Sums of squares Let $\nu > 0$, we set $\nu = \frac{1}{4}$, $P = N^{1/2}$

For $1 \le q \le P^{2-\nu}$ and $(a,q) = 1$, we set

$$M_{a/q} = \{\alpha \,;\, |\alpha - \tfrac{a}{q}| \le \frac{1}{qP^{2-\nu}}\}.$$

Note By Dir. princ: $\forall \alpha$ $\exists\, q \le P^{2-\nu}$ $(a,q)=1$ s.t. $\alpha \in M_{a/q}$

Let $M = \bigcup_{\substack{q \le P^{\nu} \\ (a,q)=1}} M_{a/q}$ "major arcs", $m = M^c$ = minor arcs.

Prop. 3.1 If $\alpha \in m$, then $|T_P(\alpha)| \lesssim P^{1-\nu/4}$ (say.)

Pf $T_P(\alpha) = \sum_{n \le P} e(\alpha n^2)$; $|\alpha - \frac{a}{q}| \le \frac{1}{qP^{2-\nu}} \le \frac{1}{q^2}$; $q \ge P^{\nu}$.

$$\Rightarrow |T_P(\alpha)| \lesssim P \cdot \log P \left(P^{-1/2} + P^{-\nu/2}\right) \lesssim P^{1-\nu/4}$$

Lemma 3.1 $\displaystyle\int_m |T_P(\alpha)|^{2\ell}\, d\alpha \lesssim P^{2\ell-2} P^{-\nu/4}$ for $\ell \ge 3$.

Pf:

$$\int_0^1 |T_P(\alpha)|^4 = \int_0^1 \Big|\sum_{n \le P} e(\alpha n^2)\Big|^4 d\alpha = \int_0^1 \sum_{n_1 - n_4} e\big(\alpha(n_1^2 - n_4^2)\big)\, d\alpha$$

$n_1^2 + n_2^2 - n_3^2 - n_4^2$

$$= \#\{n_1, \dots, n_4 \le P \,;\, n_1^2 - n_4^2 = n_3^2 - n_2^2\} \le \sum_{m \le P^2} d(m)^2$$

Let $\bar{d}(m) = \#\{n_1, n_2 \,;\, n_1^2 - n_2^2 = m\} \le d(m)$

From the fact that $d(m) \leq C_\varepsilon m^\varepsilon \ \forall \varepsilon > 0$ (HW)

we get $\int_0^1 |T_p(\alpha)|^4 d\alpha \lesssim C_\varepsilon P^{2+\varepsilon}$

Thus

$$\int_{m}^{*} |T_p(\alpha)|^6 \lesssim \left(\sup_{\alpha \in m} |T_p(\alpha)|^2\right) \int_0^1 |T_p(\alpha)|^4 d\alpha$$

$$\lesssim C P^{2-\nu/4} P^{2+\nu/8} \lesssim C P^4 P^{-\nu/8} \quad \text{(minor arcs cont.)}$$

<u>Major arcs</u> $\quad \mathcal{M}_{a/q} = \{\alpha : |\alpha - \frac{a}{q}| \leq \frac{1}{q} P^{-2+\nu}\}, \quad q \leq P^\nu$

$$|\mathcal{M}| = \left|\bigcup_{a, q \leq P^\nu} \mathcal{M}_{a/q}\right| \leq \sum_{q \leq P^\nu} \sum_{a \leq q} \frac{1}{q} P^{-2+\nu} \leq P^{-2+2\nu}$$

so trivially (using $|T_p(\alpha)| \leq P$)

$$\int_{\mathcal{M}} |T_p(\alpha)|^6 d\alpha \lesssim P^{6-2+2\nu} \leq P^{4+2\nu}$$

<u>Propos. 3.2.</u> We have $\int_{\mathcal{M}} |T_p(\alpha)|^6 d\alpha \lesssim P^4$.

From Lemma 2.1–2.2. we get

$$\int_0^1 |T_p(\alpha)|^6 d\alpha \lesssim P^4,$$

C.

To prove Prop. 3.2, we prove more, namely

<u>Lemma 3.2.</u> Let $\alpha \in \mathfrak{M}_{a/q}$ with $1 \le q \le P^{\nu}$. Then

$$T_p(\alpha) = q^{-1}\, G(a,q)\, I_p(\alpha - a/q) + O(P^{1/5})$$

where $G(a,q) = \sum_{r=0}^{q-1} e(ar^2/q)$, $I_p(\beta) = P\int_0^1 e(P^2\beta x^2)$.

<u>Pf</u>. Write $\alpha = \frac{a}{q} + \beta$, $|\beta| \le \frac{1}{q} P^{-2+\nu}$, $q \le P^{\nu}$.

Also write $m = nq + r$, $0 \le r < q$, $0 \le n \le P/q$

$$T_p(\alpha) = \sum_{r=0}^{q-1} \sum_{n=0}^{M/q} e\left(\left(\frac{a}{q}+\beta\right)\cdot(nq+r)^2\right) + O(q)$$

$$= \sum_{r=0}^{q-1} e\left(\frac{ar^2}{q}\right) \sum_{n=0}^{P/q} e(\beta(nq+r)^2) \qquad (3.1)$$

- Approximate the inner sum with an integral

$$|e(\beta(nq+r)^2) - e(\beta n^2q^2)| \le |e(\beta(2nqr + r^2)) - 1|$$
$$\lesssim |\beta|\left(\frac{P}{q}q^2 + q^2\right) \lesssim P^{-1+2\nu}$$

Summing for $n \le M/q$ and $r < q$ we get

$$T_p(\alpha) = \sum_{r=0}^{q-1} e\left(\frac{ar^2}{q}\right) \sum_{n=0}^{P/q} e(\beta n^2 q^2) + O(P^{\nu})$$

Let $0 \le t \le 1$, then

$$|e(\beta(n+t)^2q^2) - e(\beta n^2 q^2)| = |e(\beta q^2(2nt+t^2)) - 1|$$

$$\lesssim |\beta| q^2 P/q \lesssim P^{-1+2\nu} \quad (= P^{-2+2\nu} P P^{\nu})$$

Which implis (adding the errors up)

$$T_p(\alpha) = \sum_{r=0}^{q-1} e\left(\frac{ar^2}{q}\right) \sum_{n=0}^{P/q-} \int_0^1 e(\beta(n+t)^2q^2)\,dt + O(P^{2\nu})$$

$$= \sum_{r=0}^{q-1} e\left(\frac{ar^2}{q}\right) \int_0^{P/q} e(\beta q^2 x^2)\,dx = \frac{1}{q} G(a,q)\, I_p(\beta)$$

Let $y = qx$, we get $dx = \frac{1}{q}dy$

$$\int_0^{P/q} e(\beta q^2 x^2)\,dx = \frac{1}{q}\int_0^P e(\beta y^2)\,dy = \frac{P}{q}\int_0^1 e(\beta P^2 y)\,dy$$

$y = Pz$

$$= \frac{1}{q} I_p(\beta)$$

Note We have separated a sum, and an integral! □

Lemma 3.3. $|G(a,q)| \le \sqrt{2}\, q$ ($\leftarrow \sqrt{2}\, q \log q$ follows from Weyl) $(P=q \quad \alpha = \frac{a}{q})$

Pf $|G(a,q)|^2 = \sum_{r_1=0}^{q-1}\sum_{r_2=0}^{q-1} e\left(a\frac{r_1^2 - r_2^2}{q}\right)$ Let $u = r_1 - r_2$, $v = r_1 + r_2$

$(r+h)^2 - r^2 = 2rh + \ldots$

$$\le \sum_{h=0}^{q-1}\left|\sum_{r=0}^{q-1} e\left(\frac{2ah}{q}r\right)\right|$$

inner sum $= \begin{cases} q, & \text{if } 2h \equiv 0 \\ 0, & \text{otherwise} \end{cases}$

$\le 2q$ □

Lemma 3.4. For any $\lambda \geq 0$

$$I(\lambda) = \int_0^1 e^{2\pi i \lambda x^2}\, dx \leq C(1+\lambda)^{-\frac{1}{2}}$$

Pf Let $0 < \delta < 1$

$$I(\lambda) = \int_0^{\delta} e^{2\pi i \lambda x^2}\, dx + \int_{\delta}^{1} e^{2\pi i \lambda x^2}\, dx = I_1(\lambda) + I_2(\lambda)$$

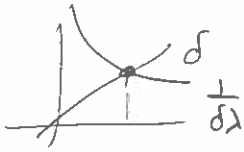
Clearly $|I_\lambda(\lambda)| \leq \delta$, for $I_2(\lambda)$ we integrate by parts:

$$|I_2(\lambda)| = \left| \int_{\delta}^{1} \overset{u}{\frac{1}{4\pi i \lambda x}} \cdot \overset{dv}{\left(\frac{d}{dx} e^{2\pi i \lambda x^2} \right)} dx \right| \leq$$

$$\leq \frac{1}{4\pi\lambda} \left(\left| \left[\frac{1}{x} e^{2\pi i \lambda x^2} \right]_{\delta}^{1} \right| + \int_{\delta}^{1} \frac{1}{x^2} e^{2\pi i \lambda x^2}\, dx \right)$$

$$\leq \frac{1}{4\pi\lambda} \left(\frac{1}{\delta} + \frac{1}{\delta} \right) \lesssim \frac{1}{\delta\lambda}$$

Thus $I(\lambda) \ll \delta + \dfrac{1}{\delta\lambda}$; optimizing



$$\delta = \frac{1}{\delta\lambda} \Rightarrow \delta = \lambda^{-\frac{1}{2}}$$

$$\ll \lambda^{-\frac{1}{2}} \cdot c$$

Also $|I(\lambda)| \leq 1 \Rightarrow |I(\lambda)| \ll \min\{1, \lambda^{-\frac{1}{2}}\} \ll (1+\lambda)^{-\frac{1}{2}}$

$\square$

Proof of Propos. 3.2.

$$|T_p(\alpha)|^6 \lesssim |q^{-1} G(a,q)|^6 \, P^6 \, |I(P^2|\alpha - \tfrac{a}{q}|)|^6$$

$$\lesssim q^{-3} P^6 (1 + P^2|\alpha - a/q|)^{-3} \quad \text{for } \alpha \in M_{a/q}$$

We have,

$$\int_0^\infty (1+P^2\beta)^{-3} \, d\beta = P^{-2}\int_0^\infty (1+\gamma)^{-3}\, d\gamma \lesssim P^{-2}$$

$$\gamma = P^2\beta$$

$$\Rightarrow \int_{M_{a/q}} |T_p(\alpha)|^6 \lesssim P^4 q^{-3}$$

$$\Rightarrow \int_M |T_p(\alpha)|^6 d\alpha \lesssim \sum_{q \leq P^\delta} \int_{M_{a/q}} |T_p(\alpha)|^6 \lesssim P^4 \sum_{q=1}^{\infty} q^{-3} \lesssim P^4$$

$\square$

From Propos. 3.1 & 3.2, we get that

$$\int_0^1 |T_p(\alpha)|^6 d\alpha \leq C P^4$$

$P^2 = N$ $\quad$ $x_1^2 + \dots + x_6^2 = n$

$r_6(N) \leq C N^2$

$$\Rightarrow \#\{n \leq N ;\ r_6(n) \geq 1\} \geq \alpha N \}$$

Note: The argument works without any changes to $r_5(n)$ but not to $r_4(n)$!