PROBLEM SESSION II

1. Let $m*$ denote the Lebesgue measure on $\mathbb{R}$.

(a) Prove that to every set $E \subseteq \mathbb{R}$ the exists a Borel set B containing E such that

$$m^*(B) = m^*(E)$$

(b) Prove that if $E \subseteq \mathbb{R}$ has the property that

$$m*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

for every set $A \subseteq \mathbb{R}$ then there exists a Borel set B such that $E = B \backslash N$ with $m^*(N) = 0$.

2. Let μ be a finite Borel measure (i.e. $\mu(\mathbb{R}) < \infty$) on the Borel subsets of $\mathbb{R}$.

(a) Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_{k+1} \subseteq F_k$ for all k, then

$$\lim_{k\to\infty} \mu(F_k) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right).$$

(b) Suppose μ has the property that $\mu(E) = 0$ whenever $m(E) = 0$. Prove that to every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \varepsilon$ whenever $m(E) < \delta$.

3. Let K be the set of numbers in [0,1] whose decimal expansions do not use the digit 4 (we use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with 399.... For example, 0.8754=0.8753999....) Show that K is a compact, nowhere dense set without isolated points, and find the Lebesgue measure $m(K)$.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{E_n\} \subseteq \mathcal{M}$ is a sequence of measurable sets such that

$$\lim_{n\to\infty} \mu(X \backslash E) = 0.$$

Let F be the set of points $x \in X$ which belong to only finitely many of the sets E_n. Prove that $\mu(F) = 0$.

5. Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set with $m(E) < \infty$. prove that the function: $f(x) := m(E \cap (E + x)$, is uniformly continuous and that $f(x) \to 0$ as $|X| \to \infty$.

6. Suppose that $E \subseteq \mathbb{R}^d$ is a Lebesgue measurable set. Prove that if $E = E_1 \cup E_2$ is a partition such that $m(E) = m^*(E_1) + m^*(E_2)$, then both E_1 and E_2 are Lebesgue measurable sets.