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$$S_k(\underline{\alpha}, X) = \sum_{x \leq X}{}' e(\alpha_k x^k + \dots + \alpha_1 x) \quad ; \quad \underline{\alpha} = (\alpha_1, \dots, \alpha_k)$$

$$J_{k,s}(X) = \int_{[0,1]^k} |S_k(\underline{\alpha}, X)|^{2s} \, d\underline{\alpha}$$

Main Conjecture

$$J_{k,s}(X) \leq C_{k,s,\varepsilon}\left(X^s + X^{2s - \frac{k(k+1)}{2}}\right) X^\varepsilon \; ; \; \forall \varepsilon > 0. \quad (1)$$

critical exponent: $s_k = \frac{k(k+1)}{2}$; then conjecture is

$\forall \varepsilon > 0 \; \exists C_{k,\varepsilon}$: $$J_{k,s_k}(X) \leq C_{k,\varepsilon} X^{s_k + \varepsilon} \quad (2)$$

Note (2) $\Rightarrow$ (1).

For $s > s_k$ this follows from $|S_k(\underline{\alpha}, X)|^{2s} \leq |S_k(\underline{\alpha}, X)|^{2s_k} X^{2s - 2s_k}$

For $s < s_k$ one may apply Hölder with $p = \frac{s_k}{s}$;

$$J_{k,s}(X) = \int_{[0,1]^k} |S_k(\underline{\alpha}, X)|^{2s} \cdot 1 \, d\alpha \leq \left(\int_{[0,1]^k} |S_k(\underline{\alpha}, X)|^{2sp} d\underline{\alpha}\right)^{\frac{1}{p}} \left(\int_{[0,1]^k} 1^{p'} d\underline{\alpha}\right)^{\frac{1}{p'}}$$

$$= \left(\int_{[0,1]^k} |S_k(\underline{\alpha}, X)|^{2s_k} d\underline{\alpha}\right)^{\frac{1}{p}} = J_{k,s_k}(X)^{\frac{s}{s_k}} \leq C_{k,\varepsilon} X^{s_k \cdot \frac{s}{s_k} + \varepsilon}$$

$$= C_{k,\varepsilon} X^{s+\varepsilon}$$

Relation to Waring's problem

Consider $x_1^k + \cdots + x_s^k = N$, $X = N^{1/k}$

$$R_{s,k}(N) = \int_0^1 e(-\alpha N)\, T_k(\alpha; X)^s\, d\alpha \quad ; \quad T_k(\alpha;X) = \sum_{1\le x\le X} e(\alpha x^k)$$

we have

$$R_{s,k}(N) = \underbrace{\int_{\mathfrak{M}} e(-\alpha N)\, T_k(\alpha, X)^s\, d\alpha}_{\text{major arcs}} + \underbrace{\int_{\mathfrak{m}} e(-\alpha N)\, T_k(\alpha; X)^s\, d\alpha}_{\text{minor arcs}}$$

$$= I_{s,k}\, \sigma_{s,k}(N)\, N^{\frac{s}{k}-1} + \mathcal{E}_{s,k}(N)$$

Want to show that the contribution of the

minor arcs

$$|\mathcal{E}_{s,k}(N)| \ll N^{\frac{s}{k}-1-\delta} = X^{s-k-\delta'} \quad \text{for some } \delta'>0$$

Suppose $S = 2s_1+1$ and (α) holds for $s_1 \geq s_k^* = \frac{1}{2}k(k+1)$

Then
$$\int_0^1 |T_k(\alpha, X)|^{2s_1}\, d\alpha = \#\{x_1^k + \cdots + x_{s_1}^k = y_1^k + \cdots + y_{s_1}^k \ ;\ 1 \le x_i, y_i \le X\}$$

$$= \sum_{|h_1|\le s_1 X} \cdots \sum_{|h_{k-1}|\le s_1 X^{k-1}} \#\left\{ \begin{array}{l} x_1^j + \cdots + x_{s_1}^j - y_1^j - \cdots - y_{s_1}^j = h_j,\ 1\le j\le k-1 \\ x_1^k + \cdots + x_{s_1}^k = y_1^k + \cdots + y_{s_1}^k \end{array} \right\}$$

$$= \sum_{|h_1|\le s_1 X} \cdots \sum_{|h_{k-1}|\le s_1 X^{k-1}} \int_{[0,1]^k} |S_k(\underline{\alpha}, X)|^{2s_1}\, e(-h_1\alpha_1 - \cdots - h_{k-1}\alpha_{k-1})\, d\underline{\alpha}$$

Thus
$$\int_0^1 |T_k(\alpha, X)|^{2s_1} d\alpha \leq c_{s,k} X^{\frac{1}{2}k(k-1)} J_{k,s}(X)$$
$$\leq c_{s,k,\varepsilon} X^{2s_1+\varepsilon} X^{2s-\frac{1}{2}k(k+1)+\frac{1}{2}k(k-1)}$$
$$\leq C_{s,k,\varepsilon} X^{2s_1-k+\varepsilon} \qquad (\forall \varepsilon > 0)$$

Using the Weyl estimate on the minor arcs;

for $\alpha \in \mathfrak{m}$: $\quad |T_k(\alpha, X)| \leq C_k X^{1-\delta}$, for some $\delta = \delta_k > 0$

on have that, with $\varepsilon = \frac{1}{2}\delta$:

$$\int_{\mathfrak{m}} |T_k(\alpha; X)|^{2s_1+1} d\alpha \ll X^{1-\delta} X^{2s_1-k+\varepsilon} \ll X^{2s_1+1-k-\delta/2}$$

So for $s = 2s_1+1$ one has

$$\int_{\mathfrak{m}} |T_k(\alpha; X)|^{s} d\alpha \ll X^{s-k-\delta'} = N^{\frac{s}{k}-1-\delta'} \quad (\text{for some } \delta' > 0).$$

In particular the Main Conjecture $\left(s_k = \frac{1}{2} k(k+1) \right)$

implies that for $s \geq 2s_k+1 = k(k+1)+1$; one has $(k \geq 3)$

$$R_{s,k}(N) = I_{s,k} \mathfrak{S}_{s,k}(N) N^{\frac{s}{k}-1} + O(N^{\frac{s}{k}-1-\delta})$$

i.e. there is an asymptotic formula for Waring's problem.

The easy case $k=2$:

$k=2$, $s_2 = \frac{2\cdot 3}{2} = 3$; want to count solutions

$$1 \le x_1, x_2, x_3, \; y_1, y_2, y_3 \le X$$

Such that

$$\left.\begin{aligned} x_1+x_2-y_3 &= y_1+y_2-x_3 \\ x_1^2+x_2^2-y_3^2 &= y_1^2+y_2^2-x_3^2 \end{aligned}\right\} \quad (3)$$

Using $(a+b-c)^2-(a^2+b^2-c^2) = 2(a-c)(b-c)$,

we get $(x_1-y_3)(x_2-y_3) = (y_1-x_3)(y_2-x_3) = m$

If $m \neq 0$, then ~~for~~ fixed x_1, x_2, y_3 and hence $m \ll X^2$

we have $d(X^2) \le C_\varepsilon X^\varepsilon$ (or $\ll X^\varepsilon$) choices

for $u_1 = y_1 - x_3$, $u_2 = y_2 - x_3$; and for fixed u, v

we have $x_1+x_2-y_3 = (y_1-x_3)+(y_2-x_3)+x_3 = u_1+u_2+x_3$

which determines x_3 and then $y_1 = u_1+x_3$, $y_2 = u_2+x_3$

$\Rightarrow$ there are at most $C_\varepsilon X^{3+\varepsilon}$ solutions to (3)

Note It is proved that $J_{2,3}(X) \sim \frac{18}{\pi^2} X^3 \log X$

So the "X^ε"-type of term cannot be omitted from the Main Conjecture!

Note: If the main conjecture is verified for

$s_k = \frac{1}{2}k(k+1)$, then an asymptotic can be derived

for $J_{k,s}(X)$, for $s > s_k$.

This similar to our treatment of Waring's problem,

writing

$$J_{k,s}(X) = \int_{[0,1]^k} |S(\underline{\alpha})|^{2s} d\alpha = \int_{\mathfrak{M}} |S(\underline{\alpha})|^{2s} d\underline{\alpha} + \int_{\mathfrak{m}} |S(\underline{\alpha})|^{2s} d\underline{\alpha}$$

with $S(\underline{\alpha}) = S_k(\underline{\alpha}, X) = \sum_{1 \le x \le X} e(\alpha_1 x + \cdots + \alpha_k x^k)$,

$\mathfrak{M}$ = family of "major arcs", $\mathfrak{m}$ = family of "minor arcs"

On the minor arcs; one uses a Weyl-type bound

$|S(\underline{\alpha})| \ll X^{1-\delta_k}$ and the mean value estimate

$$\int_{\mathfrak{m}} |S(\underline{\alpha})|^{2s_k} |S(\alpha)|^{2(s-s_k)} d\underline{\alpha} \ll X^{2(s-s_k)-\delta'} X^{s_k+\varepsilon}$$

$$\ll X^{2s - s_k - \delta'/2}$$

While we expect the main term:

$$J_{k,s}(X) \sim X^{2s - s_k} = X^{2s - \frac{1}{2}k(k+1)} \qquad (\text{why?})$$

Indeed, for any $|h_1| \ll X, \ldots, |h_k| \ll X^k$ we expect that the number 2s-tuples $1 \le x_1, \ldots, x_s, y_1, \ldots, y_s \le X$

s.t.
$$x_1 + \cdots + x_s - (y_1 + \cdots + y_s) = h_1$$
$$\vdots$$
$$x_1^j + \cdots + x_s^j - (y_1^j + \cdots + y_s^j) = h_j$$
$$\vdots$$
$$x_1^k + \cdots + x_s^k - (y_1^k + \cdots + y_s^k) = h_k$$

is about $$X^{2s} / X^{1+2+\cdots+k} = X^{2s - \frac{1}{2}k(k+1)} = X^{2s - s_k}.$$

On the major arcs $\left(\underline{\alpha} = (\alpha_1, \ldots, \alpha_k);\ |\alpha_1 - \frac{a_1}{q}| \le N^{-1+\delta}, \ldots, |\alpha_k - \frac{a_k}{q}| \le N^{-k+\delta} \text{ for some } q \le N^{\delta};\ (a_1, \ldots, a_k, q) = 1 \right)$

one can obtain an asympt formula:

$$\int_{\mathcal{M}} |S(\underline{\alpha})|^{2s} d\underline{\alpha} = \mathfrak{S}(s,k)\, I(s,k)\, X^{2s - \frac{1}{2}k(k+1)}$$

with $$\mathfrak{S}(s,k) = \sum_{q=1}^{\infty} \sum_{\substack{a_1=1 \\ (a_1, \ldots, a_k, q)=1}}^{q} \cdots \sum_{a_k=1}^{q} \left| \frac{1}{q} \sum_{r=1}^{q} e\left(\frac{a_1 r + \cdots + a_k r^k}{q} \right) \right|^{2s}$$

and $$I(s,k) = \int_{\mathbb{R}^k} \left| \int_0^1 e(\beta_1 y + \cdots + \beta_k y^k)\, dy \right|^{2s} d\beta_1 \ldots d\beta_k$$

for the singular series & singular integral.

The efficient congruencing method

Again, we consider the system

$$\sum_{i=1}^{s} x_i^j = \sum_{i=1}^{s} y_i^j \quad \text{for } 1 \le j \le k \qquad (5)$$

then, given $\lambda \neq 0$, u, $x_1, \dots, x_s, y_1, \dots, y_s$ is a solution to (5) if and only if they are solutions to

$$\sum_{i=1}^{s} (\lambda x_i + u)^j = \sum_{i=1}^{s} (\lambda y_i + u)^j \quad \text{for } 1 \le j \le k \qquad (6)$$

Note this means that $\underline{z} = (\underline{x}, \underline{y})$ is a solution

$\iff \lambda \underline{z} + \underline{u} = (\lambda \underline{x} + \underline{u}, \lambda \underline{y} + \underline{u})$ is a solution

So the system is translation-dilation invariant

Indeed, trivially $(\underline{x}, \underline{y})$ is a solution $\iff (\lambda \underline{x}, \lambda \underline{y})$ is a solution

; moreover

$$\sum_{i=1}^{s} \left((x_i+u)^j - (y_i+u)^j \right) = \sum_{l=1}^{j} \binom{j}{l} u^{j-l} \sum_{i=1}^{s} \left(x_i^l - y_i^l \right)$$

and vice-versa $\left. x_i = x_i + u - u \right) \Rightarrow (\underline{x}+u, \underline{y}+u)$ is a solution

This implies the crucial observation: the number of solutions $x_1, \dots, x_s, x_{s+1}, \dots, x_{2s}$ to $\sum_{i=1}^{s} x_i^j = \sum_{i=1}^{s} x_{i+s}^j$, for $1 \le j \le k$

subject to $x_i \equiv u \pmod q$ $\forall\, 1 \le i \le 2s$

is the number of $z_1, z_2, \dots, z_{2s}$; $z_i \le X/q$

s.t. $\sum_{i=1}^{s} z_i^j = \sum_{i=1}^{s} z_{s+i}^j$, for $1 \le j \le k$

which is $J_{k,s}(X/q)$.

Initial set-up and notation

$$S(\underline{\alpha}) = \sum_{1 \le x \le X} e(\alpha_1 x + \alpha_2 x^2 + \dots + \alpha_k x^k)$$

Given $a, \xi \in \mathbb{N}$, $p \in \mathbb{P}$, write

$$S_a(\alpha, \xi) = \sum_{\substack{1 \le x \le X \\ x \equiv \xi \pmod{p^a}}} e(\alpha_1 x + \dots + \alpha_k x^k) \qquad (7)$$

We'll fix a prime $p > k$, but will vary ξ and a.
We'll count solutions to the Vinogradov system

$$x_1^j + \dots + x_s^j = x_{s+1}^j + \dots + x_{2s}^j, \quad 1 \le j \le k \qquad (8)$$

subject to the congruence conditions, namely

$$\left.\begin{array}{l} x_i \equiv \xi \pmod{p^a} \quad \text{for } 1 \le i \le m \text{ and } s+1 \le i \le s+m \\ x_i \equiv \eta \pmod{p^b} \quad \text{for } m+1 \le i \le s \text{ and } s+m+1 \le i \le 2s \end{array}\right\} (9)$$

which we'll denote by $I_m(X; \xi, \eta, a, b)$.

Note, that

$$I_m(X; \xi, \eta, a, b) = \int_{[0,1)^k} |S_a(\underline{\alpha}, \xi)|^{2m} \, |S_b(\underline{\alpha}, \eta)|^{2(s-m)} d\underline{\alpha} \quad (10)$$

Indeed, the RHS of (10) is

$$\int_{[0,1]^k} \Big| \sum_{\substack{x \le X \\ x \equiv \xi (p^a)}} e\Big(\sum_{j=1}^{k} \alpha_j x^j\Big)\Big|^{2m} \Big| \sum_{\substack{x \le X \\ x \equiv \eta (p^b)}} e\Big(\sum \alpha_j x^j\Big)\Big|^{2m-2s} d\underline{\alpha}$$

$$= \int_{[0,1]^k} \sum_{\substack{x_1, \dots, x_m \\ x_{s+1}, \dots x_{s+m} \equiv \xi (p^a)}} \sum_{\substack{x_{m+1}, \dots, x_s \\ x_{m+s+1}, \dots, x_{2s} \equiv \eta (p^b)}} e\Big(\sum_{j=1}^{k} x_1^j + \dots + x_m^j + x_{m+1}^j + \dots + x_s^j - x_{m+1}^j \dots - x_{2s}^j\Big)$$

which is the number of solutions to (8)-(9).