Waring problem - The asymptotic formula $1 \gg \nu = \nu_k > 0$.

Recall : $\mathfrak{M} = \bigcup_{q=1}^{P^\nu} \bigcup_{(a,q)=1} \mathfrak{M}_{a/q}$, $\mathfrak{M}_{a/q} = \{\alpha ; |\alpha - \frac{a}{q}| \leq \frac{1}{qP^{k-\nu}}\}$
major arcs

$m = [0,1] \setminus \mathfrak{M}$ minor arcs

Also, for $s \geq 2^k+1$

$$R_s(N) = \#\{x_1, \dots, x_s \in \mathbb{N} ; \ x_1^k + \dots + x_s^k = N\}$$

Setting $P = N^{1/k}$ and $T(\alpha) = \sum_{x \in \mathbb{N}} e(\alpha x^k) \chi_P(x)$

we have

$$R_s(N) = \int_0^1 T(\alpha)^s e(-\alpha N)\, d\alpha = \int_{\mathfrak{M}} T(\alpha)^s e(-\alpha N) + \int_m T(\alpha)^s e(-\alpha N) d\alpha$$

$$:= M_s(N) + \mathcal{E}_s(N)$$

Here $$|\mathcal{E}_s(N)| \leq \int_m |T(\alpha)|^s d\alpha \ll N^{\frac{s}{k}-1-\delta'} \qquad (5.1)$$

Thus in order to obtain an asymptotic formula for $s \geq 2^k+1$, it is enough to show that

$$\int_{\mathfrak{M}} T(\alpha)^s e(-\alpha N) = N^{\frac{s}{k}-1} \sigma(N) J_k + O(N^{\frac{s}{k}-1-\delta})$$

with $J_k > 0$ and $0 < \alpha_1 < \sigma(N) < \beta$ for some constants $0 < \alpha < \beta$

Note

- Both $\mathfrak{S}(N)$ called the "singular series", and $\mathcal{J}_k$ called the 'singular integral", can be described explicitely ; and are quite meaningful.

Lemma 5.1. For $\alpha = \frac{a}{q} + \beta \in \mathfrak{M}_{a|q}$, we have

$$T(\alpha) = q^{-1} S_{a,q} \cdot I_P(\beta) + O(P^{2\nu}) \tag{5,3}$$

where $S_{a,q} = \sum_{r=0}^{q-1} e(ar^k/q)$, $I(\beta) = \int_0^P e(\beta y^k)\, dy$

Pf This works very similarly as for $k=2$

Write $T(\alpha) = \sum_{r=0}^{q-1} \sum_{n=0}^{P/q} \underbrace{e\left(\left(\frac{a}{q}+\beta\right)(nq+r)^k\right)}_{e\left(\frac{ar^k}{q}\right) e\left(\beta (nq+r)^k\right)}$

and use the approximations

- $e(\beta(nq+r)^k) = e(\beta n^k q^k) + O(P^{-1+2\nu})$

$$\Rightarrow T(\alpha) = \left(\sum_{r=0}^{q-1} e(ar^k/q)\right) \cdot \left(\sum_{n=0}^{P/q} e(\beta n^k q^k)\right)$$

- For $0 \le t \le 1$

$$\int_0^1 e(\beta(n+t)^k q^k)\, dt = e(\beta n^k q^k) + O(P^{-1+2\nu})$$

$$\Rightarrow \quad \sum_{n=0}^{P/q} e(\beta n^k q^k) = \int_0^{P/q} e(\beta q^k x^k)\,dx + O(P^{2\nu}/q)$$

$$y = qx;\ dx = \tfrac{1}{q}\,dy$$

$$\Rightarrow \quad T(\alpha) = \frac{1}{q} S_{a,q} \int_0^P e(\beta y^k)\,dy = \frac{1}{q} S_{a,q}\, I_P(\beta) + O(P^{2\nu})$$

$\square$

Lemma 5.2. We have, for some $\delta > 0$;

$$\int_M T(\alpha)^s e(-N\alpha)\,d\alpha = P^{s-k}\,\sigma(P^\nu, N)\, J(P^\nu) + O(P^{s-k-\delta})$$

where
$$\sigma(P^\nu, N) = \sum_{q \le P^\nu} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\frac{1}{q} S(a,q)\right)^s e\left(-\frac{a}{q} N\right) \tag{5.3}$$

$$J_k(P^\nu) = \int_{|\beta| < P^\nu} \left(\int_0^1 e(\beta y^k)\,dy\right)^s e(-\beta)\,d\beta$$

Note This is not quite an asymptotic formula yet, as $J(P^\nu)$ and $\sigma(P^\nu, N)$ may grow with P.

Pf: For $\alpha \in M_{a/q}$, $\alpha = \frac{a}{q} + \beta$ we have

$$T(\alpha) = \bar{S}(a,q)\, I_P(\beta) + O(P^{2\nu}), \quad \bar{S}(a,q) = \frac{1}{q} S(a,q)$$

$$\Rightarrow \quad T(\alpha)^s = \bar{S}(a,q)^s\, I_P(\beta)^s + O(P^{s-1+2\nu})$$

as trivially $|\bar{S}(a,q)| \le 1$ and $|I_P(\beta)| \le P$

Since $$|\mathfrak{M}| \le \sum_{q\le P^\nu}\sum_{a=1}^{q} \frac{1}{P^{k-\nu}} \le P^{-k+3\nu}$$

we have ; $\alpha = \frac{a}{q}+\beta \Rightarrow e(-N\alpha) = e(-N\frac{a}{q})\, e(-N\beta)$; $|\beta| \le P^{-k+\nu}$

$$\int_{\mathfrak{M}} T(\alpha)^s e(-N\alpha)\,d\alpha = \Big[\sum_{q\le P^\nu}\sum_{\substack{a=1\\(a,q)=1}}^{q} \bar{S}(a,q)^s e(-N\tfrac{a}{q})\Big]\Big[\int_{|\beta|\le P^{-k+\nu}} I_P(\beta)^s e(-N\beta)\Big]$$
$$+O\left(P^{-k+3\nu}\,P^{s-1+2\nu}\right)$$

The total error is $O(P^{s-k-1+5\nu}) = O(P^{s-k-\delta})$ $(\delta = \frac{1}{2}$ e.g.$)$

The integral is

$$\int_{|\beta|\le P^{-k+\nu}} \left(\int_0^P e(\beta y^k)\,dy\right)^s e(-P^k\beta)\,d\beta = P^s \int_{|\beta|\le P^{-k+\nu}} \left(\int_0^1 e(P^k\beta x^k)\,dx\right)^s e(-P^k\beta)\,d\beta$$

$x = P^{-1}y$, $\qquad \gamma = P^k\beta$,

$$= P^{s-k}\int_{|\gamma|\le P^\nu}\left(\int_0^1 e(\gamma x^k)\,dx\right) e(-\gamma)\,d\gamma = P^{s-k}\,\mathfrak{J}(P^\nu)$$

$\square$

<u>Lemma 5.3.</u> For $s \ge 2^k+1$, we have that

$$\sum_{q\ge P^\nu}\sum_{(a,q)=1} |\bar{S}(a,q)|^s \ll P^{-\nu}$$

By Weyl's inequality (with $P=q$, $\alpha=\frac{a}{q}$), $K=2^{k-1}$ we have that (as $P^k/q = q^{k-1} \geq q$)

$$|\bar{S}(a,q)| \ll q^{-\frac{1}{2^{k-1}}} \Rightarrow |\bar{S}(a,q)|^s \ll q^{-\frac{s}{2^{k-1}}} \leq q^{-2}$$

$$\Rightarrow \sum_{q \geq P^\nu} |\bar{S}(a,q)|^s \ll \sum_{q \geq P^\nu} q^{-2} \ll P^{-\nu}$$

$\square$

<u>Define</u> $\sigma(N) := \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \bar{S}(a,q)^s e(-Na/q)$,

Then $\sigma(N, P^\nu) = \sigma(N) + O(P^{-\nu})$ (5.4)

<u>Lemma 5.4.</u> Let $I_k(\gamma) := \int_0^1 e(\gamma y^k)\, dy$.

Then $I_k(\gamma) \ll (1+|\gamma|)^{-\frac{1}{k}}$

<u>Pf</u> Essentially the same as for $k=2$.

Let $0<\delta<1$, write

$$I(\gamma) = \int_0^\delta e^{2\pi i \gamma y^k} dy + \int_\delta^1 e^{2\pi i \gamma y^k} dy = I_1(\gamma) + I_2(\gamma)$$

For $I_2(\gamma)$; use that $\frac{d}{dy}\left(e^{2\pi i \gamma y^k}\right) = 2k\pi i \gamma y^{k-1} e^{2\pi i \gamma y^k}$

Thus integrating by parts give:

$$I_2(\gamma) = \left| \int_\delta^1 \frac{1}{2k\pi i \gamma y^{k-1}} \frac{d}{dy}\left(e^{2\pi i \gamma y^k}\right) dy \right|$$

$$\ll \frac{1}{\gamma} \left| \left[\frac{1}{y^{k-1}} e^{2\pi i \gamma y^k} \right]_\delta^1 \right| + \int_\delta^1 y^{-k}\, dy \ll \frac{1}{\delta^{k-1}\gamma}$$

Clearly $|I_1(\gamma)| \leq \delta$ thus $|I(\gamma)| \ll \delta + \frac{1}{\delta^{k-1}\gamma}$

choosing δ s.t. $\delta = \frac{1}{\delta^{k-1}\gamma} \Leftrightarrow \delta = \gamma^{-\frac{1}{k}}$

we get $|I(\gamma)| \ll \gamma^{-\frac{1}{k}}$ and since $|I(\gamma)| \leq 1$

$$\Rightarrow |I(\gamma)| \ll (1+|\gamma|)^{-\frac{1}{k}}$$

<u>Lemma 5.4.</u> Let $s > k$. Let $J = J_{k,s} = \int_{\mathbb{R}} I(\gamma)^s e(-\gamma)\, d\gamma$

then J is abs. convergent and and $J - J(p^\nu) = O(p^{-\nu/k})$

<u>Pf</u>: We have

$$|J - J(p^\nu)| \leq \int_{|\gamma| \geq p^\nu} |I(\gamma)|^s\, d\gamma \ll$$

$$\ll \int_{|\gamma| \geq p^\nu} |\gamma|^{-s/k}\, d\gamma \ll p^{-\nu(s/k - 1)} \ll p^{-\nu/k}$$

$\square$