Lemma 4.1. (Weyl's ineq.) Let $f(x) = \alpha x^k + \alpha_1 x^{k-1} + \dots + \alpha_k$, $\alpha \neq 0$

Assume $|\alpha - \frac{a}{q}| \leq \frac{1}{q^2}$ for some $q \geq 1$, $(a,q) = 1$.

Then $\forall \varepsilon > 0$ $\exists C_\varepsilon > 0$ s.t.

$$\left|\sum_{x=1}^{P} e(f(x))\right| \leq C_\varepsilon P^{1+\varepsilon}\left(q^{-\frac{1}{K}} + \left(\frac{P^k}{q}\right)^{-\frac{1}{K}}\right)$$

with $K = 2^{k-1}$

Pf Let $\chi_P(x) = \begin{cases} 1, & \text{if } 1 \leq x \leq P \\ 0, & \text{otherwise} \end{cases}$

Let $S = \frac{1}{P}\sum_{x=1}^{P} e(f(x)) = \mathbb{E}_x\, e(f(x))\, \chi_P(x)$

Then, as before ; $\mathbb{E}_x = \frac{1}{P}\sum_x$

$$|S|^2 = \mathbb{E}_{x,y}\, e(f(y) - f(x))\, \chi_P(y)\chi_P(x) \quad ; \quad y := x + h, \quad x := x$$

$$= \mathbb{E}_{h,x}\, e(D_h f(x))\, \Delta_h \chi_P(x) \ll \mathbb{E}_{h_1}\left|\mathbb{E}_x e(D_{h_1} f(x)) \cdot \Delta_{h_1}\chi_P(x)\right|$$

- $D_h f(x) = f(x+h) - f(x) = k\alpha x^{k-1} h + g_1(x,h)$; $\deg g_1(x,h) \leq k-2$ (in the variable x)
- $\Delta_h \chi_P$ is the indicator function of an interval of length P

By Cauchy-Schwartz

$$|S|^4 \ll \mathbb{E}_{h_1}\left|\mathbb{E}_x e(D_{h_1} f(x)) \cdot \Delta_{h_1}\chi_P(x)\right|^2 =$$

$$\ll \mathbb{E}_{h_1} \mathbb{E}_{h_2} \left| \mathbb{E}_x \, e(D_{h_2} D_{h_1} f(x)) \, \Delta_{h_2} \Delta_{h_1} \chi_p(x) \right|$$

where $\quad D_{h_2} D_{h_1} f(x) = k(k-1)\, \alpha \, x^{k-2} h_2 h_1 + g_2(x; h_2, h_1)$

with $\deg g_2(x; h_2, h_1) \leq k-3$

Repeating this procedure $k-1$-times we get.

$$|S|^{2^{k-1}} \ll \mathbb{E}_{h_1, \dots, h_{k-1}} \left| \mathbb{E}_x \, e(D_{h_{k-1}} \dots D_{h_1} f(x)) \, \Delta_{h_{k-1}} \dots \Delta_{h_1} \chi_p(x) \right|$$

$$= \mathbb{E}_{h_1 \dots h_{k-1}} \left| \mathbb{E}_x \, e(\alpha \, k! \, h_1 \dots h_{k-1} \, x) \, \Delta_{h_1} \dots \Delta_{h_{k-1}} \chi_p(x) \right|$$

as $\quad D_{h_{k-1}} \dots D_{h_1} f(x) = k! \, \alpha \, h_1 \dots h_{k-1} x + g_{k-1}(x; h_1, \dots, h_{k-1})$

with $\deg g_{k-1}(x; h_1 \dots h_{k-1}) \leq k-k = 0$

$\Rightarrow \quad g_{k-1}(x, h_1 \dots h_{k-1}) = g_{k-1}(h_1, \dots, h_{k-1})$

and $|e(g(h_1 \dots h_{k-1}))| = 1$.

Writing $\beta = k! \alpha \, h_1 \dots h_{k-1}$, and $K = 2^{k-1}$ we get

$$\left| \mathbb{E}_x \, e(\beta x) \, \chi_{p, h_1 \dots h_{k-1}}(x) \right| = \frac{1}{p} \left| \sum_x^{\mathrm{I}} e(\beta x) \, \chi_{p, h_1 \dots h_{k-1}}(x) \right|$$

$$\leq \frac{1}{p} \min \left\{ P, \frac{1}{\|\beta\|} \right\} = \frac{1}{p} \min \left\{ P, \frac{1}{\|\alpha \, k! \, h_1 \dots h_{k-1}\|} \right\}$$

write $h := h_1 \dots h_{k-1} k!$; then $|h| \le P^{k-1}$ as $|h_i| \le P$

and for given h, we have that

$$\#\{h_1, \dots, h_{k-1} : k!\, h_1 \dots h_{k-1} = h\} \le d(h)^{k-1} \ll h^{\varepsilon} \ll P^{\varepsilon}$$

$(\le C_{\varepsilon,k} h^{1+\varepsilon} \ \forall \varepsilon > 0)$

Thus we have

$$|S|^{2^{k-1}} \ll P^{-(k-1)+\varepsilon} \sum_{h \le P^{k-1}} \frac{1}{P} \min\left\{P, \frac{1}{\|\alpha h\|}\right\}$$

writing $H = P^{k-1}$ we get for the inner sum

$$\frac{1}{P}\left(\frac{P^{k-1}}{q}+1\right)\left(P + q \log q\right) = \left(\frac{P^{k-1}}{q}+1\right)\left(1+\frac{q}{P}\right)\log q$$

$$\Rightarrow |S|^{2^{k-1}} \ll P^{\varepsilon}\left(\frac{1}{q} + \frac{1}{P^{k-1}}\right)\left(1 + \frac{q}{P}\right)\log q$$

$$= P^{\varepsilon} \log q \left(\frac{1}{q} + \frac{1}{P} + \frac{q}{P^k}\right)$$

Since $|S| \ll 1$, this is trivial if $q \ge P^k$

thus WLOG $q \le P^k$, and then

$$|S|^{K} \ll P^{\varepsilon}\left(\frac{1}{q} + \frac{1}{P} + \frac{q}{P^k}\right)$$

$P \ge q$

$$\Rightarrow |S| \ll P^{\varepsilon}\left(q^{-1/K} + P^{-1/K} + \left(\frac{P^k}{q}\right)^{-1/K}\right)$$

$\square$

<u>Note</u> If $P^{\nu} \le q \le P^{k-\nu}$ then

$$|S| \ll P^{\varepsilon} P^{-\nu/K} \ll P^{-\nu/2K} \text{ if } \varepsilon \le \frac{\nu}{2K}.$$

Major / minor arcs Let $0 < \nu \ll 1$, be a small constant depending on k (choosen later)

$$\mathcal{M}_{a/q} = \{ \alpha ; |\alpha - \tfrac{a}{q}| \le \frac{1}{qP^{k-\nu}} \}, \text{ given } (a,q)=1,\ q \ge 1.$$

Note. $\forall\, 0 \le \alpha \le 1 \quad \exists\, (a,q)=1,\ 1 \le q \le P^{k-\nu}$ s.t. $|\alpha - \frac{a}{q}| \le \frac{1}{qP^k}$ by Dirichlet's principle, with level $N = P^{k-\nu}$.

We say $\mathcal{M}_{a/q}$ is a major arc if $1 \le q \le P^\nu$

and write $\mathcal{M} := \bigcup_{q=1}^{P^\nu} \bigcup_{(a,q)=1} \mathcal{M}_{a/q}$ for the union of major arcs

and $m = [0,1] \setminus \mathcal{M} \subseteq \bigcup_{P^\nu \le q \le P^{k-\nu}} \bigcup_{(a,q)=1} \mathcal{M}_{a/q}$

for the minor arcs

Cor 4.1. For $(a,q)=1$, let $S_{a,q} = \sum_{x=1}^{q} e(\frac{a}{q} x^k)$

Then $|S_{a,q}| \ll q^{1-\frac{1}{K}+\varepsilon} \quad (K = 2^{k-1})$

(i.e. $\le C_\varepsilon\, q^{1-\frac{1}{K}+\varepsilon}$, $\forall \varepsilon > 0$)

Pf Let $\alpha = \frac{a}{q}$, $P = q$ and apply Weyl's inequality

ANDE IV (5)

To estimate $\int_{\mathfrak{m}} |T(\alpha)|^s d\alpha$, one needs a mean-value theorem; we use an "easy" one given by Hua.

<u>Lemma 4.2.</u> (Hua's inequality) Let $S(\alpha) = \frac{1}{p} \sum_x e(\alpha x^k) \chi_p(x)$

Then $\int_0^1 |S(\alpha)|^{2^k} d\alpha \ll p^{-k+\varepsilon}$

<u>Pf</u> We prove by induction for $j = 1, 2, \dots, k$

$$I_j := \int_0^1 |S(\alpha)|^{2^j} d\alpha \ll p^{-j+\varepsilon}$$

$j=1$: $\int_0^1 |S(\alpha)|^2 = \frac{1}{p^2} \int_0^1 \sum_{x,y=1}^{p} e(\alpha(x^k - y^k))\, d\alpha \ll p^{-1}$

$j \mapsto j+1$: Recall our differencing procedure ($f(x) = x^k$)

$$|S(\alpha)|^2 = \mathbb{E}_{x,h_1} e(\alpha D_{h_1} f(x))\, \Delta_{h_1} \chi_p(x) \le \mathbb{E}_{h_1} |\mathbb{E}_x e(\alpha D_{h_1} f(x))\, \Delta_{h_1} \chi_p$$

$$|S(\alpha)|^4 \le \mathbb{E}_{h_1} \mathbb{E}_{h_2} \mathbb{E}_x\, e(\alpha D_{h_2} D_{h_1} f(x))\, \Delta_{h_2} \Delta_{h_1} \chi_p(x)$$

$$= p^{-3} \sum_{|h_1|, |h_2| \le p} \sum_x e(\alpha h_1 h_2 p(x, h_1, h_2))\, \chi_{p, h_1, h_2}(x)$$

<u>indeed</u>:
- $D_{h_1} f(x) = (x+h_1)^k - x^k = h_1 P_1(x, h_1) = h_1 (k x^k + c x^{k-1}$

  $D_{h_2} D_{h_1} f(x) = h_1 h_2 p(x; h_1, h_2)$ etc.

- $\Delta_h \chi_p(x) = \chi_p(x+h_1)\, \chi_p(x) = \begin{cases} 1 & \text{if } x \in [1,P] \cap [1-h_1, P- \\ 0 & \text{otherwise} \end{cases}$

ANDE

$\Rightarrow \Delta_{h_1}\chi_p(x) \equiv 0$ unless $|h_1| \le P$

since $\Delta_{h_1}\chi_p = \chi_{p,h_1}$ indicator function of an interval $I \subseteq [1,$

$\Rightarrow \Delta_{h_2}\Delta_h\chi_p \equiv 0$ unless $|h_2| \le P$ as well.

By induction we obtain

$$|S(\alpha)|^{2^j} \ll p^{-j-1} \sum_{\substack{h_1 \ldots h_{k-1} \\ |h_i| \le p}} \sum_x e(\alpha h_1 \ldots h_{k-1} \cdot p(x, h_1 \ldots h_{k-1})) \underbrace{\chi_{p,h_1 \ldots h_{k-1}}(x)}_{\substack{\text{indicator} \\ \text{function of an} \\ \text{interval } \subseteq [}}$$

$$= p^{-j-1} \sum_h e(\alpha h)\, c_h$$

where $c_h = \#\{h_1, \ldots, h_{k-1}, x,\ ;\ h_1 \ldots h_{k-1} p(x; h_1 \ldots h_{k-1}) = h\}$

$\Rightarrow |c_0| \ll p^j$, $|c_h| \ll p^\varepsilon$ (for $h \ne 0$)

But also

$$|S(\alpha)|^{2^j} = S(\alpha)^{2^{j-1}} (\overline{S(\alpha)})^{2^{j-1}} = S(\alpha)^{2^{j-1}} S(-\alpha)^{2^{j-1}} =$$

$$= p^{-2^j} \sum_{x_1, \ldots, x_j = 1}^{p} \sum_{y_1 \ldots y_j = 1}^{p} e(\alpha(x_1^k + \ldots + x_j^k - y_1^k - \ldots - y_j^k)) =$$

$$= \sum_h b_h\, e(-\alpha h) \quad \text{with } b_h = p^{-2^j} \#\{x_1, \ldots x_j, y_1, \ldots y_j \le p;\ x_1^k + \ldots + x_j^k - y_1^k - \ldots - y_j^k = h\}$$

$\Rightarrow \sum_h b_h = |S(0)|^{2^j} \ll 1$; $b_h \ge 0$.

Using Plancherel (in fact just orthogonality)

$$\int_0^1 |S(\alpha)|^{2^{j+1}} d\alpha = \int_0^1 |S(\alpha)|^{2^j} |S(\alpha)|^{2^j} \ll \int_0^1 p^{-j-1} \Big(\sum_h c_h e(\alpha h)\Big)\Big(\sum_{h'} b_{h'} e(-\alpha h$$

$$= p^{-j-1}\sum_h c_h b_h = p^{-j-1}\Big[c_0 b_0 + \sum_{h\neq 0} c_h b_h\Big]$$

$$b_0 = \#\{1\le x_1, \dots x_j, y_1 \dots y_j \le p \;:\; x_1^k + \dots + x_j^k = y_1^k + \dots + y_j^k\}\, p^{-2j}$$

$$= \int_0^1 |S(\alpha)|^{2^j} d\alpha \ll p^{-j+\varepsilon} \text{ by induction.}$$

$$\Rightarrow \int_0^1 |S(\alpha)|^{2^{j+1}} d\alpha \ll p^{-j-1}\Big[p^j p^{-j+\varepsilon} + \sum_h b_h p^{\varepsilon}\Big]$$

$$\ll p^{-j-1}\big[p^{\varepsilon} + p^{\varepsilon}\big] \ll p^{-j-1+\varepsilon}$$

$\square$

<u>Corollary</u> If $s \geq 2^k+1$, then for $T(\alpha) = \sum_{x=1}^{p} e(\alpha x^k)$

we have that
$$\int_{m} |T(\alpha)|^s d\alpha \ll p^{s-k-\delta}; \text{ for some } \delta > 0.$$

<u>Pf</u> Write $|T(\alpha)|^s = |T(\alpha)|^{2^k}\, |T(\alpha)|^{s-2^k}$

If $\alpha \in m$ then $|T(\alpha)| \ll p^{1-\nu/2^k}$ $\quad \left(\varepsilon := \frac{\nu}{2^k}\right)$

$$= p^{1-2\delta}$$

$$\Rightarrow \int_m |T(\alpha)|^s d\alpha \ll p^{(1-2\delta)(s-2^k)} \int_0^1 |T(\alpha)|^{2^k} d\alpha$$

$$\ll p^{(1-2\delta)(s-2^k)}\, p^{2^k-k+\varepsilon} \qquad p^{s-2^k} p^{2^k-k}$$

$$= p^{s-k}\, p^{-2\delta\underset{\geq 1}{(s-2^k)}+\varepsilon(s-2^k)} \leq p^{s-k-\delta} \quad \text{as } s-2^k \geq 1$$

$\square$