EMBEDDING SIMPLICES INTO SETS OF POSITIVE UPPER DENSITY IN $\mathbb{R}^d$

LAUREN HUCKABA NEIL LYALL ÁKOS MAGYAR

Abstract. We prove an extension of Bourgain's theorem on pinned distances in measurable subset of $\mathbb{R}^2$ of positive upper density, namely Theorem 1′ in [1], to pinned non-degenerate k-dimensional simplices in measurable subset of $\mathbb{R}^d$ of positive upper density whenever $d \geq k+2$ and k is any positive integer.

1. Introduction

Recall that the *upper density* $\overline{\delta}$ of a measurable set $A \subseteq \mathbb{R}^d$ is defined by

$$\overline{\delta}(A) = \limsup_{N\to\infty} \frac{|A \cap B_N|}{|B_N|},$$

where $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}^d$ and B_N denotes the cube $[-N/2, N/2]^d$.

1.1. **Existing results.** A result of Katznelson and Weiss [2] states that if A is a measurable subset of $\mathbb{R}^2$ of positive upper density, then its distance set

$$\mathrm{dist}(A) = \{|x-y| \,:\, x, y \in A\}$$

contains all large numbers. This result was later reproven using Fourier analytic techniques by Bourgain in [1]. Bourgain in fact established more, namely the following generalization and "pinned variant".

Theorem 1.1 (Theorem 2 in [1]). *Let Δ be a fixed non-degenerate k-dimensional simplex. If A is a measurable subset of $\mathbb{R}^d$ of positive upper density with $d \geq k+1$, then there exist $\lambda_0 = \lambda_0(A)$ such that for all $\lambda \geq \lambda_0$ one has*

$$x + \lambda \cdot U(\Delta) \subseteq A \tag{1}$$

for some $x \in A$ and $U \in SO(d)$.

Theorem 1.2 (Pinned distances, Theorem 1′ in [1]). *If A is a measurable subset of $\mathbb{R}^2$ of positive upper density, then there exist $\lambda_0 = \lambda_0(A)$ such that for any given $\lambda_1 \geq \lambda_0$ there is a fixed $x \in A$ such that*

$$A \cap (x + \lambda \cdot S^1) \neq \emptyset \tag{2}$$

for all $\lambda_0 \leq \lambda \leq \lambda_1$.

1.2. **New results.** Our first result is the following optimal strengthening of Theorem 1.1 above.

Theorem 1.3 (Optimal Density of Embedded Simplices). *Let $\Delta = \{0, v_1, \ldots, v_k\} \subseteq \mathbb{R}^k$ be a fixed non-degenerate k-dimensional simplex and $\varepsilon > 0$.*

If A is a measurable subset of $\mathbb{R}^d$ with $d \geq k+1$, then there exist $\lambda_0 = \lambda_0(A, \varepsilon)$ such that

$$\int_{SO(d)} \overline{\delta}(A \cap (A + \lambda \cdot U(v_1)) \cap \cdots \cap (A + \lambda \cdot U(v_k)))\, d\mu(U) > \overline{\delta}(A)^{k+1} - \varepsilon \tag{3}$$

for all $\lambda \geq \lambda_0$. In particular, for each $\lambda \geq \lambda_0$ we may conclude that there exist $U \in SO(d)$ such that

$$\overline{\delta}(A \cap (A + \lambda \cdot U(v_1)) \cap \cdots \cap (A + \lambda \cdot U(v_k))) > \overline{\delta}(A)^{k+1} - \varepsilon \tag{4}$$

and there exist $x \in A$ such that

$$\mu\big(\{U \in SO(d) \,:\, x + \lambda \cdot U(\Delta) \subseteq A\}\big) > \overline{\delta}(A)^k - \varepsilon. \tag{5}$$

While the main result of this paper is the following (optimal) extension of Bourgain's pinned distances theorem, Theorem 1.2 above, to non-degenerate k-dimensional simplices when $k \geq 2$.

2000 *Mathematics Subject Classification.* 11B30.

Theorem 1.4 (Optimal Density of Embedded Pinned Simplices)**.** *Let $\Delta = \{0, v_1, \dots, v_k\} \subseteq \mathbb{R}^k$ be a fixed non-degenerate k-dimensional simplex and $\varepsilon > 0$.*

If A is a measurable subset of $\mathbb{R}^d$ with $d \geq k+2$, then there exist $\lambda_0 = \lambda_0(A, \varepsilon)$ such that for any given $\lambda_1 \geq \lambda_0$ there is a fixed $x \in A$ such that

$$\mu\big(\{U \in SO(d) \,:\, x + \lambda \cdot U(\Delta) \subseteq A\}\big) > \overline{\delta}(A)^k - \varepsilon \quad \textit{for all} \quad \lambda_0 \leq \lambda \leq \lambda_1. \tag{6}$$

Remark. Theorem 1.4 should hold whenever $d \geq k+1$. Extending our result to this range will require an appropriate extension of *Bourgain's circular maximal function theorem* to the specific configuration spaces considered in this article. We plan to address this extension in a separate article.

Remark. Theorems 1.3 and 1.4 also hold with the notion of upper density replaced with that of upper Banach density, but we choose not to pursue this approach here.

2. Reducing Theorems 1.3 and 1.4 to Key Dichotomy Propositions

2.1. **Dichotomy Propositions.** We will adapt Bourgain's approach in [1] and deduce Theorems 1.3 and 1.4 as consequences of the following quantitative finite versions, namely Propositions 2.1 and 2.2 below.

Proposition 2.1 (Dichotomy for Theorem 1.3)**.** *Let $\Delta = \{0, v_1, \dots, v_k\} \subseteq \mathbb{R}^k$ be a fixed non-degenerate k-dimensional simplex, $\varepsilon > 0$, $0 < \eta \ll \varepsilon^{5/2}$, and $N \geq C_\Delta \eta^{-4}$.*

If $A \subseteq B_N \subseteq \mathbb{R}^d$ with $d \geq k+1$, then for any λ satisfying $1 \leq \lambda \leq \eta^4 N$ one of the following statements must hold:

(i)
$$\int_{SO(d)} \frac{|A \cap (A + \lambda \cdot U(v_1)) \cap \cdots \cap (A + \lambda \cdot U(v_k))|}{N^d} \, d\mu(U) > \left(\frac{|A|}{N^d}\right)^{k+1} - \varepsilon$$

(ii)
$$\frac{1}{|A|} \int_{\Omega_\lambda} |\widehat{1_A}(\xi)|^2 \, d\xi \gg \varepsilon^2$$

where
$$\Omega_\lambda = \Omega_\lambda(\eta) = \{\xi \in \mathbb{R}^d \,:\, \eta^2 \lambda^{-1} \leq |\xi| \leq \eta^{-2} \lambda^{-1}\}.$$

Proposition 2.2 (Dichotomy for Theorem 1.4)**.** *Let $\Delta = \{0, v_1, \dots, v_k\} \subseteq \mathbb{R}^k$ be a fixed non-degenerate k-dimensional simplex, $\varepsilon > 0$, $0 < \eta \ll \varepsilon^3$, and $N \geq C_\Delta \eta^{-4}$.*

If $A \subseteq B_N \subseteq \mathbb{R}^d$ with $d \geq k+2$, then for any pair (λ_0, λ_1) satisfying $1 \leq \lambda_0 \leq \lambda_1 \leq \eta^4 N$ one of the following statements must hold:

(i) *there exist $x \in A$ with the property that*
$$\mu\big(\{U \in SO(d) \,:\, x + \lambda \cdot U(\Delta) \subseteq A\}\big) > \left(\frac{|A|}{N^d}\right)^k - \varepsilon \quad \textit{for all} \quad \lambda_0 \leq \lambda \leq \lambda_1$$

(ii)
$$\frac{1}{|A|} \int_{\Omega_{\lambda_0,\lambda_1}} |\widehat{1_A}(\xi)|^2 \, d\xi \gg \varepsilon^2$$

where
$$\Omega_{\lambda_0,\lambda_1} = \Omega_{\lambda_0,\lambda_1}(\eta) = \{\xi \in \mathbb{R}^d \,:\, \eta^2 \lambda_1^{-1} \leq |\xi| \leq \eta^{-2} \lambda_0^{-1}\}.$$

2.2. Proof of Theorems 1.3 and 1.4.

2.2.1. *Proof that Proposition 2.1 implies Theorem 1.3.* Let $\varepsilon > 0$ and $0 < \eta \ll \varepsilon^{5/2}$. Suppose that $A \subseteq \mathbb{R}^d$ with $d \geq k+1$ is a set for which the conclusion of Theorem 1.3 fails to hold, namely that there exists arbitrarily large integers λ for which

$$\int_{SO(d)} \overline{\delta}(A \cap (A + \lambda \cdot U(v_1)) \cap \cdots \cap (A + \lambda \cdot U(v_k))) \, d\mu(U) \leq \overline{\delta}(A)^{k+1} - \varepsilon.$$

For a fixed integer $J \gg \varepsilon^{-2}$ we now choose a sequence $\{\lambda^{(j)}\}_{j=1}^J$ of such λ's with the additional property that $1 \leq \lambda^{(j)} \leq \eta^4\lambda^{(j+1)}$ for $1 \leq j < J$. We now choose N so that $\lambda^{(J)} \leq \eta^4 N$ and that we simultaneously have both that

$$\overline{\delta}(A)^{k+1} - \varepsilon/2 \leq \left(\frac{|A \cap B_N|}{N^d}\right)^{k+1} - \varepsilon/4 \tag{7}$$

and that

$$\int_{SO(d)} \frac{|A_N \cap (A_N + \lambda^{(j)} \cdot U(v_1)) \cap \cdots \cap (A_N + \lambda^{(j)} \cdot U(v_k))|}{N^d}\, d\mu(U) \leq \overline{\delta}(A)^{k+1} - \varepsilon/2$$

holds for all $1 \leq j \leq J$, where $A_N = A \cap B_N$. For the last inequality we exploited Fatou's Lemma.

Abusing notation and denoting the set $A_N = A \cap B_N$ by A, an application of Proposition 2.1, with ε replaced with $\varepsilon/4$, thus allows us to conclude that for this set one must have

$$\sum_{j=1}^J \frac{1}{|A|}\int_{\Omega_{\lambda^{(j)}}} |\widehat{1_A}(\xi)|^2\, d\xi \gg J\varepsilon^2 > 1. \tag{8}$$

On the other hand it follows from the disjointness property of the sets $\Omega_{\lambda^{(j)}}$, which we guaranteed by our initial choice of sequence $\{\lambda^{(j)}\}$, and Plancherel's Theorem that

$$\sum_{j=1}^J \frac{1}{|A|}\int_{\Omega_{\lambda^{(j)}}} |\widehat{1_A}(\xi)|^2\, d\xi \leq \frac{1}{|A|}\int_{\mathbb{T}^d} |\widehat{1_A}(\xi)|^2\, d\xi = 1 \tag{9}$$

giving a contradiction. $\square$

2.2.2. *Proof that Proposition 2.2 implies Theorem 1.4.* Let $\varepsilon > 0$ and $0 < \eta \ll \varepsilon^3$. Suppose that $A \subseteq \mathbb{R}^d$ with $d \geq k+2$ is a set for which the conclusion of Theorem 1.4 fails to hold, namely that there exists arbitrarily large pairs (λ_0, λ_1) of real numbers such that for all $x \in A$ one has

$$\mu\big(\{U \in SO(d) \,:\, x + \lambda \cdot U(\Delta) \subseteq A\}\big) \leq \overline{\delta}(A)^k - \varepsilon$$

for some $\lambda_0 \leq \lambda \leq \lambda_1$.

For a fixed integer $J \gg \varepsilon^{-2}$ we choose a sequence of such pairs $\{(\lambda_0^{(j)}, \lambda_1^{(j)}\}_{j=1}^J$ with the property that $1 \leq \lambda_0^{(j)} \leq \eta^4\lambda_1^{(j+1)}$ for $1 \leq j < J$. We now choose N so that $\lambda_1^{(J)} \leq \eta^4 N$ and

$$\overline{\delta}(A)^k - \varepsilon \leq \left(\frac{|A \cap B_N|}{N^d}\right)^k - \varepsilon/2. \tag{10}$$

Abusing notation and denoting the set $A \cap B_N$ by A, an application of Proposition 2.2 thus allows us to conclude that for this set one must have

$$\sum_{j=1}^J \frac{1}{|A|}\int_{\Omega_{\lambda_0^{(j)},\lambda_1^{(j)}}} |\widehat{1_A}(\xi)|^2\, d\xi \gg J\varepsilon^2 > 1. \tag{11}$$

On the other hand it follows from the disjointness property of the sets $\Omega_{\lambda_0^{(j)},\lambda_1^{(j)}}$, which we guaranteed by our initial choice of pair sequence $\{(\lambda_0^{(j)}, \lambda_1^{(j)})\}$, and Plancherel's Theorem that

$$\sum_{j=1}^J \frac{1}{|A|}\int_{\Omega_{\lambda_0^{(j)},\lambda_1^{(j)}}} |\widehat{1_A}(\xi)|^2\, d\xi \leq \frac{1}{|A|}\int_{\mathbb{T}^d} |\widehat{1_A}(\xi)|^2\, d\xi = 1 \tag{12}$$

giving a contradiction. $\square$

3. Preliminaries

3.1. **The multi-linear operators $\mathcal{A}_\lambda^{(j)}$.** Let $\Delta = \{0, v_1, \dots, v_k\}$ be our fixed k-dimensional simplex. Without loss of generality we may assume that $|v_1| = 1$. For each $1 \le j \le k$ we introduce the multi-linear operator $\mathcal{A}_\lambda^{(j)}$, defined initially for Schwartz functions $g_1, \dots, g_j$, by

$$\mathcal{A}_\lambda^{(j)}(g_1, \dots, g_j)(x) = \int \cdots \int g_1(x - \lambda y_1) \cdots g_j(x - \lambda y_j)\, d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}(y_j) \cdots d\sigma^{(d-1)}(y_1) \tag{13}$$

where $d\sigma^{(d-1)}$ denotes the measure on the unit sphere $S^{d-1} \subseteq \mathbb{R}^d$ induced by Lebesgue measure normalized to have total mass 1 and $d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}$ denotes, for each $2 \le j \le k$, the normalized measure on the sphere

$$S^{d-j}_{y_1,\dots,y_{j-1}} \subseteq [y_1, \dots, y_{j-1}]^\perp \simeq \mathbb{R}^{d-j+1}$$

of radius $r_j = \operatorname{dist}(v_j, [v_1, \dots, v_{j-1}])$.

The multi-linear operator $\mathcal{A}_\lambda^{(j)}$ is a natural object for us to consider in light of the observation that it could have equivalently be defined for each $1 \le j \le k$ using the formula

$$\mathcal{A}_\lambda^{(j)}(g_1, \dots, g_j)(x) := \int_{SO(d)} g_1(x - \lambda \cdot U(v_1)) \cdots g_j(x - \lambda \cdot U(v_j))\, d\mu(U) \tag{14}$$

and hence for any bounded measurable set $A \subseteq \mathbb{R}^d$, the quantity

$$\langle 1_A, \mathcal{A}_\lambda^{(k)}(1_A, \dots, 1_A)\rangle = \int_{SO(d)} |A \cap (A + \lambda \cdot U(v_1)) \cap \cdots \cap (A + \lambda \cdot U(v_k))|\, d\mu(U). \tag{15}$$

A trivial, but important, observation will be the fact that

$$\Big|\mathcal{A}_\lambda^{(j)}(g_1, \dots, g_j)(x) - g_j(x)\, \mathcal{A}_\lambda^{(j-1)}(g_1, \dots, g_{j-1})(x)\Big| \le \int \big|g_j(x - \lambda y_j) - g_j(x)\big|\, d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}(y_j). \tag{16}$$

3.2. **A second averaging operator and some basic estimates.** We now introduce a second averaging operator, which we also denote by $\mathcal{A}_\lambda^{(j)}$, defined initially for any Schwartz function g, by

$$\mathcal{A}_\lambda^{(j)}(g)(x) = \int \cdots \int \Big|\int g(x - \lambda y_j)\, d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}(y_j)\Big|\, d\sigma^{(d-j+1)}_{y_1,\dots,y_{j-2}}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1) \tag{17}$$

Note that if the functions $g_1, \dots, g_{j-1}$ are all bounded in absolute value by 1, then clearly

$$\big|\mathcal{A}_\lambda^{(j)}(g_1, \dots, g_j)(x)\big| \le \mathcal{A}_\lambda^{(j)}(g_j)(x). \tag{18}$$

Fix $1 \le j \le k$. It is easy to see, using Minkowski's inequality, that for any Schwartz functions g we have the extremely crude estimate

$$\int \big|\mathcal{A}_\lambda^{(j)}(g)(x)\big|^2\, dx \le \int |g(x)|^2\, dx. \tag{19}$$

However, arguing more carefully one can just as easily obtain, using Plancherel's identity, the estimate

$$\int \big|\mathcal{A}_\lambda^{(j)}(g)(x)\big|^2\, dx \le \int \cdots \int \Big(\int |\widehat{g}(\xi)|^2 \big|\widehat{d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}}(\lambda\,\xi)\big|^2\, d\xi\Big)\, d\sigma^{(d-j+1)}_{y_1,\dots,y_{j-2}}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1), \tag{20}$$

where as usual

$$\widehat{d\mu}(\xi) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi}\, d\mu(x) \tag{21}$$

denotes the Fourier transform of any complex-valued Borel measure $d\mu$ and $\widehat{g}(\xi)$ is the Fourier transform of the measure $d\mu = g\, dx$. In light of (20) it will come as little surprise that is the course of our arguments we will have use for the basic estimate

$$\big|\widehat{d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}}(\xi)\big| + \big|\nabla \widehat{d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}}(\xi)\big| \le C_\Delta \big(1 + \operatorname{dist}(\xi, [y_1, \dots, y_{j-1}])\big)^{-(d-j)/2}, \tag{22}$$

which is a consequence of the well-known asymptotic behavior of the Fourier transform of the measure on the unit sphere $S^{d-j} \subseteq \mathbb{R}^{d-j+1}$ induced by Lebesgue measure, see for example [4].

3.3. **A smooth cutoff function ψ and some basic properties.** Let $\psi : \mathbb{R}^d \to (0,\infty)$ be a Schwartz function that satisfies

$$1 = \widehat{\psi}(0) \geq \widehat{\psi}(\xi) \geq 0 \qquad \text{and} \qquad \widehat{\psi}(\xi) = 0 \ \text{ for } \ |\xi| > 1.$$

As usual, for any given $t > 0$, we define

$$\psi_t(x) = t^{-d}\psi(t^{-1}x). \tag{23}$$

First we record the trivial observation that

$$\int \psi_t(x)\,dx = \int \psi(x)\,dx = \widehat{\psi}(0) = 1$$

as well as the simple, but important, observation that ψ may be chosen so that

$$\big|1 - \widehat{\psi}_t(\xi)\big| = \big|1 - \widehat{\psi}(t\xi)\big| \ll \min\{1, t|\xi|\}. \tag{24}$$

Finally we record a formulation, appropriate to our needs, of the fact that for any given small parameter η, our cutoff function $\psi_t(x)$ will essentially supported where $|x| \leq \eta^{-1}t$ and is approximately constant on smaller scales. More precisely,

Lemma 3.1. *Let $\eta > 0$ and $t > 0$, then*

$$\int_{|x|\geq \eta^{-1}t} \psi_t(x)\,dx \ll \eta. \tag{25}$$

and

$$\int\int \big|\psi_t(x-\lambda y) - \psi_t(x)\big|\,d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}(y_j)\,dx \ll \eta \tag{26}$$

for any $1 \leq j \leq k$ provided $t \geq \eta^{-1}\lambda$.

Proof. Estimate (25) is easily verified using the fact that ψ is a Schwartz function on $\mathbb{R}^d$ as

$$\int_{|x|\geq \eta^{-1}t} \psi_t(x)\,dx = \int_{|x|\geq \eta^{-1}} \psi(x)\,dx \ll \int_{|x|\geq \eta^{-1}} (1+|x|)^{-d-1}\,dx \ll \eta.$$

To verify estimate (26) we make use of the fact that both ψ and its derivative are rapidly decreasing, specifically

$$\begin{aligned}\int\int \big|\psi_t(x-\lambda y) - \psi_t(x)\big|\,d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}(y_j)\,dx &\leq \int\int \big|\psi(x-\lambda y/t) - \psi(x)\big|\,d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}(y_j)\,dx \\ &\ll \frac{\lambda}{t}\int (1+|x|)^{-d-1}\,dx \ll \frac{\lambda}{t}.\end{aligned}$$

□

4. Proof of Proposition 2.1

Let $f = 1_A$ and $\delta = |A|/N^d$. Suppose that $1 \leq \lambda \leq \eta^4 N$ and that (i) does not hold, then

$$\langle f, \mathcal{A}^{(k)}_\lambda(f,\dots,f)\rangle \leq \langle f, \delta^k - \varepsilon\rangle = (\delta^k - \varepsilon)|A|. \tag{27}$$

If we let $f_1 := f * \psi_{\eta^{-1}\lambda}$, then by (16) and (26) it follows that for all $x \in \mathbb{R}^d$ and $1 \leq j \leq k$ we have

$$\Big|\mathcal{A}^{(j)}_\lambda(f,\dots,f,f_1)(x) - f_1(x)\,\mathcal{A}^{(j-1)}_\lambda(f,\dots,f)(x)\Big| \ll \eta \tag{28}$$

and consequently

$$f_1(x)^k + \sum_{j=1}^k f_1(x)^{k-j}\mathcal{A}^{(j)}_\lambda(f,\dots,f,f-f_1)(x) \ll \mathcal{A}^{(k)}_\lambda(f,\dots,f)(x) + \eta. \tag{29}$$

Together this with (27) this gives

$$\sum_{j=1}^k \big\langle f f_1^{k-j}, \mathcal{A}^{(j)}_\lambda(f,\dots,f,f-f_1)\big\rangle \leq \langle f, \delta^k - f_1^k - \varepsilon/2\rangle \tag{30}$$

provided $\eta \ll \varepsilon$. We will now combine this with the following result, which we isolate as a lemma.

Lemma 4.1. *Let $\eta > 0$ and $f_1 := f * \psi_{\eta^{-1}\lambda}$, then*

$$\langle f, \delta^k - f_1^k\rangle \ll \langle f, \eta\rangle \tag{31}$$

Combining Lemma 4.1 with (30) we see that if $\eta \ll \varepsilon$ and (27) holds, then there exist $1 \le j \le k$ such that

$$\left|\left\langle f f_1^{k-j}, \mathcal{A}_\lambda^{(j)}(f, \dots, f, f - f_1)\right\rangle\right| \gg \varepsilon |A| \tag{32}$$

and hence, using (18) and the fact that $0 \le f_1 \le 1$, that

$$\left\langle f, \mathcal{A}_\lambda^{(j)}(f - f_1)\right\rangle \gg \varepsilon |A|. \tag{33}$$

The final ingredient in the proof of Proposition 2.1 is the following

Lemma 4.2 (Error term)**.** *If $f_2 := f * \psi_{\eta^2\lambda}$, then for any $1 \le j \le k$ we have the estimate*

$$\left\langle f, \mathcal{A}_\lambda^{(j)}(f - f_2)\right\rangle \ll \eta^{2/5} |A|. \tag{34}$$

Indeed, since

$$\left\langle f, \mathcal{A}_\lambda^{(j)}(f_2 - f_1)\right\rangle \ge \left\langle f, \mathcal{A}_\lambda^{(j)}(f - f_1)\right\rangle - \left\langle f, \mathcal{A}_\lambda^{(j)}(f - f_2)\right\rangle$$

we see that (33) together with Lemma 4.2 will imply that if $\eta \ll \varepsilon^{5/2}$ and (27) holds, then there exist $1 \le j \le k$ such that

$$\left\langle f, \mathcal{A}_\lambda^{(j)}(f_2 - f_1)\right\rangle \gg \varepsilon |A|. \tag{35}$$

It then follows, via Cauchy-Schwarz and Plancherel, that

$$\int \left|\widehat{f}(\xi)\right|^2 \left|\widehat{\psi}_{\eta^2\lambda}(\xi) - \widehat{\psi}_{\eta^{-1}\lambda}(\xi)\right|^2 d\xi \gg_k \varepsilon^2 |A|, \tag{36}$$

which is essentially the estimate that we are trying to prove and since (24) implies that

$$\left|\widehat{\psi}_{\eta^2\lambda}(\xi) - \widehat{\psi}_{\eta^{-1}\lambda}(\xi)\right| \ll \eta \tag{37}$$

whenever $\xi \notin \Omega_\lambda$, it indeed sufficies and concludes the proof of Proposition 2.1. □

4.1. **Proof of Lemma 4.1.** It suffices to establish the result when $k = 1$, namely that

$$\int f(x) f_1(x)\, dx \ge (\delta - C\eta)\, |A| \tag{38}$$

since from Hölder's inequality we would then obtain

$$(\delta - C\eta)^k\, |A|^k \le \Big(\int f(x) f_1(x)\, dx\Big)^k \le |A|^{k-1} \int f(x) f_1(x)^k\, dx$$

from which the full result immediately follows. Towards establishing (38) we note that using Parserval and the fact that $0 \le \widehat{\psi} \le 1$ we have

$$\int f(x) f_1(x)\, dx = \int |\widehat{f}(\xi)|^2 \widehat{\psi}(\eta^{-1}\lambda\, \xi)\, d\xi \ge \int |\widehat{f}(\xi)|^2 |\widehat{\psi}(\eta^{-1}\lambda\, \xi)|^2\, d\xi = \int f_1(x)^2\, dx \tag{39}$$

and as such we need only show that

$$\int f_1(x)^2\, dx \ge (\delta - C\eta)\, |A|. \tag{40}$$

We now let $N' = N + \eta^{-2}\lambda$ and write

$$\int f_1(x)^2\, dx = \int_{B_N} f_1(x)^2\, dx + \int_{\mathbb{R}^d \setminus B_{N'}} f_1(x)^2\, dx + \int_{B_{N'} \setminus B_N} f_1(x)^2\, dx.$$

Cauchy-Schwarz and the fact that f is supported on B_N gives

$$\int_{B_N} f_1(x)^2\, dx \ge \frac{1}{|B_N|}\Big(\int_{B_N} f_1(x)\, dx\Big)^2 = \frac{1}{|B_N|}\Big(\int_{B_N} f(x)\, dx\Big)^2 = \delta |A|, \tag{41}$$

while the fact that $\lambda \ll \eta^4 N$ ensures that

$$\frac{|B_{N'} \setminus B_N|}{|B_N|} \ll \left(\frac{N'}{N} - 1\right) \ll \eta^{-2}\frac{\lambda}{N} \ll \eta^2$$

and hence, since $\eta \ll \delta$, that

$$\int_{B_{N'}\setminus B_N} f_1(x)^2\,dx \ll \eta^2|B_N| \leq \eta|A|.$$

Estimate (40) now follow from the discussion above since from (25) we additionally have

$$\int_{\mathbb{R}^d\setminus B_{N'}} f_1(x)^2\,dx \leq |A|\int_{|y|\gg\eta^{-2}\lambda} \psi_{\eta^{-1}\lambda}(y)\,dy \ll \eta|A|.$$

□

4.2. **Proof of Lemma 4.2.** It follows from an application of Cauchy-Schwarz and Plancherel that

$$\left\langle f, \mathcal{A}_\lambda^{(j)}(f-f_2)\right\rangle^2 \leq |A|\cdot\int |\widehat{f}(\xi)|^2|1-\widehat{\psi}(\eta^2\lambda\,\xi)|^2 I(\lambda\,\xi)\,d\xi$$

where

$$I(\xi) = \int\cdots\int \left|d\widehat{\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}}(\xi)\right|^2 d\sigma^{(d-j+1)}_{y_1,\dots,y_{j-2}}(y_{j-1})\cdots d\sigma^{(d-1)}(y_1). \tag{42}$$

While from (22), the trivial uniform bound $I(\xi) \ll 1$, and an appropriate "conical" decomposition, depending on ξ, of the configuration space over which the integral $I(\xi)$ is defined, we have

$$I(\xi) \leq C_\Delta(1+|\xi|)^{-(d-j)/2}. \tag{43}$$

Combining this observation with (24) we obtain the uniform bound

$$|1-\widehat{\psi}(\eta^2\lambda\,\xi)|^2 I(\lambda\,\xi) \ll \min\{(\lambda|\xi|)^{-1/2}, \eta^4\lambda^2|\xi|^2\} \leq \eta^{4/5} \tag{44}$$

which, after an application of Plancherel, completes the proof. □

5. Proof of Proposition 2.2

Suppose that we have a pair (λ_0, λ_1) satisfying $1 \leq \lambda_0 \leq \lambda_1 \leq \eta^4 N$, but for which (i) does not hold. It follows that for all $x \in A$ there must exist $\lambda_0 \leq \lambda \leq \lambda_1$ such that

$$\mathcal{A}_\lambda^{(k)}(f,\dots,f)(x) \leq \delta^k - \varepsilon. \tag{45}$$

We now let $f_1 = f * \psi_{\eta^{-1}\lambda_1}$, noting the slight difference from the definition of f_1 given in the proof of Proposition 2.1. It follows from (45), as in the proof of Proposition 2.1, that for all $x \in A$ there must exist $\lambda_0 \leq \lambda \leq \lambda_1$ such that

$$\sum_{j=1}^k f_1(x)^{k-j}\mathcal{A}_\lambda^{(j)}(f,\dots,f,f-f_1)(x) \leq \delta^k - f_1(x)^k - \varepsilon/2 \tag{46}$$

provided $\eta \ll \varepsilon$, and hence that

$$\sum_{j=1}^k \mathcal{A}_*^{(j)}(f-f_1)(x) \geq f_1(x)^k - \delta^k + \varepsilon/2 \tag{47}$$

for all $x \in A$, where for any Schwartz function g, $\mathcal{A}_*^{(j)}(g)$ denotes the *maximal average* defined by

$$\mathcal{A}_*^{(j)}(g)(x) := \sup_{\lambda_0\leq\lambda\leq\lambda_1} \mathcal{A}_\lambda^{(j)}(g)(x). \tag{48}$$

Consequently, provided $\eta \ll \varepsilon$ and appealing to Lemma 4.1, we may conclude that there must exist $1 \leq j \leq k$ such that

$$\left\langle f, \mathcal{A}_*^{(j)}(f-f_1)\right\rangle \gg \varepsilon|A|. \tag{49}$$

Arguing as in the proof of Proposition 2.1 we see that everything reduces to establishing the L^2-boundedness of $\mathcal{A}_*^{(j)}$ together with appropriate estimates for the "mollified" maximal operator

$$\mathcal{M}_\eta^{(j)}(f) := \mathcal{A}_*^{(j)}(f-f_2) \tag{50}$$

where $f_2 = f * \psi_{\eta^2\lambda_0}$.

Note that

$$\mathcal{M}^{(j)}_\eta(f) = \sup_{\lambda_0\le\lambda\le\lambda_1} \int\cdots\int\Big|\int f(x-\lambda y_j)\,d\mu^{(j)}_\eta(y_j)\Big|\,d\sigma^{(d-j+1)}_{y_1,\dots,y_{j-2}}(y_{j-1})\cdots d\sigma^{(d-1)}(y_1) \tag{51}$$

where

$$d\mu^{(j)}_\eta = d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}} - \psi_{\eta^2\lambda_0\lambda^{-1}} * d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}. \tag{52}$$

and hence

$$\widehat{\mu^{(j)}_\eta}(\lambda\,\xi) = \widehat{d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}}(\lambda\,\xi)\,\big(1-\widehat{\psi}(\eta^2\lambda_0\,\xi)\big). \tag{53}$$

The precise results that we need are recorded in the following two propositions.

Proposition 5.1 (L^2-Boundedness of the Maximal Averages $\mathcal{A}^{(j)}_*$)**.** *If $d\ge j+2$, then*

$$\int_{\mathbb{R}^d}|\mathcal{A}^{(j)}_*(g)(x)|^2\,dx \ll \int_{\mathbb{R}^d}|g(x)|^2\,dx. \tag{54}$$

Proposition 5.2 (L^2-decay of the "Mollified" Maximal Averages $\mathcal{M}^{(j)}_\eta$)**.** *Let $\eta>0$. If $d\ge j+2$, then*

$$\int_{\mathbb{R}^d}|\mathcal{M}^{(j)}_\eta(f)(x)|^2\,dx \ll \eta^{2/3}\int_{\mathbb{R}^d}|f(x)|^2\,dx. \tag{55}$$

The proofs of Propositions 5.1 and 5.2 are presented in Section 6 below. □

6. Proof of Propositions 5.1 and 5.2

6.1. **Proof of Propositions 5.1.** We first note that Cauchy-Schwarz ensures

$$\int_{\mathbb{R}^d}|\mathcal{A}^{(j)}_*(g)(x)|^2\,dx \le \int\cdots\int\int_{\mathbb{R}^d}\sup_{\lambda_0\le\lambda\le\lambda_1}\Big|\int g(x-\lambda y_j)\,d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}(y_j)\Big|^2\,dx\,d\sigma^{(d-j+1)}_{y_1,\dots,y_{j-2}}(y_{j-1})\cdots d\sigma^{(d-1)}(y_1).$$

Now for fixed $y_1,\dots,y_{j-1}$ we can clearly identify $[y_1,\dots,y_{j-1}]^\perp$ with $\mathbb{R}^{d-j+1}$ and $d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}$ with a constant (depending only on d and Δ) multiple of $d\sigma^{(d-j)}$, the normalized measure on the unit sphere $S^{d-j}\subseteq\mathbb{R}^{d-j+1}$ induced by Lebesgue measure. Writing $\mathbb{R}^d=\mathbb{R}^{j-1}\times\mathbb{R}^{d-j+1}$, $g(x)=g_{x'}(x'')$, and applying *Stein's spherical maximal function theorem* for functions in $L^2(\mathbb{R}^{d-j+1})$ [4], which asserts that

$$\int_{\mathbb{R}^{d-j+1}}\sup_{\lambda_0\le\lambda\le\lambda_1}\Big|\int g(x-\lambda y)\,d\sigma^{(d-j)}(y)\Big|^2\,dx \ll \int_{\mathbb{R}^{d-j+1}}|g(x)|^2\,dx \tag{56}$$

whenever $d\ge j+2$, gives

$$\begin{aligned}
&\int_{\mathbb{R}^d}\sup_{\lambda_0\le\lambda\le\lambda_1}\Big|\int g(x-\lambda y)\,d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}(y)\Big|^2\,dx\\
&\qquad= C_\Delta\int_{\mathbb{R}^{j-1}}\int_{\mathbb{R}^{d-j+1}}\sup_{\lambda_0\le\lambda\le\lambda_1}\Big|\int g_{x'}(x''-\lambda y)\,d\sigma^{(d-j)}(y)\Big|^2\,dx''\,dx'\\
&\qquad\le C\int_{\mathbb{R}^{j-1}}\int_{\mathbb{R}^{d-j+1}}|g_{x'}(x'')|^2\,dx''\,dx' = C\int_{\mathbb{R}^d}|g(x)|^2\,dx
\end{aligned}$$

with the constant C independent of the initial choice of frame $y_1,\dots,y_{j-1}$. The result follows. □

6.2. **Proof of Propositions 5.2.** We will deduce the validity of Proposition 5.2 from the following result for the slightly more general class of operators defined for any $L>0$ by

$$\mathcal{M}^{(j)}_L(f) = \sup_{\lambda_0\le\lambda\le\lambda_1}\int\cdots\int\Big|\int f(x-\lambda y)\,d\mu^{(j)}_L(y)\Big|\,d\sigma^{(d-j+1)}_{y_1,\dots,y_{j-2}}(y_{j-1})\cdots d\sigma^{(d-1)}(y_1) \tag{57}$$

where

$$\widehat{d\mu^{(j)}_L}(\lambda\,\xi) = m_L(\xi)\,\widehat{d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}}(\lambda\,\xi) \tag{58}$$

with the multiplier m_L now any smooth function that satisfies the estimate

$$|m_L(\xi)| \ll \min\{1, L|\xi|\}. \tag{59}$$

Recall that estimate (24) is precisely the statement that $|1-\widehat{\psi}(L\xi)| \ll \min\{1, L|\xi|\}$.

Theorem 6.1. *If $d \geq j+2$ and $0 < L < \lambda_0$, then*

$$\int_{\mathbb{R}^d} |\mathcal{M}_L^{(j)}(f)(x)|^2\, dx \ll \Big(\frac{L}{\lambda_0}\Big)^{1/3} \int_{\mathbb{R}^d} |f(x)|^2\, dx. \tag{60}$$

Proof. An application of Cauchy-Schwarz gives

$$\int_{\mathbb{R}^d} |\mathcal{M}_L^{(j)}(f)(x)|^2\, dx \leq \int \cdots \int \left[\int_{\mathbb{R}^d} \sup_{\lambda_0 \leq \lambda \leq \lambda_1} |M_{L,\lambda}(f)(x)|^2\, dx \right] d\sigma^{(d-j+1)}_{y_1,\dots,y_{j-2}}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1). \tag{61}$$

where $M_{L,\lambda}$ is the Fourier multiplier operator defined by

$$\widehat{M_{L,\lambda}(f)}(\xi) = \widehat{f}(\xi)\, m_L(\xi)\, \widehat{d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}}(\lambda\, \xi). \tag{62}$$

A standard application of the Fundamental Theorem of Calculus, see for example [3], gives

$$\sup_{\lambda_0 \leq \lambda \leq \lambda_1} |M_{L,\lambda}(f)(x)|^2 \leq 2 \int_{\lambda_0}^{\lambda_1} |M_{L,t}(f)(x)||\widetilde{M}_{L,t}(f)(x)|\, \frac{dt}{t} + |M_{L,\lambda_0}(f)(x)|^2 \tag{63}$$

where $\widetilde{M}_{L,t}(f) = t \dfrac{d}{dt} M_{L,t}(f)$. We further note that $\widetilde{M}_{L,t}$ is clearly also a Fourier multiplier operator, indeed

$$\widehat{\widetilde{M}_{L,t}(f)}(\xi) = \widehat{f}(\xi)\, m_L(\xi)\, \big(t\xi \cdot \nabla \widehat{d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}}(t\xi)\big). \tag{64}$$

We now immediately see that

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\mathcal{M}_L^{(j)}(f)(x)|^2\, dx \\
&\qquad \leq 2 \sum_{\ell = \lfloor \log_2 \lambda_0 \rfloor}^{\infty} \int_{2^{\ell-1}}^{2^\ell} \int \cdots \int \int_{\mathbb{R}^d} |M_{L,t}(f)(x)||\widetilde{M}_{L,t}(f)(x)|\, dx\, d\sigma^{(d-j+1)}_{y_1,\dots,y_{j-2}}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1)\, \frac{dt}{t} \\
&\qquad\qquad + \int \cdots \int \int_{\mathbb{R}^d} |M_{L,\lambda_0}(f)(x)|^2\, dx\, d\sigma^{(d-j+1)}_{y_1,\dots,y_{j-2}}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1).
\end{aligned}$$

Applying Cauchy-Schwarz to the first integral above (in the variables $x, y_1, \dots, y_{j-1}$, and t together), followed by an application of Plancherel (in two resulting integrations in x as well as in the one that appears in the second integral above), we obtain the estimate

$$\int_{\mathbb{R}^d} |\mathcal{M}_L^{(j)}(f)(x)|^2\, dx \leq 2 \sum_{\ell = \lfloor \log_2 \lambda_0 \rfloor}^{\infty} \big(\mathcal{I}_\ell\, \widetilde{\mathcal{I}}_\ell\big)^{1/2} + \mathcal{I} \tag{65}$$

with

$$\mathcal{I}_\ell = \int_{2^{\ell-1}}^{2^\ell} \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 |m_L(\xi)|^2 I(t\,\xi)\, d\xi\, \frac{dt}{t} \tag{66}$$

$$\widetilde{\mathcal{I}}_\ell = \int_{2^{\ell-1}}^{2^\ell} \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 |m_L(\xi)|^2 \widetilde{I}(t\,\xi)\, d\xi\, \frac{dt}{t} \tag{67}$$

and

$$\mathcal{I} = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 |m_L(\xi)|^2 I(\lambda_0\, \xi)\, d\xi \tag{68}$$

where, as in the proof of Proposition 4.2, we have defined

$$I(\xi) = \int \cdots \int \big|\widehat{d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}}(\xi)\big|^2\, d\sigma^{(d-j+1)}_{y_1,\dots,y_{j-2}}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1) \tag{69}$$

and analogously now also define

$$\widetilde{I}(\xi) = \int \cdots \int \big|\xi \cdot \nabla \widehat{d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}}(\xi)\big|^2\, d\sigma^{(d-j+1)}_{y_1,\dots,y_{j-2}}(y_{j-1}) \cdots d\sigma^{(d-1)}(y_1). \tag{70}$$

Combining (59) with (43), and recalling that we are assuming that $d \geq j+2$, gives

$$|m_L(\xi)|^2 I(t\,\xi) \ll \min\{(t|\xi|)^{-1}, L^2|\xi|^2\} \leq L^{2/3} t^{-2/3} \tag{71}$$

which ensures, via Plancherel, that

$$\mathcal{I}_\ell \ll \Big(\frac{L}{2^\ell}\Big)^{2/3}\|f\|_2^2 \quad \text{and} \quad \mathcal{I} \ll \Big(\frac{L}{\lambda_0}\Big)^{2/3}\|f\|_2^2. \tag{72}$$

Arguing as in the proof of estimate (43), we can see that estimate (22) for $\nabla \widehat{d\sigma^{(d-j)}_{y_1,\dots,y_{j-1}}}(\xi)$ ensures that $\widetilde{I}(\xi)$ is bounded whenever $d \geq j+2$. It follows immediately from this observation (and Plancherel) that

$$\widetilde{\mathcal{I}}_\ell \ll \|f\|_2^2. \tag{73}$$

Combining (65), (72), and (73), we get that

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathcal{M}_L^{(j)}(f)(x)|^2\,dx &\ll \left(L^{1/3}\sum_{\ell=\lfloor \log_2 \lambda_0\rfloor}^{\infty} 2^{-\ell/3} + \Big(\frac{L}{\lambda_0}\Big)^{2/3}\right)\int_{\mathbb{R}^d}|f(x)|^2\,dx \\ &\ll \Big(\frac{L}{\lambda_0}\Big)^{1/3}\int_{\mathbb{R}^d}|f(x)|^2\,dx\end{aligned}$$

as required. □

References

[1] J. Bourgain, *A Szemerdi type theorem for sets of positive density in $\mathbb{R}^k$*, Israel J. Math. 54 (1986), no. 3, 307–316.

[2] H. Furstenberg, Y. Katznelson and B. Weiss, *Ergodic theory and configurations in sets of positive density*, Israel J. Math. 54 (1986), no. 3, 307–316.

[3] L. Grafakos, *Classical Fourier Analysis*, Graduate Text in Mathematics, Volume 249, 2008.

[4] E. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, Princeton, NJ., 1993.

Department of Mathematics, The University of Georgia, Athens, GA 30602, USA

E-mail address: `lhuckaba@math.uga.edu`

Department of Mathematics, The University of Georgia, Athens, GA 30602, USA

E-mail address: `lyall@math.uga.edu`

Department of Mathematics, The University of Georgia, Athens, GA 30602, USA

E-mail address: `magyar@math.uga.edu`