PROBLEM SESSION IV

1. Let $f, g \in L^1(\mathbb{R}^n)$ such that f, g are Borel measurable.

(a) Show that the function $f(x-y)g(y)$ is Borel measurable in (x, y) and that for almost every x the function $f(x-y)g(y)$ is integrable with respect to y.

(b) Define $f * g(x) = \int f(x-y)g(y)\,dy$, and show that

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

2. Let (X, μ) be a measure space and let $f : X \to \mathbb{R}$ be a measurable function. Define, for $\lambda \geq 0$

$$\phi(\lambda) = \mu\{x : \ f(x) > \lambda\}, \quad \psi(\lambda) = \mu\{x : \ f(x) < -\lambda\}.$$

Prove that

$$\int_X |f|\,d\mu = \int_0^\infty (\phi(\lambda) + \psi(\lambda)\,d\lambda.$$

3. Let $f \in L^\infty(\mathbb{R})$. Show that

$$\lim_{n \to \infty} \int_R \frac{|f(x)|^n}{1+x^2} = \|f\|_\infty,$$

where $\|f\|_\infty = \inf\{M : \ m(|f| \geq M) = 0\}$.

4. Let $f(x, y) = \frac{xy}{(x^2+y^2)^2}$ if $(x, y) \neq (0, 0)$ and define $f(0, 0) = 0$.

Determine if $f \in L^1([0, 1] \times [0, 1])$ and justify your assertion.

5. Let $f \in L^1(\mathbb{R})$ and for $h > 0$ let $A_h f(x) := \frac{1}{2h} \int_{|y| \leq h} \int f(x-y)\, dy$.

(a) Prove that $\|A_h f\|_1 \leq \|f\|_1$ for all $h > 1$.

(b) Prove that $\|A_h f - f\|_1 \to 0$ as $h \to 0^+$.

6. Suppose f is a finite-valued measurable function on $[0,1]$ and let $g(x,y) = |f(x) - f(y)|$. Prove that if g is integrable on $[0,1]^2$ then f is integrable on $[0,1]$.

7. Suppose $f \geq 0$ and that f is integrable on $\mathbb{R}$. For $y > 0$, let

$$g(y) = y\, m(\{x : \ f(x) \geq y\}).$$

Prove that

(a) $\lim_{y \to 0_+} g(y) = \lim_{y \to \infty} g(y) = 0$.

(b) g achieves its maximum at some point y_0.