Math 8440: Analytic methods for Dioph. equations

Additive number theory: representations of a large no as sum of numbers of a spec. type

Waring problem: $N = x_1^k + \dots + x_s^k$ "sum of s kth powers" s and k fixed

Goldbach ternary: $N = p_1 + p_2 + p_3$

Golbach-Waring $N = p_1^k + p_2^k + \dots + p_s^k$

Note If $k=2$, $s=2,4,8$ there are exact formulas

i.e. for $r_2(N) = \#\{x_1, x_2 \in \mathbb{Z};\ x_1^2 + x_2^2 = N\}$

$r_4(N) = \#\{x_1, x_2, \dots, x_4 \in \mathbb{Z};\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = N\}$

If $k \neq 2^l$, in general there are no exact formula but there are asympt. fomula for $s \geq s(k)$ due to Hardy-Littlewood and Vingradov and Wooley $\rightarrow$ "Hardy-Littlewood method" (Ramanujan)

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A bit more general is the eq:

$$a_1 x_1^k + \cdots + a_s x_s^k = N \qquad (a_1, \ldots, a_s \text{ coeff's in } \mathbb{Z})$$

but a general system of dioph. eq's is of the form

$$\left.\begin{array}{l} P_1(x_1, \ldots, x_n) = N_1 \\ \vdots \\ P_r(x_1, \ldots, x_n) = N_r \end{array}\right\} \quad \begin{array}{l} P_1, \ldots, P_r \in \mathbb{Z}[x_1, \ldots, x_n] \\ d = \max \deg P_i \end{array}$$

If $n \geq n(r,d)$ and the system $P_1, \ldots, P_r$ is suff. non-singular then there are asympt. formulae for the # of int solutions.

<u>Waring problem</u> (1770) Every N can be written as the sum of 4 squares, 9 cubes, 19 biquadrates

in general $\forall k \; \exists s$ st. $\forall N \in \mathbb{N}$ can be written as $N = x_1^k + \cdots + x_s^k$ $(x_1, \ldots, x_s \in \mathbb{N})$

Write $g(k)$ to be the smallest such s.

<u>Hilbert</u>: (1909) Waring conj. is true i.e. $\exists\, g(k)$

Note. let $N = 2^k - 1$ then $N = \underbrace{1^k + \cdots + 1^k}_{2^k - 1}$ is the min. repr. as sum of k^{th} powes

$\Longrightarrow g(k) \geq 2^k - 1$

Hardy-Littlewood defines $G(k) = \{\min s;$ such that every suff. large number N can be written as $N = x_1^k + \cdots + x_s^k\}$.

The point is that

- (H-L) $\quad G(k) \leq (k-2)2^{k-1} + 5 \qquad (1)$
- (Hua) $\quad G(k) \leq 2^k + 1 \qquad (2)$
- (Vinogradov) $\quad G(k) \leq (3+\varepsilon) k \log k \qquad (k \geq k(\varepsilon)) \quad (3)$
- (Wooley) $\quad G(k) \leq k^2 + 1$
- $G(k) \geq k$; as If $x_1^k + \cdots + x_s^k \leq N$ then $x_i \leq N^{\frac{1}{k}}$

thus $\# \{x_1^k + \cdots + x_s^k ;\ 1 \leq x_i \leq N^{1/k}\} \leq N^{s/k} \leq N^{1-\frac{1}{k}}$

is $s \leq k-1 \Longrightarrow \exists \ \frac{N}{2} \leq M \leq N$ s.t. $M \neq x_1^k + \cdots + x_s^k$ $\forall x_1, \ldots, x_s \in \mathbb{N}$.

The methods to obtain (1) – (4) (except (3)) gives asymptotically:

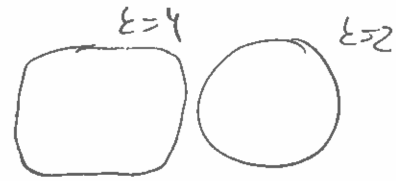
$$r(N) = r_{s,k}(N) = \# \{x_1, \ldots, x_s \in \mathbb{N};\ x_1^k + \cdots + x_s^k = N\} \quad (1.5)$$

In fact:

$$r(N) = \Omega_{k,s} N^{\frac{s}{k}-1} \mathfrak{S}(N) + O\left(N^{\frac{s}{k}-1-\delta}\right)$$

where $\delta > 0$ and $\Omega_{k,s} = \dfrac{\Gamma(1+\frac{1}{k})^s}{\Gamma(s/k)} > 0$. (s!

- $\Omega_{k,s}$ is "essentially" the surface area measure of the surface $S_{k,s} = \{x_1, \dots, x_s \in \mathbb{R}^s_+ ;\ x_1^k + \dots + x_s^k = 1\}$ called the "singular integral"



- $\mathfrak{S}(N) \geq \gamma > 0$ is a "purely arithmetical" quantity, called the "singular series"

Thus as long as $s \geq s(k)$ suff. large one has $r(N) \sim \Omega_{k,s} \mathfrak{S}(N) N^{\frac{s}{k}-1}$

Note Why $N^{\frac{s}{k}-1}$ is the right order of magnitude?

If $1 \leq x_1, \dots, x_s \leq M$ then $1 \leq x_1^k + \dots + x_s^k \leq sM^k \lessapprox M^k$

$F : [1,M]^s \to [1, sM^k]$ thus if F takes each (large $N \approx M^k$) value roughly the same number of times then each value is taken $\approx M^{s-k} \approx N^{\frac{s}{k}-1}$ times

Analytic methods

Hardy-Littlewood: generating functions; $|z|<1$

$$\left(\sum_{n=1}^{\infty} z^{n^k}\right)^s = \left(\sum_{n_1=1}^{\infty} z^{n_1^k}\right)\left(\sum_{n_2=1}^{\infty} z^{n_2^k}\right)\cdots\left(\sum_{n_s=1}^{\infty} z^{n_s^k}\right) =$$

$$= \sum_{n_1,\dots,n_s=1}^{\infty} z^{n_1^k+\dots+n_s^k} = \sum_{N=1}^{\infty} r(N)\, z^N$$

and use Cauchy's formula on $|z|=\rho<1$ to estimate $r(N)$.

Vingradov: exponential sums: $e(\alpha)=e^{2\pi i\alpha}$, $P\geq 1$

let $T(\alpha)=\sum_{x=1}^{P} e(\alpha x^k)$, then

$$T(\alpha)^s = \sum_{N} R'(N)\, e(\alpha N), \quad \text{with } R'(N) = \#\{1\leq x_1,\dots,x_s\leq P : x_1^k+\dots+x_s^k=N\}$$

Note If $P\geq N^{\frac{1}{k}}$ then $R'(N)=R(N)$

Using the basic identity:

$$\int_0^1 e(m\alpha)\,d\alpha = \int_0^1 e^{2\pi i m\alpha}\,d\alpha = \begin{cases} 1 & \text{if } m=0 \\ 0 & \text{if } m\in\mathbb{Z},\ m\neq 0 \end{cases}$$

Indeed : $\int_0^1 e^{2\pi i m\alpha}\,d\alpha = \left[\frac{e^{2\pi i m\alpha}}{2\pi i m}\right]_0^1 = 1-1=0$ if $m\neq 0$.

Thus

$$R(N) = \int_0^1 T_p(\alpha)^s e(-N\alpha)\,d\alpha \qquad (1)$$

We'll first prove the asympt. formula (1.4) for $k=2$, $s=5$ and then for $s=2^k+1$ for $k>2$ following Hua.

Thm 1.1. (Weyl ineq.) Let $f(x)=\alpha x^k+\alpha_1 x^{k-1}+\cdots+\alpha_k$ $(\alpha\neq 0)$. Let $q,P\in\mathbb{N}$. If $\exists (a,q)=1$ s.t. $\left|\alpha-\frac{a}{q}\right|\leq\frac{1}{q^2}$, then

(i) For $k=2$ we have

$$\left|\frac{1}{P}\sum_{x=1}^{P} e(f(x))\right| \leq C \log P\left(P^{-1/2}+q^{-1/2}+\left(\frac{P^2}{q}\right)^{-1/2}\right)$$

(ii) For $k\geq 2$ we have $(\forall\varepsilon>0)$

$$\left|\frac{1}{P}\sum_{x=1}^{P} e(f(x))\right| \leq C_\varepsilon P^{\varepsilon}\left(P^{-1/K}+q^{-1/K}+\left(\frac{P^k}{q}\right)^{-1/K}\right)$$

with $K=2^{k-1}$

Note If $P^\delta\leq q\leq P^{k-\delta}$ then for $f(x)=\alpha x^k$, $\left|\alpha-\frac{a}{q}\right|\leq\frac{1}{q^2}$

we have that

$$|T(\alpha)| \leq C_\varepsilon P^{-\frac{\delta}{k}+\varepsilon} \implies |T(\alpha)|^s \leq C_\varepsilon P^{s-\frac{s\delta}{k}+\varepsilon}$$
$$\leq C_\varepsilon N^{\frac{s}{k}-\frac{s\delta}{kK}+\varepsilon}$$

If $s\delta > k2^{k-1}$ $\left(s > \delta^{-1} k 2^{k-1} \right)$, then $|T(\alpha)|^s \leq N^{\frac{s}{k}-1-\varepsilon}$

(Choosing $\varepsilon > 0$ small enough)

Combining with

<u>Lemma</u> (Dirichlet's principle) Let $\alpha \in [0,1]$. Then

$$\exists\ 1 \leq q \leq P^{k-1},\ (a,q)=1 \text{ s.t. } \left|\alpha - \frac{a}{q}\right| \leq \frac{1}{qP^{k-1}} \leq \frac{1}{q^2}$$

<u>Thus</u> If $q \geq P$ as well, then $|T(\alpha)|^s \leq C_\varepsilon P^{s-\frac{s}{K}+\varepsilon}$

$$= C_\varepsilon N^{\frac{s}{k}-\frac{s}{kK}+\varepsilon}$$
$$\leq C_\varepsilon N^{\frac{s}{k}-1-\varepsilon}$$
$$\left(2\varepsilon \leq \frac{s}{kK}-1\right)$$

Write $\mathfrak{M}_{a/q} = \left\{ \alpha ;\ \left|\alpha - \frac{a}{q}\right| \leq \frac{1}{qP^{k-1}} \right\}$

$$\mathfrak{M} = \bigcup_{q \leq P} \bigcup_{(a,q)=1} \mathfrak{M}_{a/q}, \quad \mathfrak{m} = [0,1] \setminus \mathfrak{M}$$

$$\implies \int_{\mathfrak{m}} |T(\alpha)|^s \, d\alpha \leq C N^{\frac{s}{k}-1}$$

$\sum \frac{1}{p} \lesssim 1$

<u>But</u> On $\mathfrak{M}_{a/q}$ $T(\alpha)$ can be evaluated very prec., <u>note</u>: $|\mathfrak{M}| \leq \sum_{a \leq q} \sum_{q \leq P} \frac{1}{qP^{k-1}} \lesssim \sum \frac{1}{P^{k-1}}$

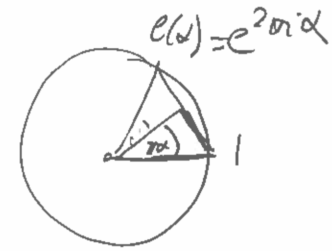
$< P^{-k+2} \lesssim N^{-1+}$[illegible]

Lemma 1.1. Let $\alpha \in \mathbb{R}$, $P \in \mathbb{N}$, then

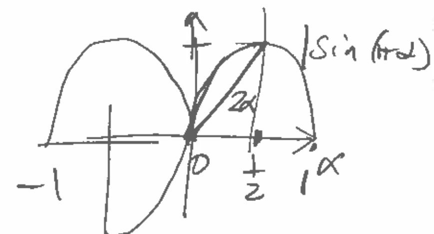
$$\left| \sum_{x=1}^{P} e(\alpha x) \right| \le \min\left\{ P, \frac{1}{2\|\alpha\|} \right\}; \quad \text{where } \|\alpha\| = \min_{m \in \mathbb{Z}} |\alpha - m|$$

Pf Write $z = e(\alpha)$ $\quad LHS = \left| \sum_{x=1}^{P} z^{x} \right| \qquad (= |\{\alpha\}|)$

$$LHS = \left| \frac{e(\alpha P) - 1}{e(\alpha) - 1} \right| = \left| \frac{\sin(\pi \alpha P)}{\sin(\pi \alpha)} \right| \le \frac{1}{|\sin(\pi\alpha)|} =$$



$$= \frac{1}{|\sin(\pi \|\alpha\|)|} \le \frac{1}{2\|\alpha\|}$$



$\square$

Note The same holds for any interval of integers $I = \{M+1, \dots, M+P\}$:

$$\left| \sum_{x \in I} e(\alpha x) \right| = \left| \sum_{x} e(\alpha x) \mathbb{1}_I(x) \right| \le$$

$$\le \min\left(P, \frac{1}{2\|\alpha\|}\right).$$

Lemma 1.2. Let $P, H \in \mathbb{N}$, $\alpha \in \mathbb{R}$ with $\left|\alpha - \frac{a}{q}\right| \le \frac{1}{q^2}$; $(a, q) = 1$.

Then
$$\sum_{h=1}^{H} \min\left(P, \frac{1}{\|\alpha h\|}\right) \le \left(\frac{H}{q} + 1\right)\left(P + q \log q\right)$$

Pf Divide the sum in h into blocks of length q (with perhaps one incomplete block); # of blocks $\lesssim \frac{H}{q}+1$

For each block one has the sum; $\alpha = \frac{a}{q}+\theta$, $|\theta| \le \frac{1}{q^2}$

$$\sum_{m=0}^{q-1} \min\left(P, \frac{1}{\|\alpha(m_0+m)\|}\right) \; ; \quad \alpha(m_0+m) = \alpha m_0 + \frac{am}{q} + \tilde{\theta}_\alpha, \ |\tilde{\theta}| = \frac{1}{q}$$

$$= \frac{am}{q} + \frac{b}{q} + o\left(\frac{1}{q}\right)$$

As $(a,q)=1$; $am+b \equiv r(q)$ has a unique solution $\forall\, 0 \le r < q$ thus, we have the sum with $\alpha m_0 = \frac{b}{q} + o\left(\frac{1}{q}\right)$

$$\sum_{s=0}^{q-1} \min\left(P, \frac{1}{\|\frac{s+\cdot}{q} + o(\frac{1}{q})\|}\right) = \sum_{s(q)} \min\left(P, \frac{1}{\|\frac{s}{q}+o(\frac{1}{q})\|}\right)$$

$$\ll P + \sum_{1 \le s \le q} \frac{q}{s} \le P + q \log q .$$

Since the no of blocks $\ll \frac{H}{q}+1$, we get

$$\left(\frac{H}{q}+1\right)\left(P + q\log q\right) \ll \left(\frac{HP}{q} + P + H + q\right) \log q$$

Pf of Thm 1.1.

(i) $p^{-1}T(\alpha) = p^{-1}\sum_{x} e(\alpha x^2 + \alpha_1 x)\ \chi_p(x) =$

$$|p^{-1}T(\alpha)|^2 \leq |\mathbb{E}_x\ e(\alpha f(x))\ \chi_p(x)|^2 =$$

$$= \mathbb{E}_x\, \mathbb{E}_y\, e(\alpha f(y))\, e(-\alpha f(x))\ \chi_p(y)\chi_p(x) \quad ;\ y = x+h$$

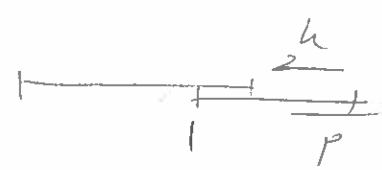
$$= \mathbb{E}_x\, \mathbb{E}_h\ e(\alpha\,[f(x+h) - f(x)])\ \chi_p(x+h)\,\chi_p(x)$$

$$\leq \mathbb{E}_h \left| \mathbb{E}_x\ e(\alpha\, D_h f(x))\ \Delta_h \chi_p(x) \right|$$

$$= p^{-2} \sum_h \left| \sum_x e(\alpha D_h f(x))\ \Delta_h \chi_p(x) \right|$$

with $\quad D_h f(x) = \alpha\,[(x+h)^2 - x^2] + \beta\,[(x+h) - x]$

$$= 2\alpha x h + \alpha h^2 + \beta h$$



$$= \mathbb{E}_h \left| \mathbb{E}_x\ e(\alpha x \cdot \underbrace{2h}_{h'})\ \Delta_h \chi_p(x) \right| \qquad 1-h \leq x \leq P-h$$

$$\Delta_h \chi_p(x) = \chi_p(x+h)\,\chi_p(x) = \begin{cases} 1 & \text{if } 1 \leq x+h \leq P,\ 1 \leq x \leq P \\ 0 & \text{o/w} \end{cases}$$

$$= \chi_{I,h}(x) \quad \text{interval of length} \leq P$$

$$\Rightarrow \quad |T(\alpha)|^2 \leq \sum_{|h| \leq 2P} \min\left\{ P, \frac{1}{\|\alpha h\|} \right\} \quad \text{by Lem 1.1.}$$

ANDE I/II – III

$$\Rightarrow |T(\alpha)|^2 \ll \left(\frac{P}{q}+1\right)(P+q)\log q$$

$$\ll \left(\frac{P^2}{q}+P+q\right)\log q = P^2\left(\frac{1}{P}+\frac{1}{q}+\frac{q}{P^2}\right)\log q$$

$$\Rightarrow |T(\alpha)| \ll P(\log q)\left(P^{-1/2}+q^{-1/2}+\left(\frac{P^2}{q}\right)^{-1/2}\right)$$

$$\ll P\log P\left(P^{-1/2}+q^{-1/2}+\left(\frac{P^2}{q}\right)^{-1/2}\right)$$ as long as $q \le P^2$

(if $q \geq P^2$ the estimate is trivial) $\square$