<u>Note</u> Strange: $\sigma(2\mathbb{N})=0$, $\sigma(2\mathbb{N}+1)=\frac{1}{2}$.
(very sensitive to small elements of a set $S\subset\mathbb{N}$)

<u>Thm</u> (Schn..) Assume $0\in S$.

(i) Let $\sigma(S)=\delta$. Then $\sigma(S+S)\geq 2\delta-\delta^2$

(ii) If $\sigma(S)>\frac{1}{2}$ then $S+S=\mathbb{N}$.

<u>thm</u> i)' If $\sigma(S)=\alpha$, $\sigma(T)=\beta$ then $\sigma(S+T)\geq\alpha+\beta-\alpha\beta$. (HW)

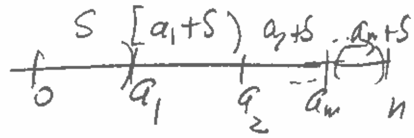
<u>Pf</u>

$1,2,\dots,n \quad n+1-1,\dots$

(ii) Let $n\in\mathbb{N}$, then $|S\cap[1,n]|>\frac{n}{2}$ let $S_1=S\cap[1,n]$
$|S_1|>\frac{n}{2}$
and let $S_2=\{n+1-m,\ \underset{m<n}{m\in S_1}\}$ then $|S_2|\geq\frac{n}{2}$

Since $S_1, S_2\subseteq[1,n]$, $|S_1|+|S_2|>n$, so

$\Rightarrow$ $S_1\cap S_2\neq\phi$ $\Rightarrow$ $\exists\ s_1,s_2\in S$
$s_1=n-s_2$



(i) Let $S\cap[1,n]=\{a_1,\dots a_m\}$ set $a_0=0$ and $a_{m+1}=n$
if $a_m<n$.

$|(S+S)\cap[a_i,a_{i+1})|\geq\#\{b\in S;\ b\leq a_{i+1}-a_i-1\}\geq\alpha(a_{i+1}-a_i-1)$

ANDE II

$$\Rightarrow |(S+S)\cap[1,n]| \geq \sum_{i=0}^{m-1} \alpha(a_{i+1}-a_i-1) + m + \alpha(n-a_m-1)$$

$$\geq \alpha n - \alpha m + m = \alpha n + (1-\alpha)\alpha n = [\alpha + (1-\alpha)\alpha]n$$

i.e. $|a_i + S \cap (a_i, a_{i+1})| \geq \alpha |(a_i, a_{i+1})|$ $\square$

$$\Rightarrow \sum_i |(a_i+s)\cap(a_i,a_{i+1})| \leq (S+S)\cap([1,N]\setminus S)$$

$$\Rightarrow (S+S)\cap[1,n] \geq \alpha n + \alpha(1-\alpha)n = (2\alpha-\alpha^2)n$$

$$= 1-(1-\alpha)^2 n \quad \square$$

Cor 2.4. If $\sigma(S) \geq \alpha$, then $\sigma(\underbrace{S+\dots+S}_{l}) \geq$

$\geq 1-(1-\alpha)^l$, for $l=2^m$.

Pf Induction on m, $m=1$ ✓ $\square$

Note The proof that $G(k)$ (and $g(k)$) exists works similarly for all k.

<u>Cor 2.2</u> $\exists\ \delta>0$ s.t. $\#\{n\le N;\ r_6(n)\geq 1\}\geq \delta N$.

<u>Pf:</u> Let $P \doteq N^{1/2}$.

$$\sum_{n\le 6N} r_6(n) \geq \sum_{n\le N} \#\{0\le m_1,\dots,m_6\le P;\ m_1^2+\dots+m_6^2=n\} = P^6 \doteq N$$

LHS:
$$\overset{\boxed{\le CN^2} = CP^4}{\le \left(\max_{n\le 6N} r_6(N)\right)} \#\{n\le N;\ r_6(n)\geq 1\}$$

$$\Longrightarrow \quad \#\{n\le N;\ r_6(n)\geq 1\} \geq \frac{N^3}{CN^2} = \delta N \qquad (*)$$

Write $S_2=\{n^2;\ n\geq 0\}$. Then $r_6(n)\geq 1 \Leftrightarrow n\in \underbrace{S_2+\dots+S_2}_{6}$

Thus writing $S:=S_2+\dots+S_2$ we have

<u>Cor 2.3</u> For all $N\geq 1$; $\left|\frac{\# S\cap[1,N]}{N}\right|\geq \delta$.

<u>Def</u> $\sigma(S)=\inf \frac{S\cap[1,N]}{N}$ Schnirelmann-density.