1)/2$. Then, for any $\lambda \in (0, \infty)$, there are linear operators $\mathcal{A}_j^\lambda = \mathcal{A}_j^{\lambda,\epsilon}$ and $\mathcal{B}_j^\lambda = \mathcal{B}_j^{\lambda,\epsilon}$ with $\mathcal{M}_j = \mathcal{A}_j^\lambda + \mathcal{B}_j^\lambda$,*

$$\Big|\Big| \sup_{j \geq 0} |\mathcal{A}_j^\lambda(f)| \Big|\Big|_{L^2(\mathbb{G}_0^\#)} \leq C_\epsilon/\lambda ||f||_{L^2(\mathbb{G}_0^\#)}, \tag{4.53}$$

and

$$\Big|\Big| \sup_{j \geq 0} |\mathcal{B}_j^\lambda(f)| \Big|\Big|_{L^p(\mathbb{G}_0^\#)} \leq C_\epsilon \lambda^\epsilon ||f||_{L^p(\mathbb{G}_0^\#)}. \tag{4.54}$$

The rest of this subsection is concerned with the proof of Lemma 4.4. In view of Lemma 4.1 with $N = 1/2, \mathcal{R}_{1/2} = \emptyset$, in proving Lemma 4.4 we may assume $\lambda \geq C_\epsilon$. With $\overline{c}$ as in Lemma 4.1, we define

$$\begin{cases} N_0 = \lambda^{1/\overline{c}}; \\ \mathcal{R}_{N_0} = \{a/N_0! : a \in \mathbb{Z}^{d'}\}; \\ J_{N_0,\mathcal{R}_{N_0}} = N_0^2. \end{cases} \tag{4.55}$$

The property (4.8) is clearly satisfied if λ is sufficiently large. For $j < J_{N_0,\mathcal{R}_{N_0}}$, let $\mathcal{A}_j^\lambda \equiv 0$, $\mathcal{B}_j^\lambda \equiv \mathcal{M}_j$. By Lemma 4.2,

$$\Big|\Big| \sup_{j \in [0, J_{N_0,\mathcal{R}_{N_0}}) \cap \mathbb{Z}} |\mathcal{B}_j^\lambda(f)| \Big|\Big|_{L^p(\mathbb{G}_0^\#)} \leq C(\ln \lambda)^2 ||f||_{L^p(\mathbb{G}_0^\#)},$$

which is better than (4.54). For $j \geq J_{N_0,\mathcal{R}_{N_0}}$, let $\mathcal{A}_j^\lambda \equiv \mathcal{M}_j - \mathcal{M}_j^{N_0,\mathcal{R}_{N_0}}$, $\mathcal{B}_j^\lambda \equiv \mathcal{M}_j^{N_0,\mathcal{R}_{N_0}}$. By Lemma 4.1 and the definition (4.55),

$$\Big|\Big| \sup_{j \geq J_{N_0,\mathcal{R}_{N_0}}} |\mathcal{A}_j^\lambda(f)| \Big|\Big|_{L^2(\mathbb{G}_0^\#)} \leq C/\lambda ||f||_{L^2(\mathbb{G}_0^\#)},$$

which gives (4.53). To complete the proof of Lemma 4.4 it suffices to show that

$$\Big|\Big| \sup_{j \geq J_{N_0,\mathcal{R}_{N_0}}} |\mathcal{M}_j^{N_0,\mathcal{R}_{N_0}}(f)| \Big|\Big|_{L^p(\mathbb{G}_0^\#)} \leq C_p(\ln N_0) ||f||_{L^p(\mathbb{G}_0^\#)}. \tag{4.56}$$

To prove (4.56) we use (7.11) and the change of coordinates (4.28). By the Fourier inversion formula, as in (4.26),

$$\begin{aligned} \mathcal{M}_j^{N_0,\mathcal{R}_{N_0}}(f)(m,u) = & \sum_{(n,v) \in \mathbb{G}_0^\#} f(n,v)\Omega_j(m-n) \\ & \eta_{2^{2j}/N_0}(u - v - R_0(m-n,n)) \sum_{r \in \mathcal{R}_{N_0} \cap [0,1)^{d'}} e^{2\pi i (u-v-R_0(m-n,n)) \cdot r}. \end{aligned} \tag{4.57}$$

Let $Q = N_0!$. The definition (4.55) shows that

$$\sum_{r \in \mathcal{R}_{N_0} \cap [0,1)^{d'}} e^{2\pi i (u-v-R_0(m-n,n)) \cdot r} = \delta_Q(u - v - R_0(m-n,n)),$$

where

$$\delta_Q : \mathbb{Z}^{d'} \to \mathbb{Z},\ \delta_Q(u) = \begin{cases} Q^{d'} & \text{if } u/Q \in \mathbb{Z}^{d'}; \\ 0 & \text{if } u/Q \notin \mathbb{Z}^{d'}. \end{cases} \tag{4.58}$$

We use the change of coordinates $\Phi_Q : \mathbb{G}_0^\# \times [Z_Q^d \times Z_{Q^2}^{d'}] \to \mathbb{G}_0^\#$ described in (4.28). Let $F((n',v'),(\nu,\beta)) = f(\Phi_Q((n',v'),(\nu,\beta)))$ and $G_j((m',u'),(\mu,\alpha)) =$

$\mathcal{M}_j^{N_0,\mathcal{R}_{N_0}}(f)(\Phi_Q((m',u'),(\mu,\alpha)))$. The formula (4.57) is equivalent to

$$G_j((m',u'),(\mu,\alpha)) = \sum_{(n',v')\in\mathbb{G}_0^{\#}} \sum_{(\nu,\beta)\in Z_Q^d\times Z_{Q^2}^{d'}} F((n',v'),(\nu,\beta))\Omega_j(Q(m'-n')+E_1)$$
$$\eta_{2^{2j}/N_0}(Q^2(u'-v'-R_0(m'-n',n'))+E_2)\cdot\delta_Q(\alpha-\beta-R_0(\mu-\nu,\nu)),$$

where $E_1=\mu-\nu$ and

$$E_2=(\alpha-\beta-R_0(\mu-\nu,\nu))+Q(R_0(\mu,m'-n')-R_0(m'-n',\nu)).$$

Clearly, $2^j\geq 2^{N_0^2}$ and $Q\leq 2^{{N_0}^{3/2}}$, thus $2^j\geq Q^{10}$. Also, $|E_1|\leq CQ$ and $|E_2|\leq C2^jQ$ if $|m'-n'|\leq C2^j/Q$. Let

$$\widetilde{G}_j((m',u'),(\mu,\alpha)) = \sum_{(n',v')\in\mathbb{G}_0^{\#}} \sum_{(\nu,\beta)\in Z_Q^d\times Z_{Q^2}^{d'}} F((n',v'),(\nu,\beta))\Omega_j(Q(m'-n'))$$
$$\eta_{2^{2j}/N_0}(Q^2(u'-v'-R_0(m'-n',n')))\cdot\delta_Q(\alpha-\beta-R_0(\mu-\nu,\nu)).$$

In view of the estimates above on $|E_1|$ and $|E_2|$, we have

$$|G_j((m',u'),(\mu,\alpha))-\widetilde{G}_j((m',u'),(\mu,\alpha))|$$
$$\leq C(N_0Q/2^j)\sum_{(n',v')\in\mathbb{G}_0^{\#}} \sum_{(\nu,\beta)\in Z_Q^d\times Z_{Q^2}^{d'}} |F((n',v'),(\nu,\beta))|Q^{-d}Q^{-2d'}$$
$$(2^j/Q)^{-d}\mathbf{1}_{[0,C2^j/Q]}(|m'-n'|)\phi_{2^{2j}/(Q^2N_0)}(u'-v'-R_0(m'-n',n')).$$

where ϕ is as in (7.7). Thus,

$$\sum_{j\geq N_0^2}||G_j-\widetilde{G}_j||_{L^p(\mathbb{G}_0^{\#}\times[Z_Q^d\times Z_{Q^2}^{d'}])}\leq C||F||_{L^p(\mathbb{G}_0^{\#}\times[Z_Q^d\times Z_{Q^2}^{d'}])}.$$

For (4.56) it remains to prove that

$$\Big|\Big|\sup_{j\geq N_0^2}|\widetilde{G}_j|\Big|\Big|_{L^p(\mathbb{G}_0^{\#}\times[Z_Q^d\times Z_{Q^2}^{d'}])}\leq C_p(\ln N_0)||F||_{L^p(\mathbb{G}_0^{\#}\times[Z_Q^d\times Z_{Q^2}^{d'}])}. \tag{4.59}$$

For this we notice that the function $\widetilde{G}_j$ is obtained as the composition of the operator

$$f\to Q^{-d}Q^{-2d'}\sum_{(\nu,\beta)\in Z_Q^d\times Z_{Q^2}^{d'}} f(\nu,\beta)\delta_Q(\alpha-\beta-R_0(\mu-\nu,\nu))$$

acting on functions $f:Z_Q^d\times Z_{Q^2}^{d'}\to\mathbb{C}$, which is clearly bounded on $L^p(Z_Q^d\times Z_{Q^2}^{d'})$, followed by an average dominated by the maximal operator $\mathcal{M}_*^{N_0}$ of Lemma 7.1. The bound (4.59) follows from (7.11).

5. The ergodic theorem

In this section we prove Theorem 1.3. We first reduce matters to proving Theorem 5.1 below (in fact, we only need this theorem for a special group $\mathbb{G}^\#$ and a special polynomial mapping P). Then we use a maximal ergodic theorem (which follows from Theorem 1.1 and a transference argument) and adapt a limiting argument of Bourgain [5].

5.1. **Preliminary reductions and a maximal ergodic theorem.** Assume (X,μ) is a finite measure space. A result equivalent to Theorem 1.3 can be formulated in terms of the action of the step 2 discrete nilpotent group $\mathbb{G}^\#$ defined in (2.2) and (2.4), corresponding to a bilinear mapping $R : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^{d'}$. Suppose $\mathbb{G}^\#$ acts on X via measure preserving transformations, and denote the action $\mathbb{G}^\# \times X \to X$ by $(g,x) \to g \cdot x$. For a polynomial map

$$P : \mathbb{Z}^d \to \mathbb{Z}^{d'} \text{ of degree at most } 2$$

and $F \in L^p(X)$, $p \in (1,\infty]$, define the averages

$$M_r(F)(x) = \frac{1}{|B_r \cap \mathbb{Z}^d|} \sum_{n \in B_r \cap \mathbb{Z}^d} F\left((n, P(n)) \cdot x\right). \tag{5.1}$$

Theorem 5.1. *For every $F \in L^p(X)$, $p \in (1,\infty)$, there exists $F_* \in L^p(X)$ such that*

$$\lim_{r\to\infty} M_r(F) = F_* \text{ almost everywhere and in } L^p. \tag{5.2}$$

Moreover, if the action of the subgroup $(q\mathbb{Z})^d \times (q\mathbb{Z})^{d'}$ is ergodic on X for every integer $q \geq 1$, then

$$F_* = \frac{1}{\mu(X)} \int_X F\, d\mu. \tag{5.3}$$

We prove now the equivalence of Theorems 1.1 and 5.1, and reduce matters to proving Theorem 5.1 on a special discrete group $\mathbb{G}^\#$ with special polynomial map P_0. We show first that Theorem 1.3 implies Theorem 5.1. Assume that $\mathbb{G}^\#$ is as in Theorem 5.1 and acts on X via measure-preserving transformations. For $g \in \mathbb{G}^\#$ define the transformation $T_g : X \to X$ by $T_g(x) = g \cdot x$. Let $\{g_i, h_j : i = 1, \dots, d,\ j = 1, \dots, d'\}$ denote the standard basis of $\mathbb{Z}^d \times \mathbb{Z}^{d'}$, and let $T_i = T_{g_i}$, $S_j = T_{h_j}$. For $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$, $m = (m_1, \dots, m_{d'}) \in \mathbb{Z}^{d'}$ it follows from the definitions that

$$\prod_{1\leq i\leq d} T_i^{n_i} \prod_{1\leq l\leq d'} S_l^{m_l} = T_{(n,\, m+Q_0(n))} \tag{5.4}$$

where $Q_0 : \mathbb{Z}^d \to \mathbb{Z}^{d'}$ is a polynomial mapping of degree 2. Thus the averages in (5.1) reduce to those in (1.12) associated to the polynomial map $Q(n) = P(n) - Q_0(n)$. Also it is clear from (5.4) that the family of transformations

$T_i^q\ _{(1\leq i\leq d)}$, $S_l^q\ _{(1\leq l\leq d')}$ generate the subgroup $\mathbb{G}_q^\# = (q\mathbb{Z})^d \times (q\mathbb{Z})^{d'}$, hence the ergodicity of the action of the subgroup implies that of the family T_i^q, S_l^q.

We start now the proof of Theorem 1.3. Notice that the coefficients of the polynomials $Q_l : \mathbb{Z}^d \to \mathbb{Z}$ of degree at most 2 must be integers or half integers. Writing $n_i = 2n_i' + \varepsilon_i$ for some fixed residue classes $\varepsilon_{i\ (1\leq i\leq d)}$ modulo 2, it follows that the average in (1.12) can be written as a linear combination of 2^d averages, where the exponents are polynomials with integer coefficients. Thus one can assume that the polynomial mapping Q in (1.11) has integer coefficients. Also, one may write

$$\prod_{1\leq l\leq d'} S_l^{Q_l(n)} = \bar{S}_0 \prod_{1\leq i\leq d} \bar{S}_i^{n_i} \prod_{1\leq i\leq j\leq d} \bar{S}_{ij}^{n_in_j} \tag{5.5}$$

by expanding $S_l^{Q_l(n)}$ into a product of factors with monomial exponents n_i and n_in_j, and collecting all the resulting factors with a given exponent. If one puts $\bar{T}_i = T_i\bar{S}_i\ _{(1\leq i\leq d)}$, then the transformations $\bar{T}_i\ _{(1\leq i\leq d)}$, $\bar{S}_{ij}\ _{(1\leq i\leq j\leq d)}$ satisfy (1.10). Moreover, the ergodicity of the family $\bar{T}_i^q\ _{(1\leq i\leq d)}$, ${\bar{S}_{ij}}^q\ _{(1\leq i\leq j\leq d)}$ implies that of the family T_i^q, $S_l^q\ _{(1\leq i\leq d,\ 1\leq l\leq d')}$. Thus it is enough to prove Theorem 1.3 for the special polynomial map

$$Q_0 : \mathbb{Z}^d \to \mathbb{Z}^{\frac{d(d+1)}{2}} \quad \text{with} \quad Q_0^{ij}(n_1,\ldots,n_d) = n_in_j \quad (1\leq i\leq j\leq d). \tag{5.6}$$

We identify the group generated by the transformations $\bar{T}_i\ _{(1\leq i\leq d)}, \bar{S}_{ij}\ _{(1\leq i\leq j\leq d)}$ as a isomorphic image of a step 2 nilpotent group $\mathbb{G}^\#$ on $\mathbb{Z}^d\times\mathbb{Z}^{d^2}$. More precisely it follows from (1.10) that

$$\prod_{1\leq i\leq d} T_i^{n_i} \prod_{1\leq j\leq d} T_j^{n_j'} = \prod_{1\leq i\leq d} T_i^{n_i+n_i'} \prod_{1\leq j<i\leq d} [T_i,T_j]^{n_in_j'} \tag{5.7}$$

This implies that the group $\bar{\mathbb{G}}_0^\#$ defined by the bilinear form $\bar{R}_0 : \mathbb{Z}^d\times\mathbb{Z}^d \to \mathbb{Z}^{d^2}$ with components

$$\bar{R}_0^{ij}(n,n') = \begin{cases} n_in_j' & \text{if } 1\leq j<i\leq d \\ 0 & \text{if } 1\leq i\leq j\leq d \end{cases} \tag{5.8}$$

acts on X via

$$(n,m)\cdot x = \prod_{1\leq i\leq d} \bar{T}_i^{n_i} \prod_{1\leq j<i\leq d} [\bar{T}_i,\bar{T}_j]^{m_{ij}} \prod_{1\leq i\leq j\leq d} \bar{S}_{ij}^{m_{ij}}(x) \tag{5.9}$$

where $n = (n_{i\ (1\leq i\leq d)})$, $m = (m_{ij\ (1\leq i,j\leq d)})$. In terms of this action the averages in (1.12) take the form

$$A_r(F)(x) = \frac{1}{|B_r\cap\mathbb{Z}^d|} \sum_{n\in B_r\cap\mathbb{Z}^d} F\left((n,0,P_0(n))\cdot x\right) \tag{5.10}$$

Thus Theorem 1.3 reduces to Theorem 5.1 in the special case $d' = d^2$,

$$\begin{cases} P(n) = \sum_{1\le i\le j\le d} n_i n_j \cdot e_{ij}; \\ R(n,n') = \sum_{1\le j<i\le d} n_i n'_j \cdot e_{ij}, \end{cases} \tag{5.11}$$

where $\{e_{ij} : i, j = 1, \dots, d\}$ denotes a standard orthonormal basis of $\mathbb{R}^{d^2}$.

We conclude this subsection with a maximal ergodic theorem, which follows from Theorem 1.1 and a general transference argument.

Theorem 5.2 (Maximal ergodic theorem). *With the notation in Theorem 5.1, let* $\mathcal{M}(F)(x) = \sup_{r>0} |M_r F(x)|$. *Then*

$$\| \mathcal{M}(F) \|_{L^p(X)} \le C_p \|F\|_{L^p(X)}. \tag{5.12}$$

Using Theorem 5.2 and the Lebesgue dominated convergence theorem, it suffices to prove the almost everywhere convergence in (5.2). We can also assume in Theorem 5.1 that F is in a suitable dense subspace of $L^p(X)$, such as $L^\infty(X)$.

5.2. **Pointwise convergence.** Assume $F \in L^\infty(X)$ and, for given $1 < \delta \le 2$, define the averages

$$\mathcal{M}_j^\delta(F)(x) = \frac{1}{\|\Omega^\delta\|_{L^1}\,\delta^j} \sum_{n\in\mathbb{Z}^d} \Omega^\delta(n/\delta^j)\, F((n, P(n))\cdot x) \tag{5.13}$$

where $\Omega^\delta : \mathbb{R}^d \to [0,1]$ is a smooth function, such that $\Omega^\delta(y) = 1$ for $|y| \le 1$ and $\Omega^\delta(y) = 0$ for $|y| \ge \delta$. For a given $r > 1$ let j be such, that $\delta^j \le r < \delta^{j+1}$ and compare the averages $M_r(F)$ and $\mathcal{M}_j^\delta(F)$. Since $F \in L^\infty$, it follows easily that for any $x \in X$

$$|M_r(F)(x) - \mathcal{M}_j^\delta(F)(x)| \le C_d\,(\delta^{-j} + \delta^d - 1)\,\|F\|_{L^\infty}.$$

Thus it suffices to show that for each $1 < \delta \le 2$ the averages $\mathcal{M}_j^\delta(F)$ converge almost everywhere as $j \to \infty$. For simplicity of notation, we drop the superscript δ and write $\mathcal{M}_j(F) = \mathcal{M}_j^\delta(F)$.

Next, we identify subspaces of $L^2(X)$ on which the convergence of $\mathcal{M}_j(F)$ is immediate. For integers $q \ge 1$ let $\mathbb{G}_q^\# = (q\mathbb{Z})^d \times (q\mathbb{Z})^{d'}$ i.e. the subgroup of points whose each coordinate is divisible by q. Define the corresponding space of invariant functions by

$$L_q^2(X) = \{F \in L^2(X) :\ T_g F = F \ \ \forall\, g \in \mathbb{G}_q^\#\} \quad L_{inv}^2(X) = \overline{\bigcup_{q\ge1} L_q^2(X)}, \tag{5.14}$$

where $T_g F(x) = F(g\cdot x)$. Notice that $L_{q_1}^2(X) \subseteq L_{q_2}^2(X)$ if q_1 divides q_2, hence $L_{inv}^2(X)$ is a closed subspace of $L^2(X)$.

Lemma 5.3. *Assume $q \geq 1$ and let $F \in L^2_q(X)$. Then, for every $x \in X$,*

$$\lim_{j\to\infty} \mathcal{M}_j(F)(x) = q^{-d} \sum_{\nu\in(\mathbb{Z}/q\mathbb{Z})^d} F((\nu, P(\nu)) \cdot x) \tag{5.15}$$

Proof of Lemma 5.3. If $n \equiv \nu$ $(mod\ q)$ then $(n, P(n)) \equiv (\nu, P(\nu))$ $(mod\ q)$ (see (5.11)), hence there is a $g \in \mathbb{G}^\#_q$ such that $(n, P(n)) = g \cdot (\nu, P(\nu))$. Thus $F((n, P(n)) \cdot x) = F((\nu, P(\nu)) \cdot x)$ since $F \in L^2_q(X)$. In view of the definitions, it is enough to show that for every $\nu \in (\mathbb{Z}/q\mathbb{Z})^d$

$$\lim_{j\to\infty} \frac{1}{\|\Omega^\delta\|_{L^1}\, \delta^j} \sum_{n\equiv\nu\ (mod\ q)} \Omega^\delta(n/\delta^j) \;=\; q^{-d},$$

which is an elementary observation. □

If for each q the action of $\mathbb{G}^\#_q$ on X is ergodic, then $L^2_{inv}(X)$ contains only constant functions. Thus, for (5.2) and (5.3), it suffices to prove that for $F \in (L^2_{inv}(X))^\perp$

$$\lim_{j\to\infty} \mathcal{M}_j(F)(x) = 0 \quad \text{for a.e. } x \in X \tag{5.16}$$

We identify now a dense subspace of the orthogonal complement of $L^2_q(X)$.

Lemma 5.4. *Assume $q \geq 1$. Then*

$$(L^2_q(X))^\perp \;=\; \overline{Span\,\{T_gH - H :\ g \in \mathbb{G}^\#_q,\ H \in L^\infty(X)\}}, \tag{5.17}$$

where Span S denotes the subspace spanned by the set S.

Proof of Lemma 5.4. Let $F \in L^2(X)$ and assume that for all $H \in L^\infty(X)$ and $g \in \mathbb{G}^\#_q$

$$\langle F, T_gH - H\rangle = 0$$

That is for every $g \in \mathbb{G}^\#_q$

$$\langle T_{g^{-1}}F - F, H\rangle = 0 \quad \forall\, H \in L^\infty(X)$$

which means $T_{g^{-1}}F = F$ for all $g \in \mathbb{G}^\#_q$, so $F \in L^2_q(X)$. This proves the lemma. □

Following an idea described in [3], we will show (5.16) by proving L^2 bounds for a family of truncated maximal functions. We will use the following construction: let $\mathcal{L}_j$ $(j \in \mathbb{N})$ be a family of bounded linear operators on $L^2(X)$, and let j_k be an increasing sequence of natural numbers. Then we define the maximal operators

$$\mathcal{L}^*_k(F)(x) \;=\; \max_{j_k\leq j<j_{k+1}} |\mathcal{L}_j(F)(x)|$$

Let $F \in (L^2_{inv}(X))^\perp$, and assume indirectly that for a set of positive measure $\lim_{j\to\infty} \mathcal{M}_j(F)(x) \neq 0$. Then there exists $\varepsilon > 0$ such that

$$\mu\{x \in X : \limsup_{j\to\infty} |\mathcal{M}_j(F)(x)| \;> \varepsilon\} \;>\; \varepsilon.$$

Then, it is easy to see that there is an increasing sequence j_k $(k \in \mathbb{Z}_+)$ such that

$$\| \mathcal{M}_k^*(F) \|_{L^2(X)}^2 \;\geq\; \varepsilon^3/2 \tag{5.18}$$

for all $k \in \mathbb{Z}_+$. Moreover the sequence j_k can be chosen to be rapidly increasing, so we may assume $j_{k+1} \geq 3j_k$.

Let $\widetilde{\chi} : \mathbb{R} \to [0,1]$ denote a smooth function supported in $[-2,2]$ and equal to 1 in $[-1,1]$. For $x \in X$ and $L \gg 1$ we define

$$f_{L,x}(g) = F(g \cdot x) \cdot \chi_L(g), \tag{5.19}$$

where $\chi_L : \mathbb{R}^d \times \mathbb{R}^{d^2} \to [0,1]$ is given by $\chi_L(m,u) = \widetilde{\chi}(|m|/L) \cdot \widetilde{\chi}(|u|/L^2)$ (recall that $\mathbb{G}^\# = \mathbb{Z}^d \times \mathbb{Z}^{d^2}$ as sets). Clearly, $\|\chi_L\|_{L^1(\mathbb{G}^\#)} \approx L^{d+2d^2}$. For $f : \mathbb{G}^\# \to \mathbb{C}$, $j \geq 0$, and $\delta \in (1,2]$, we define as in (5.13)

$$\widetilde{\mathcal{M}}_j(f)(g) \;=\; \frac{1}{\|\Omega^\delta\|_{L^1}\,\delta^j} \sum_{n \in \mathbb{Z}^d} \Omega^\delta(n/\delta^j)\, f((n, P(n)) \cdot g). \tag{5.20}$$

Using the definitions, for any $k \in \mathbb{Z}_+$ and $L \geq L_k$ large enough

$$\|\mathcal{M}_k^*(F)\|_{L^2(X)}^2 \;\leq\; \frac{C}{L^{d+2d^2}} \int_X \|\widetilde{\mathcal{M}}_k^*(f_{L,x})\|_{L^2(\mathbb{G}^\#)}^2 \, d\mu(x).$$

We assume from now on the the sequence $j_1 < j_2 < \ldots$ is fixed. To summarize, for (5.16) it suffices to prove Lemma 5.5 below.

Lemma 5.5. *Assume $F \in (L^2_{inv}(X))^\perp$ and define $f_{L,x}$ as in* (5.19)*. Then for every $\varepsilon > 0$ and $\delta \in (1,2]$ (see* (5.13)*) there exist $k = k(F, \varepsilon, \delta)$ and $L(j_{k+1}, F, \varepsilon, \delta)$ such that*

$$\frac{1}{L^{d+2d^2}} \int_X \Big\| \sup_{j_k \leq j < j_{k+1}} |\widetilde{\mathcal{M}}_j(f_{L,x})| \Big\|_{L^2(\mathbb{G}^\#)}^2 d\mu(x) \leq \varepsilon, \tag{5.21}$$

for any $L \geq L(j_{k+1}, F, \varepsilon, \delta)$.

We show now how to reduce Lemma 5.5 to Lemma 5.6 below. We may assume that $\|F\|_{L^2(X)} = 1$, so

$$\frac{1}{L^{d+2d^2}} \int_X \|f_{L,x}\|_{L^2(\mathbb{G}^\#)}^2 \, d\mu(x) \leq C \|F\|_{L^2(X)} \leq C \text{ for any } L \geq 1. \tag{5.22}$$

Also, for $f \in L^2(\mathbb{G}^\#)$, we may redefine

$$\widetilde{\mathcal{M}}_j(f)(g) \;= 2^{-dj} \sum_{n \in \mathbb{Z}^d} \Omega^\delta(n/2^j)\, f((n, P(n)) \cdot g), \tag{5.23}$$

where $\Omega^\delta : \mathbb{R}^d \to [0,1]$ is a smooth function, $\Omega^\delta(y) = 1$ for $|y| \leq c_0$ and $\Omega^\delta(y) = 0$ for $|y| \geq c_0 \cdot \delta$, $1 \leq c_0 \leq 2$.

We wil use the notation and results of subsection 4.1, especially the remark following Lemma 4.1. Assume $\varepsilon > 0$ and $\eta \in (1,2]$ are fixed. We relate now the averages $\widetilde{\mathcal{M}}_j(f)$ in (5.23) and $\mathcal{M}_j(f)$ in (4.2). We identify $\mathbb{G}^\#$ and $\mathbb{G}_0^\#$ with

$\mathbb{Z}^d \times \mathbb{Z}^{d^2}$. By taking the Fourier transform in the central variable, for θ in $\mathbb{R}^{d^2}$ we have

$$\begin{cases} \widehat{\mathcal{M}_j(f)}(m,\theta) = \sum_{n\in\mathbb{Z}^d} \Omega_j^\delta(m-n)\widehat{f}(n,\theta)e^{-2\pi i R_0(m-n,n)\cdot\theta}; \\ \widehat{\widetilde{\mathcal{M}}_j(f)}(m,\theta) = \sum_{n\in\mathbb{Z}^d} \Omega_j^\delta(m-n)\widehat{f}(n,\theta)e^{-2\pi i(-P(n-m)-R(n-m,m))\cdot\theta}, \end{cases}$$

where $\Omega_j^\delta(x) = 2^{-dj}\Omega^\delta(-x/2^j)$. For $N = N(\varepsilon,\delta)$ sufficiently large let

$$\mathcal{R}_N = \{a/q \in \mathbb{Q}^{d^2} : q \leq N \text{ and } (a,q) = 1\}.$$

For $j \geq N$, define as in (4.6) and (4.7)

$$\Psi_j^N(\theta) = \sum_{r\in\mathcal{R}_N} \psi(2^{2j}N^{-1}(\theta - r))$$

and

$$\widehat{\widetilde{\mathcal{M}}_j^N(f)}(m,\theta) = \widehat{\widetilde{\mathcal{M}}_j(f)}(m,\theta)\cdot\Psi_j^N(\theta).$$

Simple changes of variables, using (5.11), and the remark following Lemma 4.1 show that

$$\Big|\Big|\sup_{j\geq N} |\widetilde{\mathcal{M}}_j(f) - \widetilde{\mathcal{M}}_j^N(f)|\Big|\Big|_{L^2(\mathbb{G}^\#)} \leq (\varepsilon/C)\|f\|_{L^2(\mathbb{G}^\#)}$$

for any $f \in L^2(\mathbb{G}^\#)$, provided that $N = N(\varepsilon,\delta)$ is fixed sufficiently large. Thus, using (5.22),

$$\frac{1}{L^{d+2d^2}}\int_X \Big|\Big|\sup_{j\geq N} |\widetilde{\mathcal{M}}_j(f_{L,x}) - \widetilde{\mathcal{M}}_j^N(f_{L,x})|\Big|\Big|^2_{L^2(\mathbb{G}^\#)} d\mu(x) \leq \varepsilon/2. \tag{5.24}$$

Assume from now on that N is fixed. We examine the operator $\widetilde{\mathcal{M}}_j^N$, and, for $a/q \in \mathcal{R}_N$, $j \geq N$ and $f \in L^2(\mathbb{G}^\#)$, we define

$$\widehat{\widetilde{\mathcal{M}}_{j,a/q}^N(f)}(m,\theta) = \widehat{\widetilde{\mathcal{M}}_j(f)}(m,\theta)\cdot\sum_{b\in\mathbb{Z}^{d^2}} \psi(2^{2j}N^{-1}(\theta - a/q - b)). \tag{5.25}$$

Thus, for Lemma 5.5 it suffices to prove Lemma 5.6 below.

Lemma 5.6. *Assume $F \in (L^2_{inv}(X))^\perp$, $N \geq 1$, $a/q \in \mathcal{R}_N$, $\delta \in (1,2]$, and define $f_{L,x}$ as in (5.19), and $\widetilde{\mathcal{M}}_{j,a/q}^N$ as in (5.25) and (5.23). Then, for every $\varepsilon > 0$, there exist $k = k(F,N,\varepsilon,\delta)$ and $L(j_{k+1},F,N,\varepsilon,\delta)$ such that*

$$\frac{1}{L^{d+2d^2}}\int_X \Big|\Big|\sup_{j_k\leq j<j_{k+1}} |\widetilde{\mathcal{M}}_{j,a/q}^N(f_{L,x})|\Big|\Big|^2_{L^2(\mathbb{G}^\#)} d\mu(x) \leq \varepsilon, \tag{5.26}$$

for any $L \geq L(j_{k+1},F,N,\varepsilon,\delta)$.

The rest of this section is concerned with the proof of Lemma 5.6. As in (4.26), by the Fourier inversion formula,

$$\widetilde{\mathcal{M}}^N_{j,a/q}(f)(m,u) = \sum_{(n,v)\in\mathbb{Z}^d\times\mathbb{Z}^{d^2}} f(n,v)\Omega^\delta_j(m-n) \times \eta_{2^{2j}/N}(u-v+P(n-m)+R(n-m,m))e^{2\pi i(u-v+P(n-m)+R(n-m,m))\cdot a/q}, \tag{5.27}$$

where $\eta\in\mathcal{S}(\mathbb{R}^{d^2})$ is defined as in (4.26), and $\eta_r(s)=r^{-d^2}\eta(s/r)$, $r\geq 1$. As in section 7, we define $\phi:\mathbb{R}^{d^2}\to[0,1]$, $\phi(s)=(1+|s|^2)^{-(d^2+d+1)}$, and $\phi_r(s)=r^{-d^2}\phi(s/r)$. Then

$$|\widetilde{\mathcal{M}}^N_{j,a/q}(f)(m,u)| \leq C_N \sum_{(n,v)\in\mathbb{Z}^d\times\mathbb{Z}^{d^2}} |f(n,v)|\cdot\Omega^\delta_j(m-n)\phi_{2^{2j}}(u-v+R(n-m,m)), \tag{5.28}$$

so the maximal function $f\to\sup_{j_k\leq j\leq j_{k+1}}|\widetilde{\mathcal{M}}^N_{j,a/q}(f)|$ is bounded on $L^2(\mathbb{G}^\#)$ (compare with Lemma 7.1). Thus, using Lemma 5.4 and (5.22), in proving Lemma 5.6 we may assume that

$$F(x)=H(g_0\cdot x)-H(x) \text{ for some } g_0\in\mathbb{G}^\#_q \text{ and } H\in L^\infty(X) \text{ with } \|H\|_{L^\infty}=1. \tag{5.29}$$

We may also replace the function η with a smooth function $\widetilde{\eta}$ compactly supported in the set $\{s\in\mathbb{R}^{d^2}:|s|\leq N'(\epsilon,N)\}$; this is due to the fact that the bound (5.28) gains an additional small factor the right-hand side for the part of the operator corresponding to $\eta-\widetilde{\eta}$.

Using (5.19) and (5.29),

$$f_{L,x}(g)=\chi_L(g)\cdot[H(g_0g\cdot x)-H(g\cdot x)]. \tag{5.30}$$

It suffices to prove that for k, L as in Lemma 5.6 and $f_{L,x}$ as in (5.30),

$$\frac{1}{L^{d+2d^2}}\int_X \Big|\Big|\sup_{j_k\leq j<j_{k+1}}|\mathcal{M}'_j(f_{L,x})|\Big|\Big|^2_{L^2(\mathbb{G}^\#)}\,d\mu(x)\leq\varepsilon/2, \tag{5.31}$$

where $\mathcal{M}'_j(f)$ is defined as in (5.27) with $\widetilde{\eta}$ replacing η.

We define the kernels $K_j:\mathbb{G}^\#\to\mathbb{C}$,

$$K_j(n,v)=\Omega^\delta_j(-n)\widetilde{\eta}_{2^{2j}/N}(-v+P(n))e^{2\pi i(-v+P(n))\cdot a/q} \tag{5.32}$$

so, using (5.27),

$$\mathcal{M}'_j(f)(m,u)=\sum_{(n,v)\in\mathbb{G}^\#} f(n,v)\cdot K_j((n,v)\cdot(m,u)^{-1}).$$

Using (5.30) and simple changes of variables, it follows that

$$\mathcal{M}'_j(f_{L,x})(g)=\sum_{h\in\mathbb{G}^\#} H(hg\cdot x)\cdot[\chi_L(g_0^{-1}hg)K_j(g_0^{-1}h)-\chi_L(hg)K_j(h)] \tag{5.33}$$

for any $g \in \mathbb{G}^{\#}$. We use now (5.29), i.e. $\|H\|_{L^\infty} = 1$. Since $g_0^{-1} \in \mathbb{G}_q^{\#}$, the oscillatory parts of $K_j(h)$ and $K_j(g_0^{-1}h)$ agree. Simple estimates then show that (with $h = (n, v)$)

$$|\chi_L(g_0^{-1}hg)K_j(g_0^{-1}h) - \chi_L(hg)K_j(h)| \leq C(g_0, N, \varepsilon, \delta) \cdot j_k^{-1} \cdot \chi_{4L}(g) \cdot \Omega^\delta_{j+2}(n)\phi_{2^{2j}}(v),$$

if k is sufficiently large, and then L is sufficiently larger than j_{k+1}. Thus

$$|\mathcal{M}'_j(f_{L,x})(g)| \leq \frac{C(g_0, N, \varepsilon, \delta)}{j_k}\chi_{4L}(g),$$

and (5.31) follows.

6. The singular Radon transform

In this section we prove Lemma 2.4. The main ingredients are the L^2 bounds in Lemma 6.1, a super-orthogonality argument of Ionescu and Wainger [8] which reduces matters to square function estimates, and the weighted inequality in Lemma 7.4. We assume throughout this section that $d' = d^2$, and $\mathbb{G}_0^{\#}$ is the discrete nilpotent group defined in section 2.

6.1. L^2 **estimates.** Our main result in this subsection is Lemma 6.1, which is a quantitative L^2 estimate. The proof of Lemma 6.1 is based on Plancherel theorem, Lemma 3.2, Lemma 3.3, and Lemma 3.4.

Let K denote the Calderón–Zygmund kernel defined in section 1. Without loss of generality (cf. [14, p. 624]), we may assume that $K = \sum_{j=0}^{\infty} K_j$, where K_j is supported in the set $\{x : |x| \in [2^{j-1}, 2^{j+1}]\}$, and satisfies the bound

$$|x|^d|K_j(x)| + |x|^{d+1}|\nabla K_j(x)| \leq 1,\ x \in \mathbb{R}^d, j \geq 1, \tag{6.1}$$

and the cancellation condition

$$\int_{\mathbb{R}^d} K_j(x)\,dx = 0,\ j \geq 1. \tag{6.2}$$

As in section 4,

$$\mathcal{T}_0^{\#}(f) = \sum_{j=1}^{\infty} \mathcal{T}_j(f) \text{ where } \widehat{\mathcal{T}_j(f)}(m,\theta) = \sum_{n\in\mathbb{Z}^d} K_j(m-n)\widehat{f}(n,\theta)e^{-2\pi i R_0(m-n,n)\cdot\theta}. \tag{6.3}$$

As in section 4, let $\psi : \mathbb{R}^{d'} \to [0, 1]$ denote a smooth function supported in the set $\{|\xi| \leq 2\}$ and equal to 1 in the set $\{|\xi| \leq 1\}$, $N \in [1/2, \infty)$ a real number, $j \in [0, \infty) \cap \mathbb{Z}$ a nonnegative integer, and $\mathcal{R} \subseteq \mathbb{Q}^{d'}$ a discrete periodic set. As in (4.6), let

$$\Psi_j^{N,\mathcal{R}}(\theta) = \sum_{r\in\mathcal{R}} \psi(2^{2j}N^{-1}(\theta - r)),$$

and, by convention, $\Psi_j^{N,\emptyset} = 0$. For (compactly supported) functions $f : \mathbb{G}_0^{\#} \to \mathbb{C}$ we define $\mathcal{T}_j^{N,\mathcal{R}}(f)$ by

$$\widehat{\mathcal{T}_j^{N,\mathcal{R}}}(f)(m,\theta) = \widehat{\mathcal{T}_j(f)}(m,\theta) \cdot \Psi_j^{N,\mathcal{R}}(\theta). \tag{6.4}$$

Our main lemma in this section is the following L^2 estimate:

Lemma 6.1 (Strong L^2 bound). *As in Lemma 4.1, assume that $N \in [1/2, \infty)$, $\mathcal{R}_N \subseteq \mathbb{Q}^{d'}$ is a discrete periodic set, and $J_{N,\mathcal{R}_N} \in [0, \infty)$ is a real number with the properties*

$$\begin{cases} \{a/q : q \in [1, N] \text{ and } (a, q) = 1\} \subseteq \mathcal{R}_N, \\ 2^{J_{N,\mathcal{R}_N}} \geq [100 \max_{a/q \in \mathcal{R}_N \text{ and } (a,q)=1} q]^4. \end{cases} \tag{6.5}$$

Then

$$\Big|\Big| \sum_{j \geq J_{N,\mathcal{R}_N}} (\mathcal{T}_j - \mathcal{T}_j^{N,\mathcal{R}_N})(f) \Big|\Big|_{L^2(\mathbb{G}_0^{\#})} \leq C(N+1)^{-\overline{c}} ||f||_{L^2(\mathbb{G}_0^{\#})}, \tag{6.6}$$

for any $N \geq 0$, where $\overline{c} = \overline{c}(d) > 0$.

The rest of this section is concerned with the proof of Lemma 6.1. Notice that the case $N = 1/2$, $\mathcal{R}_N = \emptyset$, $J_{N,\mathcal{R}_N} = 0$, corresponds to L^2 boundedness of the operator $\mathcal{T}_0^{\#}$. For $\theta \in \mathbb{R}^{d'}$ and (compactly supported) functions $g : \mathbb{Z}^d \to \mathbb{C}$, let

$$\mathcal{U}_j^\theta(g)(m) = \sum_{n \in \mathbb{Z}^d} K_j(m-n) g(n) e^{-2\pi i R_0(m-n,n)\cdot\theta}. \tag{6.7}$$

By Plancherel theorem

$$\Big|\Big| \sum_{j \geq J_{N,\mathcal{R}_N}} (\mathcal{T}_j - \mathcal{T}_j^{N,\mathcal{R}_N})(f) \Big|\Big|^2_{L^2(\mathbb{G}_0^{\#})}$$
$$= \int_{[0,1)^{d'}} \sum_{m \in \mathbb{Z}^d} \Big| \sum_{j \geq J_{N,\mathcal{R}_N}} (1 - \Psi_j^{N,\mathcal{R}_N}(\theta)) \mathcal{U}_j^\theta(\widehat{f}(.,\theta))(m) \Big|^2 d\theta.$$

Using Plancherel theorem again, for (6.6) it suffices to prove that

$$\Big|\Big| \sum_{j \geq J_{N,\mathcal{R}_N}} (1 - \Psi_j^{N,\mathcal{R}_N}(\theta)) \mathcal{U}_j^\theta \Big|\Big|_{L^2(\mathbb{Z}^d) \to L^2(\mathbb{Z}^d)} \leq C(N+1)^{-\overline{c}} \tag{6.8}$$

for any $\theta \in \mathbb{R}^{d'}$ fixed.

Assume that $\theta \in \mathbb{R}^{d'}$ is fixed. As in section 4, for any $j \geq J_{N,\mathcal{R}_N}$ we use the approximation (4.17) with $\Lambda = 2^{(2-\delta)j}$, $\delta = \delta(d) > 0$ sufficiently small. Thus there are irreducible 1-fractions $a^j_{l_1 l_2}/q^j_{l_1 l_2}$ such that

$$1 \leq q^j_{l_1 l_2} \leq 2^{(2-\delta)j} \quad \text{and} \quad |\theta_{l_1 l_2} - a^j_{l_1 l_2}/q^j_{l_1 l_2}| \leq C/(2^{(2-\delta)j} q^j_{l_1 l_2}). \tag{6.9}$$

We fix these irreducible 1-fractions $a^j_{l_1l_2}/q^j_{l_1l_2}$ and partition the set $\mathbb{Z}\cap[J_{N,\mathcal{R}_N},\infty)$ into two subsets:

$$I_1=\{j\in\mathbb{Z}\cap[J_{N,\mathcal{R}_N},\infty):\max_{l_1,l_2=1,\ldots,d}q^j_{l_1l_2}>2^{j/(6d')}\}$$

and

$$I_2=\{j\in\mathbb{Z}\cap[J_{N,\mathcal{R}_N},\infty):\max_{l_1,l_2=1,\ldots,d}q^j_{l_1l_2}\leq 2^{j/(6d')}\}.$$

For $j\in I_1$ we use Lemma 3.2:

$$\Big|\Big|\sum_{j\in I_1}(1-\Psi_j^{N,\mathcal{R}_N}(\theta))\mathcal{U}_j^\theta\Big|\Big|_{L^2(\mathbb{Z}^d)\to L^2(\mathbb{Z}^d)}\leq\sum_{j\in I_1}2^{-\delta' j}\leq C(N+1)^{-\overline{c}},$$

as desired.

For $j\in I_2$ let a_j/q_j denote the irreducible d'-fraction with the property that $a_j/q_j=(a^j_{l_1l_2}/q^j_{l_1l_2})_{l_1,l_2=1,\ldots,d}$. In view of (6.9) and the definition of I_2,

$$1\leq q_j\leq 2^{j/6}\ \text{ and }\ |\theta-a_j/q_j|\leq C/2^{(2-\delta)j}.\tag{6.10}$$

We recall (see (4.20) that if $j,j'\in I_2$ and $j,j'\geq C$ then

$$\text{either }\ a_j/q_j=a_{j'}/q_{j'}\ \text{ or }\ |q_j/q_{j'}|\notin[1/2,2].\tag{6.11}$$

As in section 4, we further partition the set I_2,

$$I_2=\cup_{a/q}I_2^{a/q}\text{ where }I_2^{a/q}=\{j\in I_2:a_j/q_j=a/q\}.\tag{6.12}$$

For $j\in I_2^{a/q}$ we show that

$$\Big|\Big|\sum_{j\in I_2^{a/q}}(1-\Psi_j^{N,\mathcal{R}_N}(\theta))\mathcal{U}_j^\theta\Big|\Big|_{L^2(\mathbb{Z}^d)\to L^2(\mathbb{Z}^d)}\leq C(N+q)^{-\overline{c}}.\tag{6.13}$$

This would suffice to prove (6.8) since the possible denominators q form a lacunary sequence (see (6.11)). To prove (6.13) we have two cases: $q\leq N$ and $q\geq N$. If $q\leq N$, we use Lemma 3.3 together with the definitions (4.8) and (4.6). It follows that the left-hand side of (6.13) is dominated by

$$C\sum_{j\in\mathbb{Z}}\mathbf{1}_{[1,\infty)}(2^{2j}N^{-1}|\theta-a/q|)q^{-1/2}(1+2^{2j}|\theta-a/q|)^{-1/4}\leq Cq^{-1/2}N^{-1/4},$$

as desired. If $q>N$, then the left-hand side of (6.13) is dominated by

$$\begin{aligned}&\Big|\Big|\sum_{j\in I_2^{a/q},2^j\in[q^6,|\theta-a/q|^{-1/2}]}(1-\Psi_j^{N,\mathcal{R}_N}(\theta))\mathcal{U}_j^\theta\Big|\Big|_{L^2(\mathbb{Z}^d)\to L^2(\mathbb{Z}^d)}\\&+\sum_{j\in I_2^{a/q},2^j\geq|\theta-a/q|^{-1/2}}||\mathcal{U}_j^\theta||_{L^2(\mathbb{Z}^d)\to L^2(\mathbb{Z}^d)}.\end{aligned}\tag{6.14}$$

For the first term in (6.14) we use Lemma 3.4 for the kernels $(1-\Psi_j^{N,\mathcal{R}_N}(\theta))K_j(m)$. To control the second term in (6.14) we use Lemma 3.3. It follows that the expression in (6.14) is dominated by $Cq^{-1/2}$, which suffices to prove (6.13). This completes the proof of Lemma 6.1.

6.2. **An orthogonality lemma.** In this subsection we review a partition of integers and a square function estimate from [8]. The point of this construction is to find a suitable decomposition of the singular integral operator and exploit the super-orthogonality (i.e. orthogonality in L^{2r}, $r \in \mathbb{Z}_+$) of the components.

Recall that for any integer $\mu \geq 1$, $Z_\mu = \{1, \ldots, \mu\}$. Assume $\delta \in (0, 1/10]$ is given and D denotes the smallest integer $\geq 2/\delta$. Assume $N \geq 10$ is an integer. Let N' denote the smallest integer $\geq N^{\delta/2}$, and $V = \{p_1, p_2, \ldots, p_\nu\}$ the set of prime numbers between $N' + 1$ and N. For any $k \in Z_D$ let

$$W^k(N) = \{p_{i_1}^{\alpha_{i_1}} \cdot \ldots \cdot p_{i_k}^{\alpha_{i_k}} : p_{i_l} \in V \text{ distinct}, \alpha_{i_l} \in Z_D, l = 1, \ldots, k\},$$

and let $W(N) = \cup_{k \in Z_D} W^k(N)$ denote the set of products of up to D factors in V, at powers between 1 and D.

We say that a set $W' \subseteq W(N)$ has *the orthogonality property O* if there is $k \in Z_D$ and k sets $S_1, S_2, \ldots, S_k$, $S_j = \{q_{j,1}, \ldots, q_{j,\beta(j)}\}$, $j \in Z_k$, with the following properties:

(i) $q_{j,s} = p_{j,s}^{\alpha_j}$ for some $p_{j,s} \in V$, $\alpha_j \in Z_D$;

(ii) $(q_{j,s}, q_{j',s'}) = 1$ if $(j, s) \neq (j', s')$;

(iii) for any $w' \in W'$ there are (unique) numbers $q_{1,s_1} \in S_1, \ldots, q_{k,s_k} \in S_k$ with $w' = q_{1,s_1} \cdot \ldots \cdot q_{k,s_k}$.

For simplicity of notation, we say that the set $W' = \{1\}$ has the orthogonality property O with $k = 0$. The orthogonality property O is connected to Lemma 6.3 below. Notice that if a set has the orthogonality property O then all its elements have the same number of prime factors. The main result in [8, Section 3] is the following decomposition.

Lemma 6.2 (Partition of integers)**.** *With the notation above, the set $W(N) \cup \{1\}$ can be written as a disjoint union of at most $C_D(\ln N)^{D-1}$ subsets with the orthogonality property O.*

Let $Q_0 = [N'!]^D$ and define

$$Y_N = \{w \cdot Q' : w \in W(N) \cup \{1\} \text{ and } Q' | Q_0\}. \tag{6.15}$$

Notice that for any $m \in Z_N$ there is a unique decomposition $m = w \cdot Q'$, with $w \in W(N) \cup \{1\}$ and $Q' | Q_0$. In addition, $w \cdot Q' \leq N^{D^2}[N'!]^D \leq e^{N^\delta}$ if $N \geq C_\delta$. Thus, for $N \geq C_\delta$,

$$Z_N \subseteq Y_N \subseteq Z_{e^{N^\delta}}. \tag{6.16}$$

Let

$$W(N) \cup \{1\} = \cup_{s \in S} W'_s$$

denote the decomposition (guaranteed by Lemma 6.2) of $W(N)\cup\{1\}$ as a disjoint union of subsets W'_s with the orthogonality property O, where $|S|\le C_D(\ln N)^{2/\delta}$. Using this decomposition, we write $Y_N=\cup_{s\in S}Y_N^s$ (disjoint union), where

$$Y_N^s=\{w\cdot Q' : w\in W'_s \text{ and } Q'|Q_0\}. \tag{6.17}$$

This is the partition of integers we will use in section 6.3.

For any integer $q\ge 1$ let

$$P_q=\{a\in\mathbb{Z}^{d'} : (a,q)=1\},\ \widetilde{P}_q=P_q\cap[0,q)^{d'}.$$

Let $S_1,S_2,\ldots,S_k$ denote sets of integers $S_j=\{q_{j,1},\ldots,q_{j,\beta(j)}\}$, $j\in Z_k$. Assume that for some $\widetilde{Q}$

$$q_{j,s}\in[2,\widetilde{Q}] \text{ for any } j\in Z_k, s\in Z_{\beta(j)}, \tag{6.18}$$

and

$$(q_{j,s},q_{j',s'})=1 \text{ if } (j,s)\ne(j',s'). \tag{6.19}$$

For any $j\in Z_k$ let

$$T_{\{j\}}=\{a_{j,s}/q_{j,s} : s\in Z_{\beta(j)}, a_{j,s}\in P_{q_{j,s}}\}\subseteq\mathbb{Q}^{d'}$$

denote the set of irreducible fractions with denominators in S_j. Furthermore, for any set $A=\{j_1,\ldots,j_{k'}\}\subseteq Z_k$ let

$$T_A=\{r_{j_1}+\ldots+r_{j_{k'}} : r_{j_l}\in T_{\{j_l\}} \text{ for } l\in Z_{k'}\}\subseteq\mathbb{Q}^{d'}.$$

Finally, for $A=\{j_1,\ldots,j_{k'}\}\subseteq Z_k$ and any $(s_{j_1},\ldots,s_{j_{k'}})\in Z_{\beta(j_1)}\times\ldots\times Z_{\beta(j_{k'})}$ let

$$U_{A,s_{j_1},\ldots,s_{j_{k'}}}=\{a_{j_1,s_{j_1}}/q_{j_1,s_{j_1}}+\ldots+a_{j_{k'},s_{j_{k'}}}/q_{j_{k'},s_{j_{k'}}} : a_{j_l,s_{j_l}}\in P_{q_{j_l,s_{j_l}}} \text{ for } l\in Z_{k'}\},$$

that is the subset of elements of T_A with fixed denominators $q_{j_1,s_{j_1}},\ldots,q_{j_{k'},s_{j_{k'}}}$. If $A=\emptyset$ then, by definition, $T_A=U_A=\mathbb{Z}^{d'}$. Notice that the sets T_A and $U_{A,s_{j_1},\ldots,s_{j_{k'}}}$ are discrete periodic subsets of $\mathbb{Q}^{d'}$. Let $\widetilde{T}_A=T_A\cap[0,1)^{d'}$ and $\widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}=U_{A,s_{j_1},\ldots,s_{j_{k'}}}\cap[0,1)^{d'}$.

Assume that $Q\ge 1$ is an integer with the property that

$$(Q,q_{j,s})=1, \text{ for any } j\in Z_k, s\in Z_{\beta(j)}. \tag{6.20}$$

Assume $p\ge 1$ is an integer and fix

$$\gamma=(8pQ^{2p}\widetilde{Q}^{2pk})^{-1}, \tag{6.21}$$

where $\widetilde{Q}$ is as in (6.18).

For any $r\in T_{Z_k}$ let $f_r\in L^2(\mathbb{Z}^{d'})$ denote a function whose Fourier transform is supported in an γ-neighborhood of the set $\{r+a/Q : a\in\mathbb{Z}^{d'}\}$, i.e. in the set

$$\bigcup_{a\in\mathbb{Z}} r+a/Q+B(\gamma),$$

where $B(\gamma) = \{|\xi| \leq \gamma\}$. We assume that $f_r = f_{r+a}$ for any $a \in \mathbb{Z}^{d'}$. Let $(\mathbb{Z}^{d'}, dn)$ denote the set of integers with the counting measure. The main estimate in this subsection is the following lemma.

Lemma 6.3 (Square function estimate)**.** *With the notation above we have*

$$\begin{aligned}\int_{\mathbb{Z}^{d'}} \Big| \sum_{r \in \widetilde{T}_{Z_k}} f_r(u) \Big|^{2p} du \\ \leq C_{k,p} \sum_{A=\{j_1,\dots,j_{k'}\}} \sum_{s_{j_1},\dots,s_{j_{k'}}} \int_{\mathbb{Z}^{d'}} \Big(\sum_{r' \in \widetilde{T}_{c_A}} \Big| \sum_{\mu \in \widetilde{U}_{A,s_{j_1},\dots,s_{j_{k'}}}} f_{\mu+r'}(u) \Big|^2 \Big)^p du,\end{aligned} \tag{6.22}$$

where the sum in the right-hand side is taken over all sets $A = \{j_1, \dots, j_{k'}\} \subseteq Z_k$, and all $(s_{j_1}, \dots, s_{j_{k'}}) \in Z_{\beta(j_1)} \times \dots \times Z_{\beta(j_{k'})}$. The constant $C_{k,p}$ may depend only on k and p.

See [8, Section 2] for the proof.

6.3. **Proof of Lemma 2.4.** In this subsection we complete the proof of Lemma 2.4. The main ingredients are the L^2 estimate in Lemma 6.1, the partition of the integers in Lemma 6.2, the square function estimate in Lemma 6.3, and the weighted estimate in Lemma 7.4. The kernels K_j are as in (6.1) and (6.2), and the operators $\mathcal{T}_j$ are as in (6.3). Lemma 2.4 follows by interpolation (see [8, Section 7]) from the following more quantitative lemma.

Lemma 6.4. *Assume $2p \geq 4$ is an even integer and $\epsilon = 1/(2p-2)$. Then, for any $\lambda \in (0, \infty)$, there are two linear operators $\mathcal{A}_j^\lambda = \mathcal{A}_j^{\lambda,\epsilon}$ and $\mathcal{B}_j^\lambda = \mathcal{B}_j^{\lambda,\epsilon}$ with $\mathcal{T}_j = \mathcal{A}_j^\lambda + \mathcal{B}_j^\lambda$,*

$$\Big|\Big| \sum_{j \geq 1} \mathcal{A}_j^\lambda(f) \Big|\Big|_{L^2(\mathbb{G}_0^\#)} \leq C_\epsilon/\lambda ||f||_{L^2(\mathbb{G}_0^\#)}, \tag{6.23}$$

and

$$\Big|\Big| \sum_{j \geq 1} \mathcal{B}_j^\lambda(f) \Big|\Big|_{L^{2p}(\mathbb{G}_0^\#)} \leq C_\epsilon \lambda^\epsilon ||f||_{L^{2p,1}(\mathbb{G}_0^\#)}. \tag{6.24}$$

In (6.24), $L^{2p,1}(\mathbb{G}_0^\#)$ denotes the standard Lorentz space on $\mathbb{G}_0^\#$. The rest of this section is concerned with the proof of Lemma 6.4. In view of Lemma 6.1, we may assume that $\lambda \geq C_\epsilon$. With $\overline{c}$ as in Lemma 6.1, let

$$\delta = \overline{c}\epsilon/100. \tag{6.25}$$

Let

$$\begin{cases} N_0 \text{ denote the smallest integer } \geq \lambda^{1/\overline{c}}; \\ \mathcal{R}_{N_0} = \{a/q : (a,q) = 1 \text{ and } q \in Y_{N_0}\}; \\ J_{N_0,\mathcal{R}_{N_0}} = \lambda^\epsilon, \end{cases} \tag{6.26}$$

where Y_{N_0} denotes the set defined in (6.15) with $\delta = \overline{c}\epsilon/100$ as in (6.25). The property (6.5) is verified for $\lambda \geq C_\epsilon$, using (6.16). For $j < J_{N_0,\mathcal{R}_{N_0}}$ let $\mathcal{A}_j^\lambda \equiv 0$, $\mathcal{B}_j^\lambda \equiv \mathcal{T}_j$. Clearly,

$$\Big|\Big|\sum_{j\in[1,J_{N_0,\mathcal{R}_{N_0}})\cap\mathbb{Z}}\mathcal{B}_j^\lambda(f)\Big|\Big|_{L^{2p}(\mathbb{G}_0^\#)} \leq C\lambda^\epsilon||f||_{L^{2p}(\mathbb{G}_0^\#)},$$

which gives (6.24). For $j \geq J_{N_0,\mathcal{R}_{N_0}}$ let $\mathcal{A}_j^\lambda \equiv \mathcal{T}_j - \mathcal{T}_j^{N_0,\mathcal{R}_{N_0}}$, $\mathcal{B}_j^\lambda \equiv \mathcal{T}_j^{N_0,\mathcal{R}_{N_0}}$, with $\mathcal{T}_j^{N_0,\mathcal{R}_{N_0}}$ defined as in (6.4). By Lemma 6.1,

$$\Big|\Big|\sum_{j\geq J_{N_0,\mathcal{R}_{N_0}}}\mathcal{A}_j^\lambda(f)\Big|\Big|_{L^2(\mathbb{G}_0^\#)} \leq C_\epsilon/\lambda||f||_{L^2(\mathbb{G}_0^\#)},$$

which gives (6.23). To complete the proof of Lemma 6.4 it suffices to show that

$$\Big|\Big|\sum_{j\geq J_{N_0,\mathcal{R}_{N_0}}}\mathcal{T}_j^{N_0,\mathcal{R}_{N_0}}(f)\Big|\Big|_{L^{2p}(\mathbb{G}_0^\#)} \leq C_\epsilon\lambda^\epsilon||f||_{L^{2p}(\mathbb{G}_0^\#)}, \tag{6.27}$$

for any characteristic function of a bounded set f.

For simplicity of notation, let

$$J_0 = J_{N_0,\mathcal{R}_{N_0}} = \lambda^\epsilon.$$

We use the notation in section 6.2, with $\delta = \overline{c}\epsilon/100$, D the smallest integer $\geq 2/\delta$, $N = N_0$, $N' = N_0'$, and

$$Q_0 = [N_0'!]^D \leq e^{\lambda^{\epsilon/10}}. \tag{6.28}$$

Then $Y_{N_0} = \cup_{s\in S}Y_{N_0}^s$ and $\mathcal{R}_{N_0} = \cup_{s\in S}\mathcal{R}_{N_0}^{W_s'}$ (disjoint unions), where $Y_{N_0}^s$ is defined in (6.17) and

$$\mathcal{R}_{N_0}^{W_s'} = \{a'/w' + b/Q_0 : a', b \in \mathbb{Z}^{d'},\ (a', w') = 1, \text{ and } w' \in W_s'\}. \tag{6.29}$$

Clearly, for $j \geq J_0$,

$$\mathcal{T}_j^{N_0,\mathcal{R}_{N_0}}(f) = \sum_{s\in S}\mathcal{T}_j^{N_0,\mathcal{R}_{N_0}^{W_s'}}(f).$$

Since $|S| \leq C_\epsilon(\ln\lambda)^{C_\epsilon}$ (see Lemma 6.2), for (6.27) it suffices to prove that for any set $W' \subseteq W(N_0)\cup\{1\}$ with the orthogonality property O

$$\Big|\Big|\sum_{j\geq J_0}\mathcal{T}_j^{N_0,\mathcal{R}_{N_0}^{W'}}(f)\Big|\Big|_{L^{2p}(\mathbb{G}_0^\#)} \leq C_\epsilon\lambda^{\epsilon/2}||f||_{L^{2p}(\mathbb{G}_0^\#)}, \tag{6.30}$$

for any characteristic function of a bounded set f.

We fix the set W' in (6.30) and assume $W' \neq \{1\}$ (the case $W' = \{1\}$ is significantly easier). Let $S_1, \ldots, S_k$, $S_j = \{q_{j,1}, \ldots, q_{j,\beta(j)}\}$, denote the sets in the definition of the orthogonality property O. Clearly $k \leq C_\epsilon$ and

$$q_{j,s} \in [2, \lambda^{C_\epsilon}]. \tag{6.31}$$

For any $s_1 \in Z_{\beta(1)}, \ldots, s_k \in Z_{\beta(k)}$ let

$$\gamma(q_{1,s_1} \cdot \ldots \cdot q_{k,s_k}) = \begin{cases} 1 & \text{if } q_{1,s_1} \cdot \ldots \cdot q_{k,s_k} \in W'; \\ 0 & \text{if } q_{1,s_1} \cdot \ldots \cdot q_{k,s_k} \notin W'. \end{cases}$$

Any irreducible d'-fraction a'/w', $w' \in W'$, can be written in a uniquely in the form $a_{1,s_1}/q_{1,s_1} + \ldots + a_{k,s_k}/q_{k,s_k} \,(\mathrm{mod}\, \mathbb{Z}^{d'})$, with $q_{l,s_l} \in S_l$ and $a_{l,s_l} \in \widetilde{P}_{q_{l,s_l}}$, $l = 1, \ldots, k$. Conversely, if $\gamma(q_{1,s_1} \cdot \ldots \cdot q_{k,s_k}) = 1$, then any sum of the form $a_{1,s_1}/q_{1,s_1} + \ldots + a_{k,s_k}/q_{k,s_k}$, with $q_{l,s_l} \in S_l$ and $a_{l,s_l} \in \widetilde{P}_{q_{l,s_l}}$, $l = 1, \ldots, k$, belongs to the set $\{a'/w' : (a', w') = 1 \text{ and } w' \in W'\}$. Thus

$$\begin{aligned} \Psi_j^{N_0, \mathcal{R}_{N_0}^{W'}}(\theta) = & \sum_{s_1, a_{1,s_1}, \ldots, s_k, a_{k,s_k}} \sum_{b \in \mathbb{Z}^{d'}} \gamma(q_{1,s_1} \cdot \ldots \cdot q_{k,s_k}) \\ & \psi(2^{2j}(\theta - a_{1,s_1}/q_{1,s_1} - \ldots - a_{k,s_k}/q_{k,s_k} - b/Q_0)/N_0), \end{aligned} \tag{6.32}$$

where the sum is taken over all $s_l \in Z_{\beta(l)}$ and $a_{l,s_l} \in \widetilde{P}_{q_{l,s_l}}$. For any $r = a_{1,s_1}/q_{1,s_1} + \ldots + a_{k,s_k}/q_{k,s_k}$, $s_l \in Z_{\beta(l)}$ and $a_{l,s_l} \in P_{q_{l,s_l}}$ (so $r \in T_{Z_k}$ with the notation in section 6.2), we define $G_r \in L^2(\mathbb{G}_0^\#)$ by the formula

$$\widehat{G_r}(m, \theta) = \gamma(q_{1,s_1} \cdot \ldots \cdot q_{k,s_k}) \sum_{j \geq J_0} \widehat{\mathcal{T}_j(f)}(m, \theta) \sum_{b \in \mathbb{Z}^{d'}} \psi(2^{2j}(\theta - r - b/Q_0)/N_0). \tag{6.33}$$

In view of (6.32),

$$\sum_{j \geq J_0} \mathcal{T}_j^{N_0, \mathcal{R}_{N_0}^{W'}}(f) = \sum_{r \in \widetilde{T}_{Z_k}} G_r,$$

with $\widetilde{T}_{Z_k}$ defined as in section 6.2. Clearly, $\widehat{G_r}(m, .)$ is supported in a $2N_0 2^{-2J_0}$ neighborhood of the set $\{r + b/Q_0 : b \in \mathbb{Z}\}$. The condition (6.21) with $Q = Q_0$ and $\widetilde{Q} = \lambda^{C_\epsilon}$ is verified if $\lambda \geq C_\epsilon$ (see (6.26), (6.28) and (6.31)). We apply Lemma 6.3 to the functions $G_r(m, u)$, for any $m \in \mathbb{Z}^d$. With the notation in Lemma 6.3, it follows that

$$\begin{aligned} \Big|\Big| \sum_{r \in \widetilde{T}_{Z_k}} G_r \Big|\Big|_{L^{2p}(\mathbb{G}_0^\#)}^{2p} \leq C_\epsilon & \sum_{A = \{j_1, \ldots, j_{k'}\}} \\ & \sum_{s_{j_1}, \ldots, s_{j_{k'}}} \int_{\mathbb{G}_0^\#} \Big(\sum_{r' \in \widetilde{T}_{c_A}} \Big| \sum_{\mu \in \widetilde{U}_{A, s_{j_1}, \ldots, s_{j_{k'}}}} G_{r' + \mu}(m, u) \Big|^2 \Big)^p \, dm du. \end{aligned}$$

The sum over the sets $A \subseteq Z_k$ above has $2^k = C_\epsilon$ terms. To summarize, for (6.30), it suffices to prove that for any set $A = \{j_1, \dots, j_{k'}\} \subseteq Z_k$,

$$\sum_{s_{j_1},\dots,s_{j_{k'}}} \int_{\mathbb{G}_0^{\#}} \Big(\sum_{r' \in \widetilde{T}_{c_A}} \Big| \sum_{\mu \in \widetilde{U}_{A,s_{j_1},\dots,s_{j_{k'}}}} G_{r'+\mu}(m,u) \Big|^2 \Big)^p \, dmdu \le C_\epsilon \lambda^{p\epsilon} ||f||^{2p}_{L^{2p}(\mathbb{G}_0^{\#})} \tag{6.34}$$

for any characteristic function of a bounded set f.

For $(s_{j_1}, \dots, s_{j_{k'}}) \in Z_{\beta(j_1)} \times \dots \times Z_{\beta(j_{k'})}$ and $r' \in \widetilde{T}_{c_A}$, let

$$\widetilde{G}_{r',s_{j_1}\dots,s_{j_{k'}}} = \sum_{\mu \in \widetilde{U}_{A,s_{j_1},\dots,s_{j_{k'}}}} G_{r'+\mu}.$$

We also define the function $f_{r',s_{j_1}\dots,s_{j_{k'}}} \in L^2(\mathbb{G}_0^{\#})$ by the formula

$$\mathcal{F}(f_{r',s_{j_1}\dots,s_{j_{k'}}})(m,\theta) = \widehat{f}(m,\theta) \sum_{b \in \mathbb{Z}^{d'}} \sum_{\mu \in \widetilde{U}_{A,s_{j_1},\dots,s_{j_{k'}}}} \psi[2^{2J_0-1}(\theta - r' - \mu - b/Q_0)/N_0]. \tag{6.35}$$

For the bound (6.34) it suffices to prove that

$$\sum_{s_{j_1},\dots,s_{j_{k'}}} \int_{\mathbb{G}_0^{\#}} \Big(\sum_{r' \in \widetilde{T}_{c_A}} \Big| f_{r',s_{j_1}\dots,s_{j_{k'}}}(m,u) \Big|^2 \Big)^p \, dmdu \le C_\epsilon ||f||^{2p}_{L^{2p}(\mathbb{G}_0^{\#})}, \tag{6.36}$$

and

$$\begin{aligned} &\int_{\mathbb{G}_0^{\#}} \Big(\sum_{r' \in \widetilde{T}_{c_A}} \Big| \widetilde{G}_{r',s_{j_1}\dots,s_{j_{k'}}}(m,u) \Big|^2 \Big)^p \, dmdu \\ &\qquad \le C_\epsilon \lambda^{p\epsilon} \int_{\mathbb{G}_0^{\#}} \Big(\sum_{r' \in \widetilde{T}_{c_A}} \Big| f_{r',s_{j_1}\dots,s_{j_{k'}}}(m,u) \Big|^2 \Big)^p \, dmdu, \end{aligned} \tag{6.37}$$

for any $(s_{j_1}, \dots, s_{j_{k'}}) \in Z_{\beta(j_1)} \times \dots \times Z_{\beta(j_{k'})}$ fixed. The bound (6.36) follows from Lemma 6.5 below. The bound (6.37) follows from Lemma 6.6 below and the identity

$$\widetilde{G}_{r',s_{j_1}\dots,s_{j_{k'}}} = \gamma(q(r') \cdot q_{j_1,s_{j_1}} \cdot \ldots \cdot q_{j_{k'},s_{j_{k'}}}) \sum_{j \ge J_0} \mathcal{T}_j^{N_0,\mathcal{R}_{r'},Q'}(f_{r',s_{j_1}\dots,s_{j_{k'}}}), \tag{6.38}$$

where $Q' = Q_0 \cdot q_{j_1,s_{j_1}} \cdot \ldots \cdot q_{j_{k'},s_{j_{k'}}}$ and $q(r')$ is the denominator of the irreducible d'-fraction r' (see the notation in Lemma 6.6). The identity (6.38) follows from the definitions and the observation

$$\sum_{b\in\mathbb{Z}^{d'}}\sum_{\mu\in\widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}}\psi[2^{2j}(\theta-r'-\mu-b/Q_0)/N_0]=\sum_{b'\in\mathbb{Z}^{d'}}\psi[2^{2j}(\theta-r'-b'/Q')/N_0]$$
$$\times\sum_{b\in\mathbb{Z}^{d'}}\sum_{\mu\in\widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}}\psi[2^{2J_0-1}(\theta-r'-\mu-b/Q_0)/N_0].$$

Lemma 6.5. *With the notation above,*

$$\sum_{s_{j_1},\ldots,s_{j_{k'}}}\int_{\mathbb{G}_0^{\#}}\Big(\sum_{r'\in\widetilde{T}_{c_A}}\Big|f_{r',s_{j_1}\ldots,s_{j_{k'}}}(m,u)\Big|^2\Big)^p\,dmdu\le C_\epsilon||f||^{2p}_{L^{2p}(\mathbb{G}_0^{\#})}$$

for any characteristic function of a bounded set f.

Proof of Lemma 6.5. This is similar to the proof of [8, Lemma 4.3], and is inspired by the Littlewood–Paley inequality in [13]. Clearly, since $f:\mathbb{G}_0^{\#}\to\{0,1\}$, $||f||^{2p}_{L^{2p}(\mathbb{G}_0^{\#})}=||f||^2_{L^2(\mathbb{G}_0^{\#})}$. In addition, by Plancherel theorem,

$$\sum_{s_{j_1},\ldots,s_{j_{k'}}}\int_{\mathbb{G}_0^{\#}}\sum_{r'\in\widetilde{T}_{c_A}}\Big|f_{r',s_{j_1}\ldots,s_{j_{k'}}}(m,u)\Big|^2\,dmdu\le C||f||^2_{L^2(\mathbb{G}_0^{\#})},$$

since the function $\mathcal{F}[f_{r',s_{j_1}\ldots,s_{j_{k'}}}](m,.)$ is supported in a $4N_02^{-2J_0}$ neighborhood of the set

$$r'+\sum_{b\in\mathbb{Z}^{d'}}\sum_{\mu\in\widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}}b/Q_0+\mu.$$

These neighborhoods are disjoint, as $r'\in\widetilde{T}_{c_A}$ and $(s_{j_1},\ldots,s_{j_{k'}})\in Z_{\beta(j_1)}\times\ldots\times Z_{\beta(j_{k'})}$, see (6.26), (6.28), and (6.31). Therefore it suffices to prove that for any $(s_{j_1},\ldots,s_{j_{k'}})\in Z_{\beta(j_1)}\times\ldots\times Z_{\beta(j_{k'})}$ and $(m,u)\in\mathbb{Z}^d\times\mathbb{Z}^{d'}$,

$$\sum_{r'\in\widetilde{T}_{c_A}}\Big|f_{r',s_{j_1}\ldots,s_{j_{k'}}}(m,u)\Big|^2\le C_\epsilon.$$

Thus, it suffices to prove that for $(s_{j_1},\ldots,s_{j_{k'}})\in Z_{\beta(j_1)}\times\ldots\times Z_{\beta(j_{k'})}$ fixed

$$\Big|\sum_{r'\in\widetilde{T}_{c_A}}\nu(r')f_{r',s_{j_1}\ldots,s_{j_{k'}}}(m,u)\Big|\le C_\epsilon,$$

for any $(m,u)\in\mathbb{Z}^d\times\mathbb{Z}^{d'}$, and any complex numbers $\nu(r')$ with

$$\sum_{r'\in\widetilde{T}_{c_A}}|\nu(r')|^2=1. \tag{6.39}$$

Since $||f||_{L^\infty} \leq 1$, it suffices to prove that for $(s_{j_1}, \ldots, s_{j_{k'}}) \in Z_{\beta(j_1)} \times \ldots \times Z_{\beta(j_{k'})}$ fixed

$$\Big|\Big|\mathcal{F}^{-1}\Big(\theta \to \sum_{r' \in \widetilde{T}_{^cA}} \nu(r') \sum_{b \in \mathbb{Z}^{d'}} \sum_{\mu \in \widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}} \psi[2^{2J_0-1}(\theta - r' - \mu - b/Q_0)/N_0]\Big)\Big|\Big|_{L^1(\mathbb{Z}^{d'})} \leq C_\epsilon. \tag{6.40}$$

As before, let $\eta(x) = \int_{\mathbb{R}^{d'}} \psi(\xi) e^{2\pi i x\cdot\xi}\, d\xi$ denote the Euclidean inverse Fourier transform of the function ψ. An easy calculation shows that

$$\begin{aligned}
&\mathcal{F}^{-1}\Big(\theta \to \sum_{r' \in \widetilde{T}_{^cA}} \nu(r') \sum_{b \in \mathbb{Z}^{d'}} \sum_{\mu \in \widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}} \psi[2^{2J_0-1}(\theta - r' - \mu - b/Q_0)/N_0]\Big)(u) \\
&= (\sum_{r' \in \widetilde{T}_{^cA}} \nu(r') e^{2\pi i u\cdot r'}) \cdot \eta_{2^{2J_0-1}/N_0}(u) \cdot (\sum_{\mu \in \widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}} \sum_{b \in Z^{d'}_{Q_0}} e^{2\pi i u\cdot(b/Q_0+\mu)}).
\end{aligned} \tag{6.41}$$

We consider first the sum over b and μ in (6.41). For any integer $Q' \geq 1$ define the function $\delta_{Q'} : \mathbb{Z}^{d'} \to \mathbb{Z}$ as in (4.58). Clearly, $\sum_{b \in Z^{d'}_{Q'}} e^{2\pi i u\cdot b/Q'} = \delta_{Q'}(u)$. Recall from section 6.2 that $q_{j,s_j} = p^{\alpha_j}_{j,s_j}$, for some primes $p_{j,s_j} \in V$ and $\alpha_j \in [1, C_\epsilon] \cap \mathbb{Z}$. In addition, it is easy to see that if $q = p^\alpha$ and $(Q,p) = 1$ then

$$\{a/q + b/Q : b \in \mathbb{Z}^{d'},\, a \in \widetilde{P}_q\} = \{b'/(Qp^\alpha) : b' \in \mathbb{Z}^{d'}\} \setminus \{b'/(Qp^{\alpha-1}) : b' \in \mathbb{Z}^{d'}\}.$$

Thus, for $(s_{j_1}, \ldots, s_{j_{k'}}) \in Z_{\beta(j_1)} \times \ldots \times Z_{\beta(j_{k'})}$ fixed,

$$\sum_{\mu \in \widetilde{U}_{A,s_{j_1},\ldots,s_{j_{k'}}}} \sum_{b \in \mathbb{Z}^{d'}_{Q_0}} m(\mu + b/Q_0) = \sum_{\varepsilon_{j_1},\ldots,\varepsilon_{j_{k'}} \in \{0,1\}} (-1)^{\varepsilon_{j_1}+\ldots+\varepsilon_{j_{k'}}} \sum_{b \in \mathbb{Z}^{d'}_{Q'}} m(b/Q'), \tag{6.42}$$

for any periodic function $m : \mathbb{R}^{d'} \to \mathbb{C}$, where $Q' = Q_0 p^{\alpha_{j_1}-\varepsilon_{j_1}}_{j_1,s_{j_1}} \cdot \ldots \cdot p^{\alpha_{j_{k'}}-\varepsilon_{j_{k'}}}_{j_{k'},s_{j_{k'}}}$. The possible values of Q' are products of Q_0 and $p^{\alpha_l}_{j_l,s_{j_l}}$ or $p^{\alpha_l-1}_{j_l,s_{j_l}}$, $l = 1, \ldots, k'$. and the sum over $\varepsilon_{j_1}, \ldots, \varepsilon_{j_{k'}} \in \{0,1\}$ contains $2^{k'} = C_\epsilon$ terms. Thus, for (6.40), it suffices to prove that

$$||(\sum_{r' \in \widetilde{T}_{^cA}} \nu(r') e^{2\pi i u\cdot r'}) \eta_{2^{2J_0-1}/N_0}(u) \delta_{Q'}(u)||_{L^1_u(\mathbb{Z}^{d'})} \leq C_\epsilon,$$

for any Q' with (see (6.28) and (6.31))

$$Q' \in [1, e^{\lambda^{\epsilon/5}}] \cap \mathbb{Z} \text{ and } (Q', q_{j,s}) = 1 \text{ for any } j \in {}^cA, s \in Z_{\beta(j)}. \tag{6.43}$$

This is equivalent to proving that

$$||(\sum_{r'\in\widetilde{T}_{c_A}}\nu(r')e^{2\pi iQ'u\cdot r'})\eta_{2^{2J_0}/(2N_0Q')}(u)||_{L^1_u(\mathbb{Z}^{d'})}\leq C_\epsilon, \tag{6.44}$$

provided that (6.39) and (6.43) hold.

Let $\gamma_0 = 2^{-2J_0}2N_0Q' \ll 1$. The function η is a Schwartz function on $\mathbb{R}$; by Hölder's inequality, for (6.44) it suffices to prove that

$$\gamma_0^{d'/2}||(\sum_{r'\in\widetilde{T}_{c_A}}\nu(r')e^{2\pi iQ'u\cdot r'})\cdot(1+\gamma_0^2|u|^2)^{-d'}||_{L^2_u(\mathbb{Z}^{d'})}\leq C_\epsilon. \tag{6.45}$$

The left-hand side of (6.45) is equal to

$$\gamma_0^{d'/2}\Big[\sum_{r'_1,r'_2\in\widetilde{T}_{c_A}}\nu(r'_1)\overline{\nu(r'_2)}\int_{\mathbb{Z}^{d'}}(1+\gamma_0^2u^2)^{-2d'}e^{2\pi iu\cdot Q'(r'_1-r'_2)}\,du\Big]^{1/2}. \tag{6.46}$$

It remains to estimate the integrals over $\mathbb{Z}^{d'}$ in (6.46). If $r'_1 = r'_2$ then

$$\Big|\int_{\mathbb{Z}^{d'}}(1+\gamma_0^2|u|^2)^{-2d'}e^{2\pi iu\cdot Q'(r'_1-r'_2)}\,du\Big|\leq C\gamma_0^{-d'}. \tag{6.47}$$

If $r'_1\neq r'_2$ then, by (6.43), $Q'(r'_1-r'_2)\notin\mathbb{Z}^{d'}$. Let $\zeta = (\zeta_{l_1l_2})_{l_1,l_2=1,\ldots,d}$ denote the fractional part of $Q'(r'_1-r'_2)$. Since the denominators of r'_1 and r'_2 are bounded by λ^{C_ϵ}, there are $l_1,l_2\in\{1,\ldots,d'\}$ with the property that $\zeta_{l_1l_2}\in[\lambda^{-C_\epsilon},1-\lambda^{-C_\epsilon}]$. By summation by parts in the variable $u_{l_1l_2}$ corresponding to this $\zeta_{l_1l_2}$,

$$\Big|\int_{\mathbb{Z}^{d'}}(1+\gamma_0^2|u|^2)^{-2d'}e^{2\pi iu\cdot Q'(r'_1-r'_2)}\,du\Big|\leq C\gamma_0^{-d'+1}\lambda^{C_\epsilon} \tag{6.48}$$

if $r'_1\neq r'_2$. We substitute (6.47) and (6.48) in (6.46). It follows that the left-hand side of (6.45) is dominated by

$$C[\sum_{r'\in\widetilde{T}_{c_A}}|\nu(r')|^2+\gamma_0\lambda^{C_\epsilon}(\sum_{r'\in\widetilde{T}_{c_A}}|\nu(r')|)^2]^{1/2}.$$

Since $|\widetilde{T}_{c_A}|\leq\lambda^{C_\epsilon}$ and $\gamma_0\leq e^{-\lambda^{\epsilon/2}}$, the bound (6.45) follows from (6.39) and Hölder's inequality. This completes the proof of Lemma 6.5. □

Lemma 6.6. *Assume, as before, that $Q\in[1,e^{\lambda^{\epsilon/5}}]\cap\mathbb{Z}$, $J_0=\lambda^\epsilon$, $N_0\leq\lambda^C$. For any irreducible d'-fraction $r=a/q$ with $q\in[1,\lambda^{C_\epsilon}]\cap\mathbb{Z}$ and $(Q,q)=1$ let*

$$\mathcal{R}_{r,Q}=\{r+b/Q : b\in\mathbb{Z}^{d'}\},$$

and, as in (6.4),

$$\mathcal{F}[\mathcal{T}_j^{N_0,\mathcal{R}_{r,Q}}(f)](m,\theta)=\widehat{\mathcal{T}_j(f)}\sum_{b\in\mathbb{Z}^{d'}}\psi(2^{2j}(\theta-r-b/Q)/N_0).$$

Then,

$$\Big\|\Big[\sum_r\Big|\sum_{j\geq J_0}\mathcal{T}_j^{N_0,\mathcal{R}_{r,Q}}(f_r)\Big|^2\Big]^{1/2}\Big\|_{L^{2p}(\mathbb{G}_0^\#)}\leq C_\epsilon(\ln\lambda)^C\Big\|\Big[\sum_r|f_r|^2\Big]^{1/2}\Big\|_{L^{2p}(\mathbb{G}_0^\#)}, \tag{6.49}$$

for any (compactly supported) functions $f_r:\mathbb{G}_0^\#\to\mathbb{C}$, where the sums are taken over irreducible d'-fractions $r=a/q$ with $q\in[1,\lambda^{C_\epsilon}]\cap\mathbb{Z}$ and $(Q,q)=1$.

Proof of Lemma 6.6. As in (4.26), in view of the definitions and the Fourier inversion formula,

$$\mathcal{T}_j^{N_0,\mathcal{R}_{r,Q}}(f_r)(m,u)=\sum_{(n,v)\in\mathbb{G}_0^\#}f_r(n,v)K_j(m-n)$$
$$\eta_{2^{2j}/N_0}(u-v-R_0(m-n,n))e^{2\pi i(u-v-R_0(m-n,n))\cdot r}\delta_Q(u-v-R_0(m-n,n)), \tag{6.50}$$

where δ_Q is defined in (4.58). We use the change of variable $\Phi_Q:\mathbb{G}_0^\#\times[Z_Q^d\times Z_{Q^2}^{d'}]\to\mathbb{G}_0^\#$ defined in (4.28). Let $F_r((n',v'),(\nu,\beta))=f_r(\Phi_Q((n',v'),(\nu,\beta)))$ and $G_r((m',u'),(\mu,\alpha))=\sum_{j\geq J_0}\mathcal{T}_j^{N_0,\mathcal{R}_{r,Q}}(f_r)(\Phi_Q((m',u'),(\mu,\alpha)))$. Then, by (6.50),

$$\begin{aligned}G_r((m',u'),(\mu,\alpha))=&\sum_{(n',v')\in\mathbb{G}_0^\#}\ \sum_{(\nu,\beta)\in Z_Q^d\times Z_{Q^2}^{d'}}F_r((n',v'),(\nu,\beta))\\&\sum_{j\geq J_0}K_j(Q(m'-n')+E_1)\eta_{2^{2j}/N_0}(Q^2(u'-v'-R_0(m'-n',n'))+E_2)\\&\times\delta_Q(\alpha-\beta-R_0(\mu-\nu,\nu))e^{2\pi iE_3\cdot r},\end{aligned}$$

where $E_1=\mu-\nu$,

$$E_2=(\alpha-\beta-R_0(\mu-\nu,\nu))+Q(R_0(\mu,m'-n')-R_0(m'-n',\nu)),$$

and

$$E_3=Q^2(u'-v'-R_0(m'-n',n'))+E_2.$$

Clearly, $|E_1|\leq CQ$ and $|E_2|\leq C2^jQ$ if $|m'-n'|\leq C2^j/Q$. Let

$$\begin{aligned}\widetilde{G}_r((m',u'),(\mu,\alpha))=&\sum_{(n',v')\in\mathbb{G}_0^\#}\ \sum_{(\nu,\beta)\in Z_Q^d\times Z_{Q^2}^{d'}}F_r((n',v'),(\nu,\beta))\\&\sum_{j\geq J_0}K_j(Q(m'-n'))\eta_{2^{2j}/N_0}(Q^2(u'-v'-R_0(m'-n',n')))\\&\times\delta_Q(\alpha-\beta-R_0(\mu-\nu,\nu))e^{2\pi iE_3\cdot r}.\end{aligned} \tag{6.51}$$

In view of the estimates above on $|E_1|$ and $|E_2|$ and the relative sizes of Q, J_0 and N_0 (see the statement of Lemma 6.6),

$$\begin{aligned}&|G_r((m',u'),(\mu,\alpha))-\widetilde{G}_r((m',u'),(\mu,\alpha))|\\&\leq C\sum_{(n',v')\in\mathbb{G}_0^{\#}}\sum_{(\nu,\beta)\in Z_Q^d\times Z_{Q^2}^{d'}}|F_r((n',v'),(\nu,\beta))|Q^{-d}Q^{-2d'}\delta_Q(\alpha-\beta-R_0(\mu-\nu,\nu))\\&\sum_{j\geq J_0}(N_0Q/2^j)(2^j/Q)^{-d}\mathbf{1}_{[0,C2^j/Q]}(|m'-n'|)\phi_{2^{2j}/(Q^2N_0)}(u'-v'-R_0(m'-n',n')),\end{aligned}$$

where ϕ is as in (7.7). The kernel in the formula defining $|G_r((m',u'),(\mu,\alpha))-\widetilde{G}_r((m',u'),(\mu,\alpha))|$ has L^1 norm dominated by $CN_0Q/2^{J_0}\leq C$. In view of the Marcinkiewicz–Zygmund theorem,

$$\Big\|[\sum_r|G_r-\widetilde{G}_r|^2]^{1/2}\Big\|_{L^{2p}(\mathbb{G}_0^{\#}\times[Z_Q^d\times Z_{Q^2}^{d'}])}\leq C\Big\|[\sum_r|F_r|^2]^{1/2}\Big\|_{L^{2p}(\mathbb{G}_0^{\#}\times[Z_Q^d\times Z_{Q^2}^{d'}])}.$$

Thus, for (6.49), it remains to prove that

$$\Big\|[\sum_r|\widetilde{G}_r|^2]^{1/2}\Big\|_{L^{2p}(\mathbb{G}_0^{\#}\times[Z_Q^d\times Z_{Q^2}^{d'}])}\leq C_\epsilon(\ln\lambda)^C\Big\|[\sum_r|F_r|^2]^{1/2}\Big\|_{L^{2p}(\mathbb{G}_0^{\#}\times[Z_Q^d\times Z_{Q^2}^{d'}])},\tag{6.52}$$

where $\widetilde{G}_r$ is defined in (6.51).

Assume $r=a/q$, $(a,q)=1$, and for any (ν,β) fixed define

$$\begin{aligned}H_r((m',u'),(\nu,\beta))=\sup_{a_1\in\mathbb{Z}^d,a_2\in\mathbb{Z}^{d'}}\Big|\sum_{(n',v')\in\mathbb{G}_0^{\#}}F_r((n',v'),(\nu,\beta))\sum_{j\geq J_0}K_j(Q(m'-n'))\\\times Q^d\eta_{2^{2j}/(Q^2N_0)}(u'-v'-R_0(m'-n',n'))e^{2\pi i[a_1\cdot(m'-n')+a_2\cdot(u'-v'-R_0(m'-n',n'))]/q}\Big|.\end{aligned}$$

Clearly,

$$\begin{aligned}&|\widetilde{G}_r((m',u'),(\mu,\alpha))|\\&\quad\leq\sum_{(\nu,\beta)\in Z_Q^d\times Z_{Q^2}^{d'}}H_r((m',u'),(\nu,\beta))Q^{-d}Q^{-2d'}\delta_Q(\alpha-\beta-R_0(\mu-\nu,\nu)),\end{aligned}$$

so, using the Marcinkiewicz–Zygmund theorem again,

$$\Big\|[\sum_r|\widetilde{G}_r|^2]^{1/2}\Big\|_{L^{2p}(\mathbb{G}_0^{\#}\times[Z_Q^d\times Z_{Q^2}^{d'}])}\leq C\Big\|[\sum_r|H_r|^2]^{1/2}\Big\|_{L^{2p}(\mathbb{G}_0^{\#}\times[Z_Q^d\times Z_{Q^2}^{d'}])}.\tag{6.53}$$

Thus, for (6.52), it suffices to prove that

$$\Big\|[\sum_r|H_r|^2]^{1/2}\Big\|_{L^{2p}(\mathbb{G}_0^{\#}\times[Z_Q^d\times Z_{Q^2}^{d'}])}\leq C_\epsilon(\ln\lambda)^C\Big\|[\sum_r|F_r|^2]^{1/2}\Big\|_{L^{2p}(\mathbb{G}_0^{\#}\times[Z_Q^d\times Z_{Q^2}^{d'}])}.\tag{6.54}$$

To prove (6.54) we use Lemma 7.4. The connection between weighted estimates and vector-valued inequalities is well-known (see, for example, [7, Chapter V, Theorem 6.1]). In our case, let $p' \in (1, \infty]$ denote the exponent dual of p. The left-hand side of (6.54) is dominated by

$$\sup_{w:\mathbb{G}_0^\# \times [Z_Q^d \times Z_{Q^2}^{d'}] \to [0,\infty),\, ||w||_{L^{p'}}=1} \Big[\int_{\mathbb{G}_0^\# \times [Z_Q^d \times Z_{Q^2}^{d'}]} \sum_r |H_r|^2 \cdot w \Big]^{1/2}. \tag{6.55}$$

We examine the definition of the functions H_r above and notice that for fixed $(\nu, \beta) \in Z_Q^d \times Z_{Q^2}^{d'}$,

$$H_r(h, (\nu, \beta)) \leq \widetilde{\mathcal{T}}_*^{\widetilde{N}_0, q}[F_r(., (\nu, \beta))](h),\ h \in \mathbb{G}_0^\#,$$

with the notation in Lemma 7.3. The operators $\widetilde{\mathcal{T}}_*^{\widetilde{N}_0, q}$ are as in the statement of Lemma 7.3, using the kernels $\widetilde{K}_j(x) = Q^d K_{j+j_1}(Qx)$, $j \geq \lambda^\epsilon/2$, where j_1 is the smallest integer with $2^{j_1} \geq Q$, and $\widetilde{N}_0 = Q^2 N_0/2^{2j_1}$. These kernels $\widetilde{K}_j$ clearly satisfy the basic properties (6.1), (6.2), and (7.23). For fixed $(\nu, \beta) \in Z_Q^d \times Z_{Q^2}^{d'}$ we define the function $w_*^{\widetilde{N}_0}(., (\nu, \beta))$ as in (7.21) and use the bounds (7.22) and Lemma 7.4 with $\rho = C_\epsilon \ln(\widetilde{N}_0 + 1)$. The expression in (6.55) is dominated by

$$\sup_{w:\mathbb{G}_0^\# \times [Z_Q^d \times Z_{Q^2}^{d'}] \to [0,\infty),\, ||w||_{L^{p'}}=1} C_\epsilon (\ln \lambda)^C \Big[\int_{\mathbb{G}_0^\# \times [Z_Q^d \times Z_{Q^2}^{d'}]} \sum_r |F_r|^2 \cdot w_*^{\widetilde{N}_0} \Big]^{1/2},$$

which easily leads to (6.54) (using again the bounds (7.22)). □

7. Real-variable theory on the group $\mathbb{G}_0^\#$

In this section, which is self-contained, we discuss some features of the real-variable theory on the group $\mathbb{G}_0^\#$. Our basic reference is [14, Chapters I, II, and V]. The main results in this section are the bound (7.11), which is used in section 4.3, and Lemma 7.4 which is used in section 6.3. We assume throughout this section that $d' = d^2$, and $\mathbb{G}_0^\#$ is the discrete nilpotent group defined in section 2.

7.1. **Weighted maximal functions.** We define the "distance" function $d : \mathbb{G}_0^\# \times \mathbb{G}_0^\# \to [0, \infty)$,

$$d(0, (m, u)) = \max(|m|, |u|^{1/2}),\ d(h, h') = d(0, h' \cdot h^{-1}) \text{ if } h, h' \in \mathbb{G}_0^\#. \tag{7.1}$$

It is easy to see that $d(h, h') \approx d(h', h)$ and $d(h, h'') \leq C(d(h, h') + d(h', h''))$ for any $h, h', h'' \in \mathbb{G}_0^\#$. We define the family of nonisotropic balls on $\mathbb{G}_0^\#$

$$\mathbb{B} = \{B(h, r) = \{g \cdot h : d(0, g) \leq r\},\ h \in \mathbb{G}_0^\#,\ r \geq 1/2\}, \tag{7.2}$$

and notice that we have the basic properties

$$\begin{cases} \text{if } B(h, r) \cap B(h', r) \neq \emptyset \text{ then } B(h', r) \subseteq B(h, C_1 r) \\ |B(h, C_1 r)| \leq C_2 |B(h, r)|, \end{cases} \tag{7.3}$$

for any $h, h' \in \mathbb{G}_0^\#$ and $r \geq 1/2$. As a consequence, we have the Whitney decomposition (see [14, p. 15]): if $O \subseteq \mathbb{G}_0^\#$ is a finite set then there are balls $B_k \in \mathbb{B}$, $k = 1, \dots, K$, with the properties

$$\begin{cases} B_k \cap B_{k'} = \emptyset \text{ for any } k \neq k'; \\ O = \cup_k B_k^*; \\ B_k^{**} \cap {}^cO \neq \emptyset, \end{cases} \tag{7.4}$$

where, if $B = B(h, r)$ then $B^* = B(h, c^*r)$ and $B^{**} = B(h, (c^*)^2 r)$ for a sufficiently large constant c^*. In addition, there are pairwise disjoint Whitney "cubes" Q_k with the properties $\cup_k Q_k = O$ and $B_k \subseteq Q_k \subseteq B_k^*$.

For any set $E \subseteq \mathbb{G}_0^\#$ and any function $w : \mathbb{G}_0^\# \to [0, \infty)$ let $w(E) = \int_E w(h)\, dh$. If $w : \mathbb{G}_0^\# \to [0, \infty)$ is a nonnegative function we define $L^p(w)$, $p \in [1, \infty]$, and $L^{1,\infty}(w)$ the corresponding weighted spaces on $\mathbb{G}_0^\#$. It follows from (7.3) that the standard non-centered maximal function

$$\widetilde{\mathcal{M}}(f)(h) = \sup_{h \in B \in \mathbb{B}} \frac{1}{|B|} \int_B |f(g)|\, dg, \tag{7.5}$$

extends to a bounded operator from $L^1(w)$ to $L^{1,\infty}(\widetilde{\mathcal{M}}(w))$:

$$\alpha \cdot w(\{h : \widetilde{\mathcal{M}}(f)(h) > \alpha\}) \leq C \int_{\mathbb{G}_0^\#} |f(h)| \widetilde{\mathcal{M}}(w)(h)\, dh, \tag{7.6}$$

for any $f : \mathbb{G}_0^\# \to \mathbb{C}$ and $\alpha \in (0, \infty)$ (see [14, p. 53]).

Let Ω, Ω_j be defined as in (4.1). In this section we assume, in addition, that $\Omega(x) = 1$ if $|x| \leq 2$. Let $\phi, \phi_r : \mathbb{R}^{d'} \to [0, 1]$ denote the functions

$$\phi(s) = (1 + |s|^2)^{-(d'+d+1)/2} \text{ and } \phi_r(s) = r^{-d'} \phi(s/r),\ r \geq 1. \tag{7.7}$$

Assume $N \geq 1$ is a real number. For integers $j \geq \log_2 N$ we define the kernels $A_j^N, {A'}_j^N : \mathbb{G}_0^\# \to [0, \infty)$,

$$A_j^N(m, u) = \Omega_j(m) \phi_{2^{2j}/N}(u) \text{ and } {A'}_j^N(g) = A_j^N(g^{-1}),\ g \in \mathbb{G}_0^\#.$$

For $N \geq 1$ and $f : \mathbb{G}_0^\# \to \mathbb{C}$ let

$$\mathcal{M}_*^N(f)(h) = \sup_{j \geq \log_2 N} |f * (A_j^N + {A'}_j^N)(h)| + \sup_{j \geq 0} |f * (A_j^1 + {A'}_j^1)(h)|. \tag{7.8}$$

We start with a weighted maximal inequality.

Lemma 7.1. *Assume $N, \rho \in [1, \infty)$, and $w : \mathbb{G}_0^\# \to (0, \infty)$ is a function with the property that*

$$\mathcal{M}_*^N(w)(h) \leq \rho \cdot w(h) \text{ for any } h \in \mathbb{G}_0^\#. \tag{7.9}$$

Then, for any compactly supported function $f:\mathbb{G}_0^{\#}\to\mathbb{C}$*,*

$$\begin{cases} ||\mathcal{M}_*^N(f)||_{L^{1,\infty}(w)}\leq C\rho^2\cdot\ln(N+1)||f||_{L^1(w)}; \\ ||\mathcal{M}_*^N(f)||_{L^p(w)}\leq C_p\rho^2\cdot\ln(N+1)||f||_{L^p(w)},\ p\in(1,\infty]. \end{cases} \tag{7.10}$$

In particular, if $w\equiv 1$*,*

$$\begin{cases} ||\mathcal{M}_*^N(f)||_{L^{1,\infty}(\mathbb{G}_0^{\#})}\leq C\ln(N+1)||f||_{L^1(\mathbb{G}_0^{\#})}; \\ ||\mathcal{M}_*^N(f)||_{L^p(\mathbb{G}_0^{\#})}\leq C_p\ln(N+1)||f||_{L^p(\mathbb{G}_0^{\#})},\ p\in(1,\infty]. \end{cases} \tag{7.11}$$

Proof of Lemma 7.1. The main issue is to prove that there is only a logarithmic loss in N in (7.10) and (7.11). Since the non-centered maximal operator $\widetilde{\mathcal{M}}$ in (7.5) is dominated by $C\mathcal{M}_*^1$, it follows from (7.9) that

$$w(B)/|B|\leq C\rho\cdot\min_{h\in B}w(h)\text{ for any ball }B\in\mathbb{B}. \tag{7.12}$$

We recall the Calderón–Zygmund decomposition of functions on $\mathbb{G}_0^{\#}$: if $f\in L^1(\mathbb{G}_0^{\#})$ and $\alpha\in(0,\infty)$ is a given "height", let $E_\alpha=\{h:\widetilde{\mathcal{M}}(f)(h)>\alpha\}$ and $E_\alpha=\cup_k B_k^*=\cup_k Q_k$ the Whitney decomposition of the set E_α (see (7.4)). Let

$$f_0(h)=\mathbf{1}_{{}^cE_\alpha}(h)f(h)+\sum_{k=1}^K\mathbf{1}_{Q_k}(h)\frac{1}{|Q_k|}\int_{Q_k}f(h')\,dh';$$
$$b_k(h)=\mathbf{1}_{Q_k}(h)\Big[f(h)-\frac{1}{|Q_k|}\int_{Q_k}f(h')\,dh'\Big].$$

Clearly, $f=f_0+\sum_{k=1}^K b_k$; in addition, directly from the definitions,

$$\begin{cases} |f_0(h)|\leq C\alpha\text{ for any }h\in\mathbb{G}_0^{\#}; \\ b_k\text{ is supported in }Q_k,\ \int_{\mathbb{G}_0^{\#}}b_k(h)\,dh=0. \end{cases} \tag{7.13}$$

Also, using (7.12) for the balls B_k^* and the definition of b_k,

$$\int_{\mathbb{G}_0^{\#}}|b_k(h)|w(h)\,dh\leq C\rho||f\cdot\mathbf{1}_{Q_k}||_{L^1(w)}. \tag{7.14}$$

By interpolation, we only need to prove the $L^1(w)\to L^{1,\infty}(w)$ bound in (7.10). Assume $f:\mathbb{G}_0^{\#}\to\mathbb{C}$ is a compactly supported function and fix $\alpha\in(0,\infty)$. It suffices to prove that

$$\alpha\cdot w(\{h:\mathcal{M}_*^N(f)(h)>\alpha\})\leq C\rho^2\cdot\ln(N+1)||f||_{L^1(w)}.$$

We use the Calderón–Zygmund decomposition $f=f_0+\sum_{k=1}^K b_k=f_0+b$ at height α/C, C sufficiently large. It suffices to prove that

$$\alpha\cdot w(\{h:\mathcal{M}_*^N(b)(h)>\alpha/2\})\leq C\rho^2\cdot\ln(N+1)||f||_{L^1(w)}. \tag{7.15}$$

For (7.15) it suffices to prove that

$$\alpha \sum_{k=1}^{K} w(B_k^{**}) \leq C\rho^2 ||f||_{L^1(w)}, \tag{7.16}$$

and

$$\sum_{k=1}^{K} \int_{{}^cB_k^{**}} \mathcal{M}_*^N(b_k)(h)w(h)\,dh \leq C\rho^2 \cdot \ln(N+1)||f||_{L^1(w)}, \tag{7.17}$$

where B_k^{**} are sufficiently large dilates of the balls B_k that appear in the Whitney decomposition of the set $E_{\alpha/C}$.

To prove (7.16) we use (7.12) and (7.6):

$$\begin{aligned}\alpha \sum_{k=1}^{K} w(B_k^{**}) &\leq C\rho\alpha \sum_{k=1}^{K} |B_k^{**}| \min_{h\in B_k^{**}} w(h) \leq C\rho\alpha \cdot w(\{h : \widetilde{\mathcal{M}}(f)(h) > \alpha/C\}) \\ &\leq C\rho \int_{\mathbb{G}_0^\#} |f(h)|\widetilde{\mathcal{M}}(w)(h)\,dh \leq C\rho^2||f||_{L^1(w)},\end{aligned}$$

as desired.

To prove (7.17) we use (7.14) and the fact that the cubes Q_k are pairwise disjoint. By translation invariance, it suffices to prove that if $B = B(0,r)$ is a ball centered at 0 and $f : \mathbb{G}_0^\# \to \mathbb{C}$ is a function supported in the ball B with the property that $\int_{\mathbb{G}_0^\#} f(g)\,dg = 0$, then

$$\begin{aligned}&\sum_{j\geq \log_2 N} \int_{{}^cB^*} |f * (A_j^N + A'^N_j)(h)|w(h)\,dh \\ &\quad + \sum_{j\geq 0} \int_{{}^cB^*} |f * (A_j^1 + A'^1_j)(h)|w(h)\,dh \leq C\rho \cdot \ln(N+1)||f||_{L^1(w)},\end{aligned} \tag{7.18}$$

where, as before, $B^* = B(0, c^*r)$, c^* sufficiently large. To prove (7.18) it suffices to control the first sum in the left-hand side (the second sum corresponds to the particular case $N = 1$). Since $r \geq 1/2$, fix $k_0 \in \mathbb{Z} \cap [-1,\infty)$ such that $2^{k_0} \leq r < 2^{k_0+1}$. We divide the sum in j in (7.18) into three parts: $j \leq k_0$, $j \in [k_0, k_0 + 2\ln(N+1)]$, and $j \geq k_0 + 2\ln(N+1)$.

For $\log_2 N \leq j \leq k_0$, ignoring the condition $\int_{\mathbb{G}_0^\#} f(g)\,dg = 0$, we notice that if $h \in {}^cB^*$, $g \in B$, and c^* is sufficiently large, then $\min(d(0, h\cdot g^{-1}), d(0, g\cdot h^{-1})) \geq (c^*/2)2^{k_0}$. From the definitions,

$$(A_j^N + A'^N_j)(hg^{-1}) \leq C2^{j-k_0}(A_{k_0+2}^N + A'^N_{k_0+2})(gh^{-1}).$$

Thus, using (7.9),

$$\int_{{}^cB^*} |f*(A_j^N + {A'}_j^N)(h)|w(h)\,dh \\ \leq C2^{j-k_0}\int_B |f(g)|\cdot[w*(A_{k_0+2}^N + {A'}_{k_0+2}^N)(g)]\,dg \leq C\rho 2^{j-k_0}||f||_{L^1(w)}, \tag{7.19}$$

which suffices to prove (7.18) for this part of the sum.

For $j \geq \log_2 N$ and $j \in [k_0, k_0 + 2\ln(N+1)]$, we use (7.9) as before and notice that the sum contains at most $C\ln(N+1)$ terms.

For $j \geq k_0 + 2\ln(N+1)$ we use the condition $\int_{\mathbb{G}_0^\#} f(g)\,dg = 0$ and write

$$|f*(A_j^N + {A'}_j^N)(h)| \leq \int_B |f(g)|\cdot|(A_j^N + {A'}_j^N)(hg^{-1}) - (A_j^N + {A'}_j^N)(h)|\,dg.$$

Assume $h = (n, v) \in {}^cB^*$ and $g = (m, u) \in B$. Then $hg^{-1} = (n - m, v - u - R_0(n - m, m))$ and

$$\begin{aligned} |A_j^N(hg^{-1}) - A_j^N(h)| &\leq |\Omega_j(n-m) - \Omega_j(n)|\phi_{2^{2j}/N}(v) \\ &\quad + \Omega_j(n-m)|\phi_{2^{2j}/N}(v - u - R_0(n-m,m)) - \phi_{2^{2j}/N}(v)| \\ &\leq C(N+1)2^{k_0-j}[2^{-dj}\mathbf{1}_{[0,2^{j+3}]}(n)]\phi_{2^{2j}/N}(v) \\ &\leq C(N+1)2^{k_0-j}A_{j+3}^N(hg^{-1}). \end{aligned} \tag{7.20}$$

Similar estimates show that

$$|{A'}_j^N(hg^{-1}) - {A'}_j^N(h)| \leq C(N+1)2^{k_0-j}{A'}_{j+3}^N(hg^{-1})$$

The estimate (7.18) for this part of the sum follows using (7.9), as in (7.19). This completes the proof of Lemma 7.1. □

We explain now how to construct weights with the property (7.9). Assume $p \in (1, \infty]$, $w : \mathbb{G}_0^\# \to [0, \infty)$, and $w \in L^p(\mathbb{G}_0^*)$. For $N \geq 1$ let

$$w_*^N = \sum_{k=0}^\infty [C_p \ln(N+1)]^{-k}[\mathcal{M}_*^N]^k(w), \tag{7.21}$$

where C_p is a sufficiently large constant. Then, using (7.11),

$$\begin{cases} w(h) \leq w_*^N(h) \text{ for any } h \in \mathbb{G}_0^\# \text{ and } ||w_*^N||_{L^p(\mathbb{G}_0^\#)} \leq C||w||_{L^p(\mathbb{G}_0^\#)}; \\ \mathcal{M}_*^N(w_*^N)(h) \leq C_p \ln(N+1)w_*^N(h) \text{ for any } h \in \mathbb{G}_0^\#. \end{cases} \tag{7.22}$$

In particular, (7.9) holds for the function w_*^N with $\rho = C_p\ln(N+1)$. We use this construction in the proof of Lemma 6.6 in section 6.3.

7.2. **Maximal oscillatory singular integrals.** We consider now singular integrals on the group $\mathbb{G}_0^\#$. The main result in this subsection is Lemma 7.4. Let $K_j : \mathbb{R}^d \to \mathbb{C}$, $j = 0, 1, \ldots$, denote a family of kernels on $\mathbb{R}^d$ with the properties (6.1) and (6.2). In this section it is convenient to make a slightly less restrictive assumption on the supports of K_j, namely

$$K_j \text{ is supported in the set } \{x : |x| \in [c_0 2^{j-1}, c_0 2^{j+1}]\} \text{ for some } c_0 \in [1/2, 2]. \tag{7.23}$$

Assume $\eta \in \mathcal{S}(\mathbb{R}^{d'})$ is a fixed Schwartz function and let

$$\eta_r(s) = r^{-d'}\eta(s/r),\ s \in \mathbb{R}^{d'},\ r \geq 1.$$

Assume $N \geq 1$ is a real number. For integers $j \geq \log_2 N$ we define the kernels $L_j^N : \mathbb{G}_0^\# \to \mathbb{C}$,

$$L_j^N(m, u) = K_j(m)\eta_{2^{2j}/N}(u).$$

For (compactly supported) functions $f : \mathbb{G}_0^\# \to \mathbb{C}$ let $\mathcal{T}_j^N(f) = f * L_j^N$ and $\mathcal{T}_{\geq j}^N(f) = \sum_{j'=j}^\infty \mathcal{T}_{j'}^N(f)$.

Lemma 7.2 (Maximal singular integrals)**.** *Assume $N \in [1, \infty)$. The maximal singular integral operator*

$$\mathcal{T}_*^N(f)(h) = \sup_{j \geq \ln N} |\mathcal{T}_{\geq j}^N(f)(h)|$$

extends to a bounded (subadditive) operator on $L^p(\mathbb{G}_0^\#)$, $p \in (1, \infty)$, with

$$||\mathcal{T}_*^N||_{L^p \to L^p} \leq C_p[\ln(N+1)]^2. \tag{7.24}$$

Proof of Lemma 7.2. As in Lemma 7.1, the main issue is to prove that there is only a logarithmic loss in N in (7.24). We show first that

$$\Big|\Big| \sum_{j \geq \ln N} \mathcal{T}_j^N \Big|\Big|_{L^2 \to L^2} \leq C \ln(N+1). \tag{7.25}$$

In the proof of (7.25) we assume that the kernels K_j satisfy the slightly different cancellation condition $\sum_{m \in \mathbb{Z}^d} K_j(m) = 0$ instead of (6.2). The two cancellation conditions are equivalent (at least in the proof of (7.25)) by replacing K_j with $K_j - C_j 2^{-j}\varphi_j$ for suitable constants $|C_j| \leq C$, where $\varphi : \mathbb{R}^d \to [0, 1]$ is a smooth function supported in $\{|x| \in [1/2, 2]\}$ and $\varphi_j(x) = (c_0 2^j)^{-d}\varphi(x/(c_0 2^j))$. By abuse of notation, in the proof of (7.25) we continue to denote by $\mathcal{T}_j^N$, L_j^N etc the operators and the kernels corresponding to these modified kernels K_j. Clearly, $||\mathcal{T}_j^N||_{L^2 \to L^2} \leq C$ for any $j \geq \log_2 N$. By the Cotlar–Stein lemma, it suffices to prove that

$$||\mathcal{T}_i^N[\mathcal{T}_j^N]^*||_{L^2 \to L^2} + ||\,[\mathcal{T}_i^N]^*\mathcal{T}_j^N||_{L^2 \to L^2} \leq C(N+1)2^{-|i-j|} \tag{7.26}$$

for any $i, j \geq \log_2 N$ with $|i-j| \geq 2\ln(N+1)$. Assume that $i \geq j$. The kernel of the operator $\mathcal{T}_i^N[\mathcal{T}_j^N]^*$ is

$$L_{i,j}^N(g) = \int_{\mathbb{G}_0^\#} \overline{L}_j^N(h) L_i^N(gh)\,dh.$$

Using the cancellation condition (6.2), with $h = (n, v)$,

$$\begin{aligned}|L_{i,j}^N(g)| &\leq \int_{|v|\leq 2^{i+j}} |L_j^N(h)| \cdot |L_i^N(gh) - L_i^N(g)|\,dh + \int_{|v|\geq 2^{i+j}} |L_j^N(h)| \cdot |L_i^N(gh)|\,dh \\ &= I_1(g) + I_2(g).\end{aligned}$$

An estimate similar to (7.20) shows that

$$I_1(m,u) \leq C(N+1)2^{-|i-j|}[2^{-di}\mathbf{1}_{[0,2^{i+3}]}(m)]\phi_{2^{2i}/N}(u).$$

Also, by integrating the variable g first, it is easy to see that $||I_2||_{L^1(\mathbb{G}_0^\#)} \leq C(N+1)2^{-|i-j|}$. The bound for the first term in (7.26) follows. The bound for the second term in (7.26) is similar, which completes the proof of (7.25).

The proof of (7.20) shows that

$$\sum_{j\geq \log_2 N} \int_{{}^cB(0,c^*r)} |L_j^N(hg^{-1}) - L_j^N(h)|\,dh \leq C\ln(N+1),$$

for any $r > 0$ and $g \in B(0,r)$. Let $\mathcal{T}^N(f) = \sum_{j\geq \ln N} \mathcal{T}_j^N$. It follows from (7.25) and standard Calderón–Zygmund theory that

$$||\mathcal{T}^N||_{L^1\to L^{1,\infty}} \leq C\ln(N+1) \text{ and } ||\mathcal{T}^N||_{L^p\to L^p} \leq C_p\ln(N+1),\ p\in(1,\infty). \tag{7.27}$$

We turn now to the proof of (7.24). In view of (7.11) and (7.27), it suffices to prove the pointwise bound

$$\mathcal{T}_*^N(f)(h) \leq C\ln(N+1)[\widetilde{\mathcal{M}}(\mathcal{M}_*^N(|f|))(h) + \widetilde{\mathcal{M}}(|\mathcal{T}^N(f)|)(h)] \tag{7.28}$$

for any $h \in \mathbb{G}_0^\#$, where $\widetilde{\mathcal{M}}$ is the non-centered maximal operator defined in (7.5). By translation invariance, it suffices to prove this bound for $h = 0$. Thus, it suffices to prove that for any $k_0 \geq \log_2 N$,

$$\Big|\sum_{j\geq k_0} \mathcal{T}_j^N(f)(0)\Big| \leq C\ln(N+1)[\widetilde{\mathcal{M}}(\mathcal{M}_*^N(|f|))(0) + \widetilde{\mathcal{M}}(|\mathcal{T}^N(f)|)(0)]. \tag{7.29}$$

Assume k_0 fixed and let $f_1 = f\cdot\mathbf{1}_{B(0,2^{k_0-2})}$ and $f_2 = f - f_1$. It follows from the definitions that $\sum_{j\geq k_0}\mathcal{T}_j^N(f)(0) = \sum_{j\geq k_0}\mathcal{T}_j^N(f_2)(0)$.

We show first that for any $h \in B(0, c2^{k_0})$, c sufficiently small,

$$\Big|\sum_{j\geq k_0} \mathcal{T}_j^N(f_2)(0) - \mathcal{T}^N(f_2)(h)\Big| \leq C\ln(N+1)[\mathcal{M}_*^N(|f|)(0) + \mathcal{M}_*^N(|f|)(h)]. \tag{7.30}$$

To prove (7.30) we notice first that

$$\Big|\sum_{j\in[k_0,k_0+2\ln(N+1))}\mathcal{T}_j^N(f_2)(0)-\sum_{j\in[k_0,k_0+2\ln(N+1))}\mathcal{T}_j^N(f_2)(h)\Big|$$

is clearly controlled by the right-hand side of (7.30). In addition,

$$\begin{aligned}\Big|\sum_{j\geq k_0+2\ln(N+1)}&\mathcal{T}_j^N(f_2)(0)-\sum_{j\geq k_0+2\ln(N+1)}\mathcal{T}_j^N(f_2)(h)\Big|\\&\leq\sum_{j\geq k_0+2\ln(N+1)}\int_{\mathbb{G}_0^\#}|f_2(g^{-1})|\cdot|L_j^N(g)-L_j^N(hg)|\,dg\\&\leq C\sum_{j\geq k_0+2\ln(N+1)}(N+1)2^{k_0-j}\mathcal{M}_*^N(|f_2|)(0),\end{aligned}$$

using an estimate on the difference $|L_j^N(g)-L_j^N(hg)|$ similar to (7.20). Finally,

$$\begin{aligned}\Big|\sum_{j\in[\log_2 N,k_0)}\mathcal{T}_j^N(f_2)(h)\Big|&\leq\int_{\mathbb{G}_0^\#}|f_2(g^{-1})|\cdot\Big(\sum_{j\in[\log_2 N,k_0]}|L_j^N(hg)|\Big)\,dg\\&\leq C\int_{\mathbb{G}_0^\#}|f_2(g^{-1})|\cdot A_{k_0}^N(hg)\,dg\leq C\mathcal{M}_*^N(|f_2|)(h).\end{aligned}$$

The bound (7.30) follows. Thus, for any $h\in B(0,c2^{k_0})$,

$$\begin{aligned}&\Big|\sum_{j\geq k_0}\mathcal{T}_j^N(f)(0)\Big|\\&\leq C\ln(N+1)[\mathcal{M}_*^N(|f|)(0)+\mathcal{M}_*^N(|f|)(h)]+|\mathcal{T}^N(f)(h)|+|\mathcal{T}^N(f_1)(h)|.\end{aligned}$$

The proof of (7.29) now follows easily as in [14, Chapter I, Section 7.3], using (7.11) and (7.27). This completes the proof of the lemma. □

In the proof of Lemma 6.6 we need bounds on more general oscillatory singular integral operators. Assume $q\geq 1$ is an integer, $N\geq 1$ is a real number as before, $a_1\in\mathbb{Z}^d$, and $a_2\in\mathbb{Z}^{d'}$. For integers $j\geq\log_2(2Nq)$ and K_j as in (6.1), (6.2), and (7.23), we define the kernels $L_{j,a_1,a_2}^{N,q}:\mathbb{G}_0^\#\to\mathbb{C}$,

$$L_{j,a_1,a_2}^{N,q}(m,u)=K_j(m)\eta_{2^{2j}/N}(u)e^{2\pi i(a_1\cdot m+a_2\cdot u)/q}.$$

For (compactly supported) functions $f:\mathbb{G}_0^\#\to\mathbb{C}$ let

$$\mathcal{T}_{j,a_1,a_2}^{N,q}(f)=f*L_{j,a_1,a_2}^{N,q}\text{ and }\mathcal{T}_{\geq j,a_1,a_2}^{N,q}(f)=\sum_{j'=j}^{\infty}\mathcal{T}_{j',a_1,a_2}^{N,q}(f).$$

Lemma 7.3 (Maximal oscillatory singular integrals)**.** *Assume $N \in [1,\infty)$. The maximal oscillatory singular integral operator*

$$\mathcal{T}_*^{N,q}(f)(h) = \sup_{a_1\in\mathbb{Z}^d, a_2\in\mathbb{Z}^{d'}} \sup_{j\geq \log_2(2Nq)} |\mathcal{T}^{N,q}_{\geq j,a_1,a_2}(f)(h)|$$

extends to a bounded (subadditive) operator on $L^p(\mathbb{G}_0^\#)$, $p\in(1,\infty)$, with

$$||\mathcal{T}_*^{N,q}||_{L^p\to L^p} \leq C_p[\ln(N+1)]^2. \tag{7.31}$$

Proof of Lemma 7.3. Notice first that the case $q=1$ follows from Lemma 7.2 since $L^{N,1}_{j,a_1,a_2} = L^N_j$. To deal with the case $q\geq 2$, we use the coordinates (4.28) on $\mathbb{G}_0^\#$ adapted to the factor q,

$$\begin{cases} \Phi_q : \mathbb{G}_0^\# \times [Z_q^d \times Z_{q^2}^{d'}] \to \mathbb{G}_0^\#, \\ \Phi_q((m',u'),(\mu,\alpha)) = (qm'+\mu, q^2u'+\alpha+qR_0(\mu,m')). \end{cases}$$

Let $F((n',v'),(\nu,\beta)) = f(\Phi_q((n',v'),(\nu,\beta)))$ and

$$G_{j,a_1,a_2}((m',u'),(\mu,\alpha)) = \mathcal{T}^{N,q}_{\geq j,a_1,a_2}(f)(\Phi_q((m',u'),(\mu,\alpha))).$$

The definitions show that

$$\begin{aligned} G_{j,a_1,a_2}((m',u'),(\mu,\alpha)) &= \sum_{(n',v')\in\mathbb{G}_0^\#} \sum_{(\nu,\beta)\in Z_q^d\times Z_{q^2}^{d'}} F((n',v'),(\nu,\beta)) \\ &\sum_{j'=j}^{\infty} K_{j'}(q(m'-n')+E_1)\eta_{2^{2j'}/N}(q^2(u'-v'-R_0(m'-n',n'))+E_2) \\ &\times e^{2\pi i[a_1\cdot(\mu-\nu)+a_2\cdot(\alpha-\beta-R_0(\mu-\nu,\nu))]/q}, \end{aligned}$$

where $E_1 = \mu-\nu$ and

$$E_2 = (\alpha-\beta-R_0(\mu-\nu,\nu)) + q(R_0(\mu,m'-n')-R_0(m'-n',\nu)).$$

Clearly, $|E_1|\leq Cq$ and $|E_2|\leq C2^{j'}q$ if $|m'-n'|\leq C2^{j'}/q$. Let

$$\begin{aligned} \widetilde{G}_{j,a_1,a_2}((m',u'),(\mu,\alpha)) &= \sum_{(n',v')\in\mathbb{G}_0^\#} \sum_{(\nu,\beta)\in Z_q^d\times Z_{q^2}^{d'}} F((n',v'),(\nu,\beta)) \\ &\sum_{j'=j}^{\infty} q^dK_{j'}(q(m'-n'))\cdot q^{2d'}\eta_{2^{2j'}/N}(q^2(u'-v'-R_0(m'-n',n'))) \\ &\times q^{-d}q^{-2d'}e^{2\pi i[a_1\cdot(\mu-\nu)+a_2\cdot(\alpha-\beta-R_0(\mu-\nu,\nu))]/q}. \end{aligned} \tag{7.32}$$

In view of the estimates above on $|E_1|$ and $|E_2|$ we have

$$
\begin{aligned}
&|G_{j,a_1,a_2}((m',u'),(\mu,\alpha)) - \widetilde{G}_{j,a_1,a_2}((m',u'),(\mu,\alpha))| \\
&\leq C \sum_{(n',v')\in\mathbb{G}_0^\#} \sum_{(\nu,\beta)\in Z_q^d\times Z_{q^2}^{d'}} |F((n',v'),(\nu,\beta))| q^{-d} q^{-2d'} \\
&\sum_{j'=j}^{\infty} (qN/2^{j'})(2^{j'}/q)^{-d} \mathbf{1}_{[0,C2^{j'}/q]}(|m'-n'|)\phi_{2^{2j'}/(Nq^2)}(u'-v'-R_0(m'-n',n')),
\end{aligned}
$$

where ϕ is as in (7.7). Thus,

$$
\Big|\Big| \sup_{a_1,a_2,j\geq \log_2(2Nq)} |G_j - \widetilde{G}_j| \Big|\Big|_{L^p(\mathbb{G}_0^\#\times[Z_q^d\times Z_{q^2}^{d'}])} \leq C||F||_{L^p(\mathbb{G}_0^\#\times[Z_q^d\times Z_{q^2}^{d'}])}.
$$

For (7.31) it suffices to prove that

$$
\Big|\Big| \sup_{a_1,a_2,j\geq \log_2(2Nq)} |\widetilde{G}_j| \Big|\Big|_{L^p(\mathbb{G}_0^\#\times[Z_q^d\times Z_{q^2}^{d'}])} \leq C_p[\ln(N+1)]^2||F||_{L^p(\mathbb{G}_0^\#\times[Z_q^d\times Z_{q^2}^{d'}])}, \tag{7.33}
$$

where $\widetilde{G}_j$ is defined in (7.32). We examine the definition (7.32) and notice first that $q^{2d'}\eta_{2^{2j'}/N}(q^2(u'-v'-R_0(m'-n',n'))) = \eta_{(2^{j'}/q)^2/N}(u'-v'-R_0(m'-n',n'))$. Fix j_0 the smallest integer with the property that $c_0 2^{j_0}/q = \widetilde{c}_0 \in [1/2,2]$. The kernels $\widetilde{K}_j(x) = q^d K_{j+j_0}(qx)$, $j \geq \log_2 N$, have the properties (6.1), (6.2), and (7.23). Let $\widetilde{N} = q^2N/2^{2j_0}$, and define $\widetilde{L}_j^{\widetilde{N}}$ and $\widetilde{T}_*^{\widetilde{N}}$ as before, using the kernels $\widetilde{K}_j$. Then, from the definition (7.32),

$$
|G_{j,a_1,a_2}((m',u'),(\mu,\alpha))| \leq \sum_{(\nu,\beta)\in Z_q^d\times Z_{q^2}^{d'}} q^{-d}q^{-2d'}\widetilde{T}_*^{\widetilde{N}}(F((.,.),(\nu,\beta))(m',u').
$$

The bound (7.33) follows from Lemma 7.2. □

Finally, we prove a weighted version of Lemma 7.3.

Lemma 7.4 (Weighted maximal oscillatory singular integrals). *Assume that* $w \in L^\infty(\mathbb{G}_0^\#)$, $w : \mathbb{G}_0^\# \to (0,\infty)$, *satisfies* (7.9), *i.e.*

$$
\mathcal{M}_*^N(w)(h) \leq \rho \cdot w(h) \text{ for any } h \in \mathbb{G}_0^\#.
$$

Then, for any compactly supported function $f : \mathbb{G}_0 \to \mathbb{C}$,

$$
||\mathcal{T}_*^{N,q}(f)||_{L^p(w)} \leq C_p\rho^6[\ln(N+1)]^3||f||_{L^p(w)},\ p\in(1,\infty), \tag{7.34}
$$

where $\mathcal{T}_*^{N,q}$ *is the maximal operator defined in Lemma 7.3.*

Proof of Lemma 7.4. We use the method of distributional inequalities, as in [14, Chapter V]. Fix $p_1 = (p+1)/2 \in (1,p)$ and assume we could prove the distributional inequality

$$\begin{aligned} w\Big(\{h : \mathcal{T}_*^{N,q}(f)(h) > \alpha \text{ and } \widetilde{\mathcal{M}}[(\mathcal{M}_*^N(|f|))^{p_1}]^{1/p_1}(h) \leq \gamma_1 \cdot \alpha\}\Big) \\ \leq (1-\gamma_3) \cdot w(\{h : \mathcal{T}_*^{N,q}(f)(h) > (1-\gamma_2)\cdot\alpha\}) \end{aligned} \tag{7.35}$$

for any $\alpha \in (0,\infty)$, for some small constants $\gamma_1, \gamma_2, \gamma_3 > 0$ depending on p, ρ and $\ln(N+1)$ with the property

$$1 - \gamma_3/2 < (1-\gamma_2)^p. \tag{7.36}$$

By integrating and using the assumptions that f is compactly supported and $w \in L^\infty(\mathbb{G}_0^\#)$ (so $\mathcal{T}_*^{N,q}(f) \in L^p(w)$, $p \in (1,\infty]$), it would follow that

$$\begin{aligned} ||\mathcal{T}_*^{N,q}(f)||_{L^p(w)} &\leq \frac{1}{\gamma_1[1-(1-\gamma_3)(1-\gamma_2)^{-p}]^{1/p}} ||\widetilde{\mathcal{M}}[(\mathcal{M}_*^N(|f|))^{p_1}]^{1/p_1}||_{L^p(w)} \\ &\leq C_p(\gamma_1\gamma_3)^{-1}\rho^4 \ln(N+1)||f||_{L^p(w)}, \end{aligned} \tag{7.37}$$

using Lemma 7.1. Thus, for (7.34), it suffices to prove the distributional inequality (7.35) with (7.36) satisfied and control over $(\gamma_1\gamma_3)^{-1}$.

The bound (7.12) shows easily that if Q is a "cube" (i.e. $B \subseteq Q \subseteq B^*$ for some ball $B \in \mathbb{B}$) and $F \subseteq Q$ then

$$w(F)/w(Q) \leq 1 - (C\rho)^{-1}(1 - |F|/|Q|). \tag{7.38}$$

Indeed, the bound (7.38) is equivalent to $|G|/|Q| \leq (C\rho)w(G)/w(Q)$ for any $G \subseteq Q$, which follows from (7.12). To prove (7.35) we fix $\gamma_3 = (C\rho)^{-1}$, $\gamma_2 = (C_p\rho)^{-1}$, such that (7.36) holds. Let E denote the bounded set

$$E = \{h : \mathcal{T}_*^{N,q}(f)(h) > (1-\gamma_2)\cdot\alpha\},$$

and $E = \cup_{k=1}^K Q_k$ its Whitney decomposition in disjoint cubes (see (7.4)). For (7.35) it suffices to prove that

$$\begin{aligned} w\Big(\{h \in Q_k : \mathcal{T}_*^{N,q}(f)(h) > \alpha \text{ and } \widetilde{\mathcal{M}}[(\mathcal{M}_*^N(|f|))^{p_1}]^{1/p_1}(h) \leq \gamma_1 \cdot \alpha\}\Big) \\ \leq (1-\gamma_3)w(Q_k), \end{aligned}$$

for $k = 1, \ldots, K$. In view of (7.38), it suffices to prove that

$$|\{h \in Q_k : \mathcal{T}_*^{N,q}(f)(h) > \alpha \text{ and } \widetilde{\mathcal{M}}[(\mathcal{M}_*^N(|f|))^{p_1}]^{1/p_1}(h) \leq \gamma_1 \cdot \alpha\}| \leq (1/2)|Q_k|, \tag{7.39}$$

for $k = 1, \ldots, K$ and some constant $\gamma_1 > 0$.

Since Q_k is a Whitney cube,

$$\mathcal{T}_*^{N,q}(f)(h_0) \leq (1-\gamma_2)\cdot\alpha \text{ for some } h_0 \in B_k^{**}. \tag{7.40}$$

In addition, either the inequality (7.39) is trivial or

$$\widetilde{\mathcal{M}}(|f|^{p_1})^{1/p_1}(h_1) \leq \gamma_1 \cdot \alpha \text{ for some } h_1 \in B_k^*, \tag{7.41}$$

since $|f(h)| \leq \mathcal{M}_*^N(|f|)(h)$ for any $h \in \mathbb{G}_0^\#$. Let $f_1 = f \cdot \mathbf{1}_{B_k^{**}}$ and $f_2 = f \cdot \mathbf{1}_{{}^cB_k^{**}}$, $f = f_1 + f_2$. The left-hand side of (7.39) is dominated by

$$\begin{aligned} &|\{h : \mathcal{T}_*^{N,q}(f_1)(h) > (\gamma_2/2) \cdot \alpha\}| \\ &+|\{h \in B_k^* : \mathcal{T}_*^{N,q}(f_2)(h) > (1-\gamma_2/2) \cdot \alpha \text{ and } \mathcal{M}_*^N(|f_2|)(h) \leq \gamma_1 \cdot \alpha\}|. \end{aligned} \tag{7.42}$$

However, using Lemma 7.3, the definition $\gamma_2 = (C_p\rho)^{-1}$, and (7.41),

$$\begin{aligned} |\{h : \mathcal{T}_*^{N,q}(f_1)(h) >&(\gamma_2/2) \cdot \alpha\}| \leq \frac{C_p}{(\gamma_2 \cdot \alpha)^{p_1}} ||\mathcal{T}_*^{N,q}(f_1)||_{L^{p_1}}^{p_1} \\ &\leq C_p \alpha^{-p_1} \rho^{p_1} [\ln(N+1)]^{2p_1} \int_{B_k^{**}} |f(h)|^{p_1} \, dh \\ &\leq C_p [\gamma_1 \rho [\ln(N+1)]^2]^{p_1} |Q_k|. \end{aligned} \tag{7.43}$$

We fix now $\gamma_1 = (C_p\rho(\ln(N+1))^2)^{-1}$, C_p sufficiently large, and show that the set in the second line of (7.42) is empty. Assuming this, the bound (7.39) follows and Lemma 7.4 follows from (7.37).

It remains to show that the set in the second line of (7.42) is empty. We will use the property (7.40) and the definition of the operators $\mathcal{T}_*^{N,q}$. Assume that the ball B_k^* has radius $r \in [2^{k_0}, 2^{k_0+1})$, $k_0 \in [-1, \infty) \cap \mathbb{Z}$. We notice that if $h \in B_k^*$ and $g \in {}^cB_k^{**}$ then $d(0, hg^{-1}) \geq (c^*/2)2^{k_0}$. If, in addition, $\log_2 N \leq j \leq k_0$ then

$$|L_{j,a_1,a_2}^{N,q}(hg^{-1})| \leq C \cdot A_j^N(hg^{-1}) \leq C2^{j-k_0} A_{k_0+2}^N(hg^{-1}),$$

thus, for any $j \in [\log_2 N, k_0] \cap \mathbb{Z}$, $a_1 \in \mathbb{Z}^d$, $a_2 \in \mathbb{Z}^{d'}$,

$$|\mathcal{T}_{j,a_1,a_2}^{N,q}(f_2)(h)| \leq C2^{j-k_0} \mathcal{M}_*^N(|f_2|)(h).$$

Since $|\mathcal{T}_{j,a_1,a_2}^{N,q}(f_2)(h)| \leq C\mathcal{M}_*^N(|f_2|)(h)$ for any $j \geq \log_2 N$ and $j \in [k_0, k_0 + \ln N + C]$, for any $h \in B_k^*$ we have

$$\sup_{a_1 \in \mathbb{Z}^d, a_2 \in \mathbb{Z}^{d'}} \sum_{j \in [\log_2 N, k_0 + \ln N + C]} |\mathcal{T}_{j,a_1,a_2}^{N,q}(f_2)(h)| \leq C \ln(N+1) \mathcal{M}_*^N(|f_2|)(h). \tag{7.44}$$

Assume now that $j \geq \min(\log_2(2Nq), k_0 + \ln N + C)$, $a_1 \in \mathbb{Z}^d$, $a_2 \in \mathbb{Z}^{d'}$, and $h = (n, v) \in B_k^*$. With $h_0 = (n_0, v_0)$ as in (7.40), let $a_{1,0} \in \mathbb{Z}^d$ be such that

$$a_{1,0} \cdot m = a_1 \cdot m + a_2 \cdot R_0(n - n_0, m) \text{ for any } m \in \mathbb{Z}^d.$$

Then, from the definitions,

$$
\begin{aligned}
\Big|\sum_{j'=j}^{\infty}\mathcal{T}^{N,q}_{j',a_1,a_2}(f_2)(h)\Big| &- \Big|\sum_{j'=j}^{\infty}\mathcal{T}^{N,q}_{j',a_{1,0},a_2}(f_2)(h_0)\Big| \\
&\leq \Big|\int_{{}^cB_k^{**}} f_2(m,u)e^{-2\pi i(a_1\cdot m+a_2\cdot u+a_2\cdot R_0(n-m,m))/q} \\
&\qquad \sum_{j'=j}^{\infty}[L^N_{j'}((n,v)\cdot(m,u)^{-1}) - L^N_{j'}((n_0,v_0)\cdot(m,u)^{-1})]\,dmdu\Big| \\
&\leq \int_{{}^cB_k^{**}}|f_2(g)|\sum_{j\geq k_0+\ln N+C}^{\infty}|L^N_j(hg^{-1}) - L^N_j(h_0g^{-1})|\,dg.
\end{aligned}
\tag{7.45}
$$

An estimate similar to (7.20) shows that

$$|L^N_j(hg^{-1}) - L^N_j(h_0g^{-1})| \leq C(N+1)2^{k_0-j}A^N_{j+3}(hg^{-1}),$$

since $h, h_0 \in B_k^{**}$ and $j \geq k_0 + \ln N + C$. In addition, for $j' \geq j \geq k_0 + \ln N + C$,

$$\mathcal{T}^{N,q}_{j',a_{1,0},a_2}(f_2)(h_0) = \mathcal{T}^{N,q}_{j',a_{1,0},a_2}(f)(h_0) - \mathcal{T}^{N,q}_{j',a_{1,0},a_2}(f_1)(h_0) = \mathcal{T}^{N,q}_{j',a_{1,0},a_2}(f)(h_0).$$

Thus, from (7.44) and (7.45), for any $h \in B_k^*$,

$$\mathcal{T}^{N,q}_*(f_2)(h) \leq \mathcal{T}^{N,q}_*(f_2)(h_0) + C\ln(N+1)\mathcal{M}^N_*(|f_2|)(h),$$

so the set in the second line of (7.42) is empty, as desired. This completes the proof of Lemma 7.4. □

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