

Weyl estimates

$$S(\alpha) = \sum_m e^{2\pi i m^2 \alpha} \varphi_M(m)$$

$$|S(\alpha)|^2 = \sum_{m,n} e^{2\pi i (m^2 - n^2)} \varphi_M(m) \varphi_M(n) ; \text{ let } m = n+h$$

$$\leq \sum_h \left| \sum_n e^{2\pi i n(2h\alpha)} \varphi_M(n+h) \varphi_M(n) \right| \quad m^2 - n^2 = h^2 + 2nh$$

Note . $0 < n < M$ on $\text{supp } \varphi_M(n)$

• $0 < n+h < M \Rightarrow -M < h < M$

• $\varphi\left(\frac{n+h}{M}\right) - \varphi\left(\frac{n}{M}\right) = \psi_{h'}\left(\frac{n}{M}\right) \quad \text{with} \quad h' = \frac{h}{M}; |h'| \leq |$

$\psi_{h'} \in C^\infty$ with bounded derivatives unif. for $|h'| \leq |$.

Is it possible to change var's $h := 2h$ ~~then $h' = \frac{h}{2}$~~ -2 .

Poisson-Summatuon

$$\begin{aligned} \sum_n e^{2\pi i n \beta} \varphi_M(n) &= \sum_m \widehat{\varphi}_M(n-\beta) = M \sum_m \widehat{\varphi}(M(n-\beta)) \\ &\leq M \sum_m (1 + M|\beta|)^{-2} \leq M (1 + M\|\beta\|)^{-2} + M^{-1} \\ &\leq 2M (1 + M\|\beta\|)^{-2} \end{aligned}$$

$$\begin{aligned} \text{Thus } |S(\alpha)|^2 &\leq 2M \sum_{|h| \leq 2M} (1 + M\|h\alpha\|)^{-2} ; \quad \|\beta\| = \min_{n \in \mathbb{Z}} |\beta - n| \\ &\leq 2M + 4M \sum_{h=1}^{2M} (1 + M\|h\alpha\|)^{-2} \end{aligned}$$

• We use the fact that $|\alpha - \frac{\alpha}{q}| \leq \frac{1}{q^2}$.

Let $\{\beta\} = \beta - n$ when $n \in \mathbb{Z}$ is s.t. $|\beta - n|$ is minimal.

Then $-\frac{1}{2} \leq \{\beta\} \leq \frac{1}{2}$ and $\|\beta\| = |\{\beta\}|$.

Fix an interval I of length $\frac{q}{2}$. If $h_1, h_2 \in I$

$$\text{then } |\{h_2\alpha\} - \{h_1\alpha\}| = |(h_2\alpha - b_2) - (h_1\alpha - b_1)| = |(h_2 - h_1)\alpha - b|.$$

Write $h = h_2 - h_1$, $a = \frac{\alpha}{q} + \beta$ with $|\beta| \leq \frac{1}{q^2} \geq \|h\alpha\|$, $b = h_2 - h_1$

$$h\alpha = \frac{ha}{q} + h\beta \quad ; \quad \frac{ha}{q} = \frac{l}{q} + k, \quad 0 \leq l \leq \frac{q}{2}, \quad k \in \mathbb{Z} \quad \text{as } (a, q) = 1$$

$$|h\beta| \leq \frac{q}{2} \cdot \frac{1}{q^2} \leq \frac{1}{2q} \quad 1 \leq h < q$$

$$\Rightarrow \|h\alpha\| \geq \frac{1}{q} - \frac{1}{2q} \geq \frac{1}{2q}$$

Claim: $\sum_{h \in I} (1 + M\|h\alpha\|)^{-2} \leq \begin{cases} 10 & \text{if } q \leq M \\ \frac{16q}{M} & \text{if } q > M \end{cases}$

Pf Let $\alpha_n := \{h\alpha\} \subseteq [-\frac{1}{2}, \frac{1}{2}]$; then $|\alpha_{n_1} - \alpha_{n_2}| \geq \frac{1}{2q}$.

Consider $\alpha_n \geq 0$ and arrange them in increasing order.

Order $0 \leq \alpha_0 < \alpha_1 < \dots < \alpha_k \quad (k \leq \frac{q}{2}) \Rightarrow \alpha_k \geq \frac{k}{2q}$

Case 1: $2q \leq M$; $\sum_{k=0}^{\frac{q}{2}} \left(1 + \frac{Mk}{2q}\right)^{-2} \leq 1 + 4 \frac{q^2}{M^2} \leq 5$

Case 2 $2q > M$,

$$\sum_{k=0}^l \left(1 + \frac{Mk}{2q}\right)^{-2} = \sum_{b=0}^{2q/M} 1 + \sum_{b \geq 2q/M} \frac{4q^2}{M^2} b^{-2}$$

25/3

$$\leq \frac{2q}{M} + \frac{4q^2}{M^2} \times \frac{M}{2q} \leq \frac{4q}{M}$$

The same estimate holds for α'_k 's where $\alpha_k \leq 0$.

Cor 1. $S = \sum_{\|h\| \leq M} (1 + M \|h\| \alpha\|)^{-2} \leq 40 \left(\frac{M}{q} + \frac{q}{M}\right) = 40M \left(\frac{1}{q} + \frac{q^2}{M}\right)$

Pf: If $q \leq 2M$ then divide the sum into $\frac{2M}{q}$ groups of size $\frac{q}{2} \Rightarrow S \leq \frac{2M}{q} \cdot 10$

If $q > 2M$ then divide into ≤ 4 groups of length $\frac{q}{2}$

$$\Rightarrow S \leq \frac{40q}{M}$$

Cor 2. $|\tilde{S}_M(\alpha)| \leq 10M \left(\frac{1}{q} + \frac{q^2}{M}\right)^{-\frac{1}{2}}$

Pf $|\tilde{S}_M(\alpha)|^2 \leq M \cdot \sum_{\|h\| \leq M} (1 + M \|h\| \alpha\|)^{-2} \leq 40M^2 \left(\frac{1}{q} + \frac{q^2}{M}\right)$

□

Note • Usual Weyl sum $S_M(\alpha) = \sum_{1 \leq m \leq M} e^{2\pi i m^2 \alpha}$, $|S_M(\alpha)| \leq CM \log M \left(\frac{1}{q} + \frac{q^2}{M} \right)^{\frac{1}{2}}$

$$\text{b/c } \sum_{h \in I} (1 + M \|h\alpha\|)^{-1} \leq C \log q \leq C \log M \text{ if } q \leq M$$

- The method works for any integral polynomial

$$P(m) = a_k m^k + \dots + a_0$$

$$S_{M,P}(\alpha) = \sum_m e^{2\pi i P(m)\alpha} \psi_M(m)$$

$$\begin{aligned} |S_{M,P}(\alpha)|^2 &\leq \sum_{m, h_1} e^{2\pi i (P(m+h_1) - P(m))\alpha} \underbrace{\Delta_{h_1} \psi_M(m)}_{\psi_M(m+h_1) \psi_M(m)} \\ &\leq \sum_{h_1} \left| \sum_m e^{2\pi i \alpha D_{h_1} P(m)} \Delta_{h_1} \psi_M(m) \right|, \quad D_{h_1} P(m) = P(m+h_1) - P(m) - P(h_1) \end{aligned}$$

$$\deg D_{h_1} P = k-1$$

$$|S_{M,P}(\alpha)|^{2^{k-1}} \leq \sum_{h_1, \dots, h_{k-1}} \left| \sum_m e^{2\pi i \alpha D_{h_1, \dots, h_{k-1}} P(m)} \Delta_{h_1, \dots, h_{k-1}} \psi_M(m) \right|$$

where $D_{h_1, \dots, h_{k-1}} P(m) = a_k h_1 \dots h_{k-1} m^\alpha = h \cdot \alpha$ [multiplicity] $d(h)^k \leq M^\varepsilon$

then the inner sum again is

$$\sum_m e^{2\pi i m^\alpha} \psi_M(m)$$

$$\Rightarrow |S_{M,P}(\alpha)| \leq C_\varepsilon M^{1+\varepsilon} \left(\frac{1}{q} + \frac{q^2}{M} \right)^{\frac{1}{2^{k-1}}} ; \quad \forall \varepsilon > 0$$