Efficient congruencing: Model Case $k=2$.

Want to prove: $J(X) = J_{3,2}(X) \ll_\varepsilon X^{3+\varepsilon}$.

We have reduced this to proving the lemmas

Lemma 0. If $a, b \geq 1$, $p^b \leq X$, then $I_0(X; a, b) \leq J(2X/p^b)$. (15.0)

Lemma 1. If $p \leq X$, then

$$J(X) \ll p\, J(2X/p) + p^6 I_1(X; 1, 1) \qquad (15.1)$$

Lemma 2. If $1 \leq a \leq 2b$, then

$$I_1(X; a, b) \leq p^{2b-a} I_1(X; 2b, b) \qquad (5.2)$$

Lemma 3. If $1 \leq a \leq 2b$, then

$$I_1(X; a, b) \leq p^{2b-a} I_1(X; b, 2b)^{\frac{1}{2}} J(2X/p^b)^{\frac{1}{2}} \qquad (5.3)$$

Idea: Apply Lemma 2 - Lemma 3 recursively, to compare $J(X)$ to $I(X; b, 2b)$ for large b.

Pf. Lemma 2.0. (Works for general k and s).

$$I_0(X;a,b) = I_0(X;b) = \max_{\eta \pmod{p^b}} \#\{1 \le x_1, \dots, x_{2s} ;\ \text{sol. to } \boxed{V} \text{ s.t. } x_i \equiv \eta \pmod{p^b}\ \forall 1 \le i \le 2s\}$$

Writing $x_i = p^b y_i + \eta$; $1 \le y_i \le 2X/p^b$, we get $y_1, \dots, y_s$ is a sol. of $\boxed{V}$ $\Rightarrow$ $I_0(X;a,b) \le J(2X/p^b)$

Pf. Lemma 2.1 Consider a solution to $\boxed{V}$ for $k=2, s=3$

i.e. to
$$x_1 + x_2 + x_3 = x_4 + x_5 + x_6$$
$$x_1^2 + x_2^2 + x_3^2 = x_4^2 + x_5^2 + x_6^2$$

such that;

I $x_1 \equiv \dots \equiv x_6 \equiv \eta \pmod{p}$.

By Lemma 0, the # of such solutions $\le p\, J(2X/p)$

II. $\exists\, i,j$: $x_i \equiv \xi \pmod{p}$, $x_j \equiv \eta \pmod{p}$; $\xi \not\equiv \eta \pmod{p}$.

Recall
$$S_1(\alpha, \xi) = \sum_{\substack{1 \le x \le X \\ x \equiv \xi\ (p)}} e(\alpha_1 x + \alpha_2 x^2)$$

Thus the number of such solutions are bdd by.

$$\binom{6}{2} \sum_{\substack{\xi \neq \eta \\ (\mathrm{mod}\, p)}} \underbrace{\int_{[0,1]^2} |S_1(\underline{\alpha},\xi)|\,|S_1(\underline{\alpha},\eta)|\,|S(\underline{\alpha})|^4 \, d\alpha}_{J_2(\xi,\eta)} \qquad (15.5)$$

For fixed ξ, η apply Hölder's ineq. to get : $\frac{1}{6}+\frac{1}{6}+\frac{2}{3}=1$

$$|J_2(\xi,\eta)| \leq \left(\int |S_1(\underline{\alpha},\xi)|^2 |S_1(\alpha,\eta)|^4 d\underline{\alpha}\right)^{\frac{1}{6}} \left(\int |S_1(\underline{\alpha},\xi)|^4 |S_1(\alpha,\eta)|^2 d\underline{\alpha}\right)^{\frac{1}{6}}$$

$$\times \left(\int |S(\alpha)|^6 d\alpha\right)^{2/3}$$

Using $\int fgh \leq \left(\int |f|^6\right)^{1/6} \left(\int |g|^6\right)^{1/6} \left(\int |h|^{3/2}\right)^{2/3}$ as $\frac{1}{6}+\frac{1}{6}+\frac{2}{3}=1$

with $f = |S_1(\alpha,\xi)|^{\frac{1}{3}} |S_1(\alpha,\eta)|^{2/3}$, $g = |S_1(\alpha,\xi)|^{2/3} |S_1(\underline{\alpha},\eta)|^{1/3}$

$h = |S(\underline{\alpha})|^4$

Thus

$$|J_2(\xi,\eta)| \leq I_1(X;1,1)^{1/3} J(X)^{2/3}$$

$$\Rightarrow J(X) = J_1(X) + J_2(X) \qquad (15.6)$$

$$\ll p^{-} J(2X/p) + p^2 I_1(X,1,1)^{1/3} J(X)^{2/3}$$

If $p\,J(2X/p)$ dominates the R.H.S. then (15.1) holds.

otherwise

$$\therefore\ J(X) \ll p\, I_1(X; 1, 1)^{1/3} J(X)^{2/3}$$

$$\Rightarrow\ J(X) \ll p^6 I_1(X; 1, 1)$$

Pf. Lemma 2.

Recall that $I_1(X; \xi, \eta, a, b)$ is the # of solutions $x_1, \ldots, x_6$ to V, satisfying

$$x_1 = \xi + p^a y_1, \quad x_4 = \xi - p^a y_2$$

$$x_i = \eta + p^b y_i \quad \text{for } i = 2, 3, 5, 6. \qquad (15.7)$$

by translation invariance, writing $\nu = \xi - \eta$ ($\not\equiv 0 \pmod p$)

(15.7) is equiv. to:

$$z_i = \nu + p^a y_i \quad (i = 1, 4)$$

$$z_i = p^b y_i \quad (i = 2, 3, 5, 6) \qquad (15.8)$$

but then

$$z_1 \equiv z_4 \pmod{p^b}, \ \& \ z_1^2 \equiv z_4^2 \pmod{p^{2b}}$$

$$\Rightarrow (\nu + p^a y_1)^2 \equiv (\nu + p^a y_4)^2 \pmod{p^{2b}}$$

as $p > 2$ $\Rightarrow$

$$\nu + p^a y_1 \equiv \nu + p^a y_4 \pmod{p^{2b}}$$

$$\Rightarrow\ y_1 \equiv y_4 \pmod{p^{2b-a}}$$

Suppose y_4 is fixed $(\mathrm{mod}\ p^{2b-a}) \Rightarrow y_1$ is fixed $(\mathrm{mod}\ p^{2b-a})$

$\Rightarrow$ x_1, x_4 fixed $(\mathrm{mod}\ p^{2b})$ ie $x_1 \equiv x_4 \equiv \bar{\xi}\ (\mathrm{mod}\ p^{2b})$

$$\Rightarrow I_1(X; \xi, \eta, a, b) \leq p^{2b-a}\, I_1(X; 2b, b)$$

(as there are p^{2b-a} choices for y_1 $(\mathrm{mod}\ p^{2b-a})$.

Pf. Lemma 3. We have

$$I_1(X; \xi, \eta, 2b, b) = \int_{[0,1]^2} |S_{2b}(\underline{\alpha}; \xi)|^2 |S_b(\underline{\alpha}; \xi)|^4 d\underline{\alpha}$$

$|S_b| \cdot |S_b|^3$

$$\leq \left(\int_{[0,1]^2} |S_{2b}(\underline{\alpha}; \xi)|^4 |S_b(\underline{\alpha}; \xi)|^2 d\underline{\alpha}\right)^{\frac{1}{2}} \left(\int |S_b(\underline{\alpha}; \eta)|^6\right)^{\frac{1}{2}}$$

$$= I_1(X; \eta, \xi, b, 2b)^{\frac{1}{2}}\, I_0(X; \eta, \underline{b})^{\frac{1}{2}}$$

$$\leq I_1(X; b, 2b)^{\frac{1}{2}}\, J(2X/p^b)^{\frac{1}{2}}$$

Thus by Lemma 2,

$$I_1(X; a, b) \ll p^{2b-a}\, I_1(X; 2b, b) \ll p^{2b-a}\, I_1(X; b, 2b)^{\frac{1}{2}}\, J(2X/p^b)^{\frac{1}{2}}$$

Efficient congruencing for $k=3$

Thm $J_{6,3}(X) \ll_\varepsilon X^{6+\varepsilon}$, for all $X \geq 1$, $\varepsilon > 0$.

Note
- This is a highly non-triv. result, eventually proved by Wooley in 2014?
- Write $J(X)$ for $J_{6,3}(X)$
- We follow Heath-Brown's approach, [HB] which simplifies Wooley's proof.

Lemma 3.0. For any $a, b \geq 1$ if $p^b \leq X$, then

$$I_0(X; a, b) \leq J(2X/p^b) \qquad (16.0)$$

Note This lemma shows that counting with congruence conditions can be reduced to counting *without* congr. cond on a smaller range.

Proof (this is Lemm

Lemma 3.6 If $p \le X$, then $J(X) \ll p\, J(2X/p) + p^{12} I_2(X; 1, 1)$ (16.1)

Note This is used to introduce congruence conditions

Proof Is essentially the same as the proof of Lemma 2.1.

We have, for some $\xi \neq \eta$

$$J(X) \le p\, J(2X/p) + \binom{12}{2} p(p-1) \int_{[0,1]^3} |S_1(\underline{\alpha}, \xi)\, S_1(\underline{\alpha}, \eta)\, S(\underline{\alpha})^{10}|\, d\underline{\alpha}$$

Using Hölder with $\frac{1}{12} + \frac{1}{12} + \frac{5}{6} = 1$, we get

$$J(X) \ll p\, J(2X/p) + p^2 I_2(X; 1, 1)^{\frac{1}{12}} \cdot I_2(X; 1, 1)^{\frac{1}{12}} J(X)^{\frac{5}{6}} \quad (16.2)$$

as

$$I_2(X; 1, 1) = \max_{\xi \not\equiv \eta \ (\mathrm{mod}\ p)} \int_{[0,1)^3} |S_1(\underline{\alpha}, \xi)|^4\, |S_1(\underline{\alpha}, \eta)|^2\, d\underline{\alpha}$$

As before, this implies

$$J(X) \ll_p J(2X/p) + p^{12} I_2(X; 1, 1)$$

$\square$

We'll get recursive estimates by comparing $I_1(X;a,b)$ and $I_2(X;a,b)$ different ways.

Lemma 3.2 $\quad I_2(X;a,b) \leq I_2(X;b,a)^{1/3}\, I_1(X;a,b)^{2/3}$ (16.3)

Lemma 3.3 $\quad I_1(X;a,b) \leq p^{3b-a}\, I_2(X;b,a)^{1/4}\, J(2X/p)^{3/4}$

Pf of Lemma 3.2. $\quad \exists\, \xi \neq \eta$ s.t.

$$I_2(X;a,b) = \int |S_a(\alpha,\xi)|^4\, |S_b(\alpha,\eta)|^8\, d\alpha$$

Using Hölder with $\frac{1}{3}+\frac{2}{3}$ with $f = |S_a(\,,\,)|^{8/3} \cdot |S_b(\,,\,)|^{4/3}$

$g = |S_a(\,,\,)|^{4/3} \cdot |S_b(\,,\,)|^{20/3}$

$$\Rightarrow |I_2(X;a,b)| \leq \left(\int |S_a(\alpha,\xi)|^8 |S_b(\alpha,\eta)|^4\, d\alpha\right)^{1/3} \times$$

$$\times \left(\int |S_a(\alpha,\xi)|^2 |S_b(\alpha,\eta)|^{10}\, d\alpha\right)^{2/3}$$

$$= I_2(X;b,a)^{1/3}\, I_1(X;a,b)^{2/3}$$

$\square$