Singular solutions: $y_1 \equiv y_2 \equiv y_7 \equiv y_8 \pmod p$

so $y_i = \beta + p u_i$ and let $S_i' = u_1^i + u_2^i - u_7^i - u_8^i$ $(i = 1,2,3)$

Then, we have: $2\gamma S_1 + p^a S_2 = 2(\gamma + \beta p^a)pS_1' + p^{a+2} S_2'$ (1)'

as $S_1 = pS_1'$, $S_2 = 2\beta p S_1' + p^2 S_2'$, $S_3 = 3\beta^2 p S_1' + 3\beta p^2 S_2' + p^3 S_3'$

thus $3\gamma S_2 + 2p^a S_3 = 6\beta(\gamma + \beta p^a)pS_1' + 3(\gamma + 2\beta p^a)p^2 S_2' + 2p^{a+3} S_3'$.

(2)'

Writing $\gamma' = \gamma + \beta p^a$ ($\equiv \gamma \not\equiv 0 \pmod p$).

Eliminating S_1' from (2)', as before we get

$$2\gamma' S_1' + p^{a+2} S_2' \equiv 0 \pmod{p^{c-1}} \qquad (3')$$

$$3\gamma' S_2' + 2p^{a+1} S_3' \equiv 0 \pmod{p^{c-2}} \qquad (4')$$

First, replace (3') with the same equation $\pmod{p^{c-2}}$

i.e. $2\gamma' S_1' + p^{a+2} S_2' \equiv 0 \pmod{p^{c-2}}$ (3'')

We'd like to count solutions $y_1, \dots, y_7 \pmod{p^c}$ so $u_1, u_2, u_7, u_8 \pmod{p^{c-1}}$; for that by induction we have that $\#\{v_1, v_2, v_7, v_8$ sol. to (3''), (4') $\pmod{p^{c-2}}\}$

$$\le (c-1)p^{2c-4}$$

Then $\#\{u_1, u_2, u_7, u_8$ sol. to (3'), (4') $\pmod{p^{c-1}}\} \le (c-1)p^{2c-1}$

There are choices for $\beta \pmod p$ $\Longrightarrow$

$\#\{y_1, y_2, y_3, y_4$ sol. to (1), (2) $\pmod{p^c}$, with $y_1 \equiv y_2 \equiv y_3 \equiv y_4 \pmod p\}$

$$\leq (c-1)p^{2c}.$$

Thus $N(p; a, c) \leq 2p^{2c} + (c-1)p^{2c} = (c+1)p^{2c}$. $\square$

<u>Note</u> This finishes the proof of

<u>Lemma 3.5</u> $\quad I_2(X; a, b) \leq 2b\, p^{4(b-a)}\, I_2(X; 2b-a, b)$

<u>Prop 1.</u> If $1 \leq a \leq b$, $p^b \leq X$, then

$$I_2(X; a, b) \leq 2b\, p^{-\frac{10a}{3} + \frac{16b}{3}}\, I_2(X; b, 2b-a)^{\frac{1}{3}}\, I_2(X; b, 3b)^{\frac{1}{6}}\, J(2X/p^b)^{\frac{1}{2}} \tag{5}$$

<u>Pf:</u> We apply Lemmas 1–4,

$$I_2(X; a, b) \leq 2b\, p^{4(b-a)} I_2(X; 2b-a, b) \leq 2b\, p^{4(b-a)} I_2(X; b, 2b-a)^{\frac{1}{3}}\, I_1(X; 2b-a, b)^{\frac{2}{3}}$$

$$\leq 2b\, p^{4(b-a)} I_2(X; 2b-a, b)^{\frac{1}{3}} \{p^{3b-(2b-a)}\, I_1(X; 3b, b)\}^{\frac{2}{3}}$$

$$\leq 2b\, p^{4(b-a)+2(b+a)/3}\, I_2(X; 2b-a, b)^{\frac{1}{3}}\, I_2(X; b, 3b)^{\frac{1}{6}}\, J(2X/p^b)^{\frac{1}{2}}$$

, which is valid as $2b-a \leq 3b$.

$\square$

Note, that $X^6 \leq \mathcal{J}(X) \leq X^{12}$ and again, define

$$\delta_* = \inf \{ \delta > 0 ;\ \mathcal{J}(X) \ll X^{6+\delta} \text{ for } X \geq 1 \}.$$

Then, we have $\mathcal{J}(X) \ll_\varepsilon X^{6+\delta_*+\varepsilon}$ $\forall \varepsilon > 0$.

Also clearly: $I_2(X;a,b) \leq \mathcal{J}(X) \ll_\varepsilon X^{6+\delta_*+\varepsilon}$ $(\forall \varepsilon > 0)$.

If $1 \leq a \leq b$ then $3(3b-a) \geq 2a+2b$ $(\Leftrightarrow 5b \geq 5a)$

thus
$$I_2(X;a,b) \ll_\varepsilon X^{6+\delta_*+\varepsilon} p^{-2a-4b} p^{3(3b-a)} \tag{6}$$

<u>Note</u> This looks "strange", but it is the "starting point" of the following "crucial" estimate

<u>Proposition 2.</u> Let $n \in \mathbb{N}$, $1 \leq a \leq b$ and $p^{3^n b} \leq X$.

Then
$$I_2(X;a,b) \ll_{\varepsilon,n,a,b} X^{6+\delta^*+\varepsilon} p^{-2a-4b} p^{(3-n\delta^*/6)(3b-a)} \tag{7}$$

<u>Proof</u> (induction on n)

- $n=0$, by (6) ✓
- $n \to n+1$: A plug (7) into (5) and after a "brutal calculation" we get

ANDE XUN

(4)

$$I_2(X; a, 2b-a) \ll X^{6+\delta^* + \varepsilon} p^{-2b-4(2b-a)} p^{(3-n\delta_+/6)(3(2b-a)-b)}$$

$$= X^{6+\delta^*+\varepsilon} p^{4a-10b} p^{(3-n\delta_+/6)(5b-3a)}$$

(here we have $p^{3^n(2b-a)} \le p^{3^{n+1}b} \le X$). Also

$$I_2(X; b, 3b) \ll X^{6+\delta^*+\varepsilon} p^{-2b-12b} p^{(3-n\delta_+/6)(9b-b)}$$

$$= X^{6+\delta^*+\varepsilon} p^{-14b} p^{(3-n\delta_+/6)\,8b}$$

Finally, $\mathcal{J}(2X/p^b) \ll_\varepsilon X^{6+\delta_++\varepsilon} p^{-6b-\delta_+ b}$

By Prop. 1,

$$I_2(X; a, b) \ll_{\varepsilon, n, a, b} X^{6+\Delta+\varepsilon} p^{-2a-4b} p^{(3-(n+1)\delta^*/6)(3b-a)}$$

$\square$

<u>Note</u> This "magicly worked out" but seems unmotivated;

Indeed we have:

$$p^{(3-n\delta_+/6)[(5b-3a)\cdot\frac{1}{3} + 8b\cdot\frac{1}{6}] - \delta_+ b/2]} = p^{(3-n\delta_+/6)[3b-a] - \delta_+ 3b/6)}$$

$$\leq p^{3-(n+1)\delta_+/6\,(3b-a)}$$

<u>Thm 3.1</u> $\mathcal{J}(X) = J_{3,6}(X) \ll_\varepsilon X^{6+\varepsilon}$, $\forall \varepsilon > 0$. (8)

<u>Pf</u> Assume, indirectly, that $\delta_+ > 0$.

Let $n \in \mathbb{N}$, s.t. $n\delta_2 \geq 30$, assume X is suff. large

i.e. $X \geq 10^{3^n}$ and choose $p > 3$ s.t. $\frac{1}{2} X^{\frac{1}{3^n}} \leq p \leq X^{\frac{1}{3^n}}$.

Using Lemma 3.1 and Prop. 2, we have

$$
\begin{aligned}
J(X) &\ll p\, J(2X/p) + p^{12} I_2(X;1,1) \\
&\ll_\varepsilon p\,(2X/p)^{6+\delta_n+\varepsilon} + p^{12} X^{6+\delta_n+\varepsilon} p^{-6} p^{(6-n\delta^*/3)} \\
&\ll_\varepsilon X^{6+\delta_n+\varepsilon}\left(p^{-5} + p^{12-n\delta^*/3}\right) \ll_\varepsilon X^{6+\delta^*+\varepsilon} p^{-1} \\
&\ll_\varepsilon X^{6+\delta^*-3^{-n}+\varepsilon} \ll X^{6+\delta^*-3^{-(n+1)}}
\end{aligned}
$$

↯ definition δ^*

∎

Remark. It was not at all transparent how to use Prop1. to get Prop 2. and Thm 3.1.

The point is, if we have $J(X) \ll X^{\theta+\varepsilon}$

and $I_2(X;a,b) \ll X^{\theta} p^{\alpha a + \beta b}$ then

Prop1. gives $I_2(X;a,b) \ll X^{\theta} p^{\alpha' a + \beta' b}$

with $\alpha' = -\frac{10}{3} - \frac{1}{3}\beta$, $\beta' = \frac{14}{3} - \frac{\theta}{2} + \frac{1}{2}\alpha + \frac{7}{6}\beta$