MATH 8100 ASSIGNMENT 4
CONVERGENCE AND FUBINI'S THEOREM
DUE: OCT. 15TH

I. **Stein-Sakarchi, Chapter 2:** 5, 8, 21, 22, 24

Problem 6. Complete the following steps to give alternative proofs of the basic convergence theorems (without using the topology of $\mathbb{R}^d$).

(a) Suppose that $0 \leq f_n \leq f_{n+1} \leq \ldots$ is a sequence of measure functions. Let $0 \leq g = \sum_{i=1}^k a_i \chi_{E_i}$ be a simple function. Prove directly that if $g(x) \leq \lim_{n\to\infty} f_n(x)$ for a.e. x, then

$$\int g \leq \lim_{n\to\infty} \int f_n.$$

Hint: Given $\varepsilon > 0$, consider the sets $A_{i,\varepsilon} = \{x : \; f_n(x) \geq (1-\varepsilon)a_i$.
Explain how the Monotone Convergence Theorem follows from this!

(b) Deduce Fatou's Lemma directly from the Monotone Convergence Theorem.

(c) Deduce the Lebesgue Dominated Convergence Theorem from Fatou's Lemma.
Hint: Assume f_n is real, and $-g \leq f_n \leq g$ for $g \in L^1$. Apply Fatou's lemma to both the sequence $f_n + g$ and $g - f_n$.

Problem 7. Assume the both $f(x)$ and $xf(x)$ are integrable functions on $\mathbb{R}$. Prove that the Fourier transform

$$\hat{f}(t) := \int_{\mathbb{R}} f(x)\, e^{-2\pi i x t}\, dx,$$

is differentiable and calculate its derivative.

Problem 8. Let $f, g \in L^1([0,1])$ and for $0 \leq x \leq 1$, define

$$F(x) = \int_0^x f(t)\, dt, \quad G(x) = \int_0^x g(t)\, dt.$$

Prove that,

$$\int_0^1 F(x)g(x)\, dx = F(1)G(1) - \int_0^1 f(x)G(x)\, dx.$$